Balance in W*-dynamical systems

by

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DECLARATION

I, Mathys Machiel Snyman, declare that the thesis, which I hereby submit for the degree Doctor of Philosophy : Physics at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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DATE: 30 April 2019

Abstract. Using a study of the connection between entanglement and quantum detailed balance as motivation, we define and study the concept of balance between two W^*-dynamical systems. Balance is defined in terms of certain correlated states (couplings), with entangled states as a specific case. Basic properties of balance are derived, and a connection with correspondences in the sense of Connes is discussed. The characterization, and possible generalizations of a quantum detailed balance condition is explored. A characterization of ergodicity in terms of balance is also given.

Contents

Introduction

In this thesis we define and study a general notion of balance between two W^*-dynamical systems, as motivated by a study of the connection between quantum detailed balance and entanglement in the finite dimensional case.

Entanglement is a central aspect of quantum physics and is important in several areas of physics, such as quantum information and statistical mechanics, whereas detailed balance is a form of microscopic reversibility that is closely related to equilibrium in statistical mechanics. The quantum version of detailed balance for open systems has been studied for many years, and research continues to the present day (see [**5**, **15**, **38**] for some of the early papers, and [**27**, **53**] for examples of more recent work). Ideas related to quantum detailed balance continue to play an important role in studying certain aspects of non-equilibrium statistical mechanics, in particular non-equilibrium steady states (see for example [**2**, **3**, **4**]). There are several different approaches to quantum detailed balance as illustrated in the mentioned papers, corresponding to different types of quantum detailed balance.

Connections between detailed balance and entangled states have in fact already been exploited in [**12**, **29**, **30**] with regards to entropy production for quantum Markov semigroups. However our interest will be more explicit, specifically how precisely the connection arises. We'll do this by deriving a characterization of quantum detailed balance conditions in terms of an entangled state. Balancing behaviour inherent to this characterization will then act as our motivation to define and study a more general and abstract notion of balance.

Due to the above connection between detailed balance and non-equilibrium statistical mechanics, the notion of balance is therefore potentially applicable to non-equilibrium statistical mechanics. However, in this thesis we will only develop the basics of a theory of balance as a foundation for further work.

In Chapter 1 we study the connection between quantum detailed balance and entanglement in finite dimensions, closely following [**24**], and culminating in characterizations of two quantum detailed balance conditions in terms of an entangled state. We conclude the chapter with a brief motivation as to how the "balancing behaviour" exhibited in the characterizations can be investigated more abstractly.

In Chapter 2 we cover mathematical background that will be used in Chapters 3 and 4. Various notations and conventions will also be discussed here.

In Chapter 3 we define the notion of balance between two W^*-dynamical systems in terms of a dual of a system, and a coupling of two states on von Neumann algebras. With the main definitions in place we then proceed with a study of the concept, investigating trivial examples, symmetry, transitivity and characterizations of ergodicity and quantum detailed balance in terms of balance. A connection with correspondences in the sense of Connes is also discussed. This chapter closely follows [**25**].

The thesis is concluded in Chapter 4 with a simple example from [**25**] that illustrates some of the ideas in Chapter 3. We define two W^*-dynamical systems on the same algebra and state pair, but with different possible dynamics, and then identify a "family" of possible couplings of the system's state with itself. Conditions for balance in terms of these couplings are then derived. We conclude the example by investigating a quantum detailed balance condition defined in Chapter 3, and ways in which the condition can be strictly weakened in terms of balance.

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Index of Symbols and Conventions

$\mathbb{N}$: Natural numbers $\{1, 2, 3, ...\}$.
$\mathbb{R}$: Field of real numbers.
$\mathbb{C}$: Field of complex numbers.
M_n : Shorthand notation for $M_{n\times n}(\mathbb{C})$, the space of all $n \times n$ matrices with $\mathbb{C}$-valued entries.
u.c.p. : Short for unital and completely positive.

$L(X)$: Space of linear operators $X \to X$.
$\mathscr{L}(X)$: Space of bounded linear operators $X \to X$.
$L^1(X)$: Space of trace-class operators $X \to X$.
X^* : Dual space of a normed space X, i.e. the space of all bounded linear functionals on X equipped with the operator norm.
$(A)_+$: The positive elements of a C^*-algebra A.
A' : The commutant of an algebra A.

id_A : The identity operator on an operator space restricted to a subspace A of an operators space. That is, $\mathrm{id}_A : a \mapsto a$ for all $a \in A$ where, e.g. $A \subset \mathscr{L}(H)$.
1_A : The unit of an algebra A, also sometimes denoted without the subscript if the context is clear.
$x \bowtie y$: The operator defined on an inner product space X by

$$x \bowtie y(z) = \langle y, z\rangle x$$

for all $z \in X$ and some fixed $x, y \in X$.
$[\cdot,\cdot]$: The commutator on an algebra A. That is, $[a, b] = ab - ba$ for all $a, b \in A$.
$\ominus$: The symmetric difference of two sets.
δ_μ : The diagonal coupling of μ with itself. See Eq. (28).
$\mu \odot \nu$: The product/trivial coupling of μ and ν. See Eq. (29).

(i) Unless explicitly stated otherwise, any vector space or algebra will be over the field of complex numbers.
(ii) An inner product of an inner product space will always be taken to be linear in the second argument and conjugate linear in the first argument.
(iii) In Chapter 1 only, we will denote inner products with the somewhat less standard notation $(\cdot, \cdot)$ so as to avoid confusion with the standard notation for an expected value, $\langle\cdot\rangle$. In all subsequent chapters we will use $\langle\cdot, \cdot\rangle$ to denote inner products.

CHAPTER 1

Detailed balance in finite dimensions

The main aim of this chapter is to show how two quantum detailed balance conditions can be characterized in terms of an entangled state ω. In Section 1.1 we derive ω as the purification of a mixed state of a quantum system with a finite dimensional Hilbert space, and in Section 1.2 we define two types of quantum detailed balance: detailed balance II, and standard quantum detailed balance w.r.t a reversing operation Θ. In Section 1.3 we then investigate the connection between detailed balance and entanglement, and derive characterizations of the these two types of quantum detailed balance in terms of ω .

The main results of this chapter are Theorem 1.3.3, which is a characterization of detailed balance II in terms of ω, and Proposition 1.3.4, which is a characterization of Θ standard quantum detailed balance in terms of ω. We conclude the chapter with a short heuristic argument as to how one may formulate and investigate the "balancing behaviour" implicit in the characterizations more abstractly.

1.1. Purifying with an entangled state

Here we set up a representation of the purification of a state, which will be convenient when we study the connection between detailed balance and entanglement in Section 1.3.

Consider a quantum system with an $n \geq 2$ dimensional Hilbert space H whose state is given by a density matrix ρ. We assume the state is mixed, i.e. $\mathrm{Tr}(\rho^2) < 1$. If A is an observable of the system, i.e. if A is a hermitian operator in $\mathscr{L}(H)$ then its expected value is given by

$$< A >= \mathrm{Tr}(\rho A). \tag{1}$$

For convenience we extend Eq. (1) to all operators $A \in M_n$ instead of just the hermitian ones

$$\omega_{H,\rho} : M_n \to \mathbb{C} : A \mapsto \mathrm{Tr}(\rho A). \tag{2}$$

Eq. (2) may be viewed as a representation of the state of the system since if $\mathrm{Tr}(\rho_1 A) = \mathrm{Tr}(\rho_2 A)$ for all $A \in M_n$ then $\rho_1 = \rho_2$. That is, ρ can be uniquely recovered from Eq. (2).

Consider next the composite system consisting of a copy of the system and itself. The composite system has state space $H \otimes H$ which we take to be the Kronecker product $H \otimes H = M_n$ where $x \otimes y = xy^{\intercal}$

for all $x, y \in H$, viewed as column vectors. The inner product on $H \otimes H$ then becomes the Hilbert-Schmidt inner product:

$$\begin{aligned}(x_1 \otimes y_1, x_2 \otimes y_2) &:= (x_1, x_2)(y_1, y_2) \\ &= x_1^\dagger x_2 \; y_1^\dagger y_2 \\ &= \operatorname{Tr}\left((x_1 y_1^\intercal)^\dagger (x_2 y_2^\intercal)\right)\end{aligned}$$

where $(\cdot)^\dagger$ is used to denote both the conjugate transpose of a column vector in $\mathbb{C}$ and the hermitian adjoint in M_n, which is also given by the conjugate transpose. Since the elementary tensors span $H \otimes H$ it therefore follows that

$$(\psi, \phi) = \operatorname{Tr}(\psi^\dagger \phi) \tag{3}$$

for all $\psi, \phi \in H \otimes H$.

A somewhat subtle but important thing to note here is that Eq. (3) is independent of the choice of basis for H. This follows since $(x, y) = x^\dagger y$ for all representations of two elements $x, y \in H$ as column vectors in $\mathbb{C}_n$

Suppose the composite system is in the pure state $\psi \in H \otimes H$. If T is an observable of the composite system then its expected value is given by

$$< T >= (\psi, T\psi).$$

Just as before we view the following extended linear map as a representation of the state of the composite system

$$\omega_{H \otimes H, \psi} : \mathscr{L}(H \otimes H) \to \mathbb{C} : T \mapsto (\psi, T\psi)\,.$$

If $T = A \otimes B$ where A, B are observables of the original system, and if $\psi = x \otimes y$ for some $x, y \in H$ then

$$\begin{aligned}T\psi &= (A \otimes B)x \otimes y \\ &= Ax \otimes By \\ &= Axy^\intercal B^\intercal \\ &= A\psi B^\intercal.\end{aligned} \tag{4}$$

Hence, since the elementary tensors span $H \otimes H$ it follows by the linearity of T that Eq. (4) holds for all $\psi \in H \otimes H$. It now follows from Eq. (3) that

$$\begin{aligned}\omega_{H \otimes H, \psi}(A \otimes B) &= (\psi, (A \otimes B)\psi) \\ &= (\psi, A\psi B^\intercal) \\ &= \operatorname{Tr}\left(\psi^\dagger A \psi B^\intercal\right)\end{aligned} \tag{5}$$

for all $A, B \in M_n$.

Note that $\omega_{H \otimes H, \psi}$ does not depend on our choice of basis for H either. Comparing Eq. (2) and (5) we see that if we were to identify an $r \in M_n$ such that $r^2 = \rho$ and $r^\dagger = r$ then

$$\omega_{H\otimes H,r}(A \otimes I) = \mathrm{Tr}(rAr) = \mathrm{Tr}(r^2 A) = \omega_{H,\rho}(A), \text{ and}$$
$$\omega_{H\otimes H,r}(I \otimes B) = \mathrm{Tr}(r^2 B^\intercal) = \mathrm{Tr}(Br^2) = \omega_{H,\rho}(B)$$

for all $A, B \in M_n$. That is, $\omega_{H\otimes H,r}$ would be a purification of $\omega_{H,\rho} = \langle\cdot\rangle$. However, since ρ is a positive operator in $\mathscr{L}(H)$, it has a unique positive square root $\rho^{1/2} \in \mathscr{L}(H)$ which is hermitian, so in particular it satisfies ${\rho^{1/2}}^\dagger = \rho^{1/2}$. For $\rho^{1/2}$ to be a pure state of the composite system it also needs to be unital, which follows from (3) since $(\rho^{1/2}, \rho^{1/2}) = \mathrm{Tr}({\rho^{1/2}}^\dagger \rho^{1/2}) = \mathrm{Tr}(\rho) = 1$.

For the remainder of this chapter we will denote $\omega_{H\otimes H,\rho^{1/2}}$ simply by ω, so

$$\omega(A \otimes B) = \mathrm{Tr}(\rho^{1/2} A \rho^{1/2} B^\intercal), \tag{6}$$

and since we are free to choose any basis in H we will assume w.l.o.g that the basis in H is such that ρ is diagonal.

That ω is an entangled state follows from our assumption that the state of the original system is mixed, i.e. that $\mathrm{Tr}(\rho^2) < 1$. If r is not an entangled state, in other words $r = x \otimes y$ for some $x, y \in H$, then it would follow from $\mathrm{Tr}(r^2) = \mathrm{Tr}(\rho) = 1$ that $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(r^4) = \mathrm{Tr}(r^2)^2 = 1$. This would contradict the assumption $\mathrm{Tr}(\rho^2) < 1$, i.e. that our original state is mixed. To see why, note that $r = x \otimes y = xy^\intercal \in M_n$ is the matrix representation of the operator $x \bowtie \overline{y} \in \mathscr{L}(H)$, from which it is easy to see that r^n is the matrix representation of the operator $(x, \overline{y})^{n-1} x \bowtie \overline{y}$. Hence

$$\mathrm{Tr}(r^n) = (x, \overline{y})^{n-1}\, \mathrm{Tr}(x \bowtie \overline{y}) = (x, \overline{y})^n = \mathrm{Tr}(r)^n$$

To summarize, ω is a pure state of the 2-system whose reduced states to both systems are given by $\langle\cdot\rangle$, i.e. by ρ, and since in statistical mechanics we are particularly interested in cases where ρ is not pure, it follows then that ω is an entangled state.

For the remainder of the chapter we will assume that ρ is invertible, i.e. all its eigenvalues are strictly positive.

1.2. Quantum detailed balance definitions

We now describe two definitions of quantum detailed balance for which the connection to the entangled state ω from Section 1.1 can be made in a particularly clear way.

For a simple and clear discussion of how one can rewrite the classical definition of detailed balance in a form that suggests the basic form of the definitions of quantum detailed balance presented below, please refer to [**29, 54**]. This gives some intuition regarding the origins of these definitions. For the origins of quantum detailed balance, see [**5**], [**7**], [**15**], [**38**] and [**41**].

As before we consider a system with n dimensional Hilbert space. We allow the system to interact with its environment, i.e. it is an open system. A standard approach to this situation is to model the time-evolution of the system in the Heisenberg picture as a quantum Markov semigroup (QMS) τ_t on the algebra M_n, where we take the time variable to be either continuous, i.e. $t \geq 0$, or discrete, i.e. $t = 0, 1, 2, 3, \ldots$. This means that for each t the corresponding τ_t is a completely positive linear map from M_n to itself which is also unital, i.e. $\tau_t(I) = I$, and furthermore the semigroup property $\tau_s \tau_t = \tau_{s+t}$ is satisfied. Extensive discussions as to when a QMS is a good approximation to the physical time-evolution is given for example in the books [**8**] and [**13**], but also see [**18**] for one of the original papers. A brief overview of the definition of completely positive maps and some related results are given in Section 2.3.

It turns out that for the framework presented in this section and the results discussed in the next, the semigroup property is not needed, so this assumption can in fact be dropped, which may be relevant when studying non-Markovian dynamics. We do however keep the rest of the above mentioned assumptions regarding τ_t, in which case we simply refer to τ_t as *dynamics*. The literature on detailed balance which is related to our approach typically assumes the semigroup property.

The first definition of quantum detailed balance we consider is from [**43**], and is called detailed balance II. In [**43**] the dynamics is only assumed to be positive, rather than completely positive, and they only consider the case of discrete time. We therefore adapt their approach to completely positive maps and also to include continuous time. Our results in the next section in fact still hold when working with positivity instead of complete positivity, but as is well known [**39**] there are convincing physical reasons to assume complete positivity, and this also happens to be mathematically convenient in many cases. In this regard also see again the books [**8**] and [**13**]. The above mentioned extension from discrete to continuous time on the other hand is a minor mathematical issue in our setup in this section. All our arguments in this section and the next work for both the case of continuous time and the case of discrete time.

We are going to define detailed balance of the dynamics τ_t of the system relative to a given fixed density matrix ρ of the system. The key mathematical idea to define and study detailed balance is to consider certain duals or adjoints of τ_t. In particular for detailed balance II we need the following:

With $\langle \cdot \rangle$ the expectation functional given by ρ as in the previous section, we can define the *dual* (relative to ρ) of any linear map $\alpha : M_n \to M_n$ as the linear map $\alpha' : M_n \to M_n$ such that

$$\langle \alpha'(A)B \rangle = \langle A\alpha(B) \rangle$$

for all $n \times n$ matrices A and B. Note that since ρ is invertible, such an α' necessarily exists and is unique, since it can be obtained from the Hermitian adjoint of α with respect to the inner product $(A, B)_\rho := \mathrm{Tr}(\rho A^\dagger B) = \langle A^\dagger B \rangle$. Indeed, denoting this Hermitian adjoint by α^ρ, it is easy to check that $\alpha'(A) = \alpha^\rho(A^\dagger)^\dagger$.

Definition 1.2.1. We say that τ_t as given above satisfies *detailed balance II with respect to* ρ if τ'_t is a completely positive unital linear map for every t.

As a general remark, note that if τ_t has the semigroup property, then τ'_t automatically has it as well, since

$$\langle \tau'_{s+t}(A)B \rangle = \langle A\tau_{s+t}(B) \rangle = \langle A\tau_s[\tau_t(B)] \rangle = \langle \tau'_t[\tau'_s(A)]B \rangle .$$

Note that roughly speaking detailed balance II boils down to requiring that the dual τ'_t is a sensible physical time-evolution.

Next we consider a type of standard quantum detailed balance (see[**19**], and also [**46**] for related work). The particular form of standard quantum detailed balance considered below was studied in [**30, 27**]. It will immediately be seen that it is defined in a form directly related to the entangled state ω, a point we come back to in the next section. It is defined in terms of a *reversing operation* $\Theta : M_n \to M_n$, meaning that Θ is a $*$-anti-automorphism, i.e. it is linear, $\Theta(A^\dagger) = \Theta(A)^\dagger$ and $\Theta(AB) = \Theta(B)\Theta(A)$, and we furthermore assume that Θ^2 is the identity map on M_n. Note that some form of time reversal plays a central role in a number of approaches to detailed balance; see for example [**5, 42**], and also the discussion in [**43**].

For any linear $\alpha : M_n \to M_n$ we define $\alpha^{(1/2)} : M_n \to M_n$ (relative to ρ) by

$$\mathrm{Tr}(\rho^{1/2}\alpha^{(1/2)}(A)\rho^{1/2}B) = \mathrm{Tr}(\rho^{1/2}A\rho^{1/2}\alpha(B))$$

for all $n \times n$ matrices A and B. We note that $\alpha^{(1/2)}$ exists and is uniquely determined. In fact it is easily seen to be given by

$$\alpha^{(1/2)}(A) = \rho^{-1/2}\alpha^\dagger(\rho^{1/2}A^\dagger\rho^{1/2})^\dagger\rho^{-1/2}$$

where $\alpha^\dagger$ is the Hermitian adjoint of α with respect to the Hilbert-Schmidt inner product. From this formula it also follows that $\alpha^{(1/2)}$ is positive if α is, and completely positive if α is. Furthermore, if τ_t is a QMS, it can be seen that $\alpha_t^{(1/2)}$ is as well. However, the semigroup property will again not be essential for our work.

The dual $\alpha^{(1/2)}$ is known as the *KMS-dual* of α, however we will not explore KMS-theory in this thesis.

Definition 1.2.2. We say that τ_t on M_n satisfies *standard quantum detailed balance with respect to the reversing operation* Θ *and the density matrix* ρ, abbreviated as Θ-*sqdb with respect to* ρ, if

$$\tau_t^{(1/2)} = \Theta \circ \tau_t \circ \Theta.$$

As the preceding references and introduction show, there are also a number of other definitions of quantum detailed balance in the literature. For remarks comparing some of these, refer to [**27**, **43**] in particular.

1.3. Detailed balance and entanglement

In this section we turn to the main goal of this chapter, namely to characterize quantum detailed balance in terms of the entangled state ω introduced in Section 1.1. As mentioned in Section 1.1, ρ is an invertible density matrix throughout and we have chosen some fixed basis in which ρ is diagonal to define the transposition. Furthermore, the term *dynamics* is as defined in the previous section.

The central tool towards our goal is the modular operator Δ defined by

$$\Delta(A) = \rho A \rho^{-1}$$

for all $n \times n$ matrices A. This operator is part of a very general theory, namely modular theory or Tomita-Takesaki theory, which is discussed for example in [**14**]. Section 2.2 contains a brief summary of the main operators as well as a principal result of the theory, but since we are still only working in finite dimensions we don't need to delve into the general theory here.

We start with a technical result regarding the modular operator in our finite dimensional context, which we will use to prove the main results in this section.

Lemma 1.3.1. *For any $z \in \mathbb{C}$, Δ^z is well defined and given by*

$$\Delta^z(A) = \rho^{-z} A \rho^z, A \in M_n. \tag{7}$$

Moreover, if $\alpha : M_n \mapsto M_n$ is linear then $\Delta\alpha = \alpha\Delta$ if and only if $\Delta^z\alpha = \alpha\Delta^z$ for some nonzero $z \in \mathbb{C}$.

Proof. From the definition of Δ, it is easily verified that $\Delta^\dagger = \Delta$, where again the Hermitian adjoint $\Delta^\dagger$ is taken with respect to the Hilbert-Schmidt inner product. I.e. Δ is self-adjoint, and similarly $\Delta^{1/2} := \rho^{1/2}(\cdot)\rho^{-1/2}$ is self-adjoint. The latter means that $\Delta = \Delta^{1/2}\Delta^{1/2} \geq 0$. Furthermore, $\Delta^{-1} = \rho^{-1}(\cdot)\rho$ exists so all of the eigenvalues of Δ are strictly positive, so in fact

$$\Delta > 0$$

as an operator on the Hilbert space M_n with the Hilbert-Schmidt norm. This means that Δ^z is well-defined for all $z \in \mathbb{C}$.

A convenient and standard representation of a linear map $\alpha : M_n \to M_n$, for example Δ above, is to arrange the columns of an $n \times n$ matrix in order below one another in an n^2 dimensional column, in which case α can be written as an $n^2 \times n^2$ matrix. This is just a choice of basis, and is essentially an explicit case of the GNS construction with respect to the trace (see for example [**14**] for the general GNS construction). In this representation $\alpha^\dagger$ is then easily seen to be represented by the Hermitian adjoint of the $n^2 \times n^2$ matrix (i.e. transpose and complex conjugation).

Since we are working in a basis in which ρ is diagonal it follows that in the above mentioned representation,

$$\Delta = \begin{bmatrix} \begin{bmatrix} \rho_1\rho_1^{-1} & & \\ & \ddots & \\ & & \rho_n\rho_1^{-1} \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} \rho_1\rho_n^{-1} & & \\ & \ddots & \\ & & \rho_n\rho_n^{-1} \end{bmatrix} \end{bmatrix} \tag{8}$$

where we have indicated $n \times n$ blocks for clarity. From this we see that

$$\Delta^z(A) = \rho^{-z} A \rho^z. \tag{9}$$

The final assertion follows from (8) and the easily verified observation that, if $A = [a_{ij}], B = [b_{ij}] \in M_n$ are the matrix representations of linear maps $\alpha_A, \alpha_B : \mathbb{C}^n \mapsto \mathbb{C}^n$ and B is diagonal, then $\alpha_A\alpha_B = \alpha_B\alpha_A$ if and only if $a_{ij}b_{jj} = a_{ji}b_{jj}$ for all $i, j = 1, ..., n$. □

We now proceed with the following characterization of detailed balance II in terms of the modular operator.

THEOREM 1.3.2. *The dynamics τ_t satisfies detailed balance II with respect to ρ if and only if it commutes with the modular operator, i.e.*

$$\tau_t\Delta = \Delta\tau_t, \tag{10}$$

and it leaves the state ρ invariant in the sense that

$$\langle \tau_t(A) \rangle = \langle A \rangle \tag{11}$$

for all $n \times n$ matrices A.

PROOF. To simplify the technical steps to follow we start by introducing another type of dual operator. Given any linear map $\alpha : M_n \to M_n$ we define the linear map $\alpha^\ddagger : M_n \to M_n$ by

$$\mathrm{Tr}[\alpha^\ddagger(A)B] = \mathrm{Tr}[A\alpha(B)].$$

Similar to α', $\alpha^\ddagger$ can be obtained from the usual Hermitian adjoint $\alpha^\dagger$ of the operator α with respect to the Hilbert-Schmidt inner product by the formula

$$\alpha^\ddagger(A) = \alpha^\dagger(A^\dagger)^\dagger$$

where the 2^{nd} and 3^{rd} $\dagger$ refer to the Hermitian adjoint of the $n \times n$ matrices A and $\alpha^\dagger(A^\dagger)$. Note that $\alpha(I) = I$ if and only if $\mathrm{Tr} \circ \alpha^\ddagger = \mathrm{Tr}$. It is similarly easy to see that $\langle \alpha(A) \rangle = \langle A \rangle$ for all A if and only if

$\alpha^{\ddagger}(\rho) = \rho$. In the case that α is a Hermitian map, i.e. it satisfies $\alpha(A^{\dagger}) = \alpha(A)^{\dagger}$, we see that $\alpha^{\dagger}$ is also Hermitian, since

$$\begin{aligned}\mathrm{Tr}\left[\alpha^{\dagger}(A^{\dagger})B\right] &= \mathrm{Tr}\left[A^{\dagger}\alpha(B)\right] = \{\mathrm{Tr}[\alpha(B^{\dagger})A]\}^* = \{\mathrm{Tr}[B^{\dagger}\alpha^{\dagger}(A)]\}^* \\ &= \mathrm{Tr}\left[\alpha^{\dagger}(A)^{\dagger}B\right].\end{aligned}$$

Therefore

$$\alpha^{\ddagger} = \alpha^{\dagger}$$

if α is Hermitian.

Assume that the dynamics τ_t satisfies detailed balance II with respect to ρ. That is, τ_t and τ_t' are both completely positive and unital. τ_t and τ_τ' are therefore in particular positive, and hence both are Hermitian maps.

Consider any linear $\alpha : M_n \to M_n$. It follows that $\langle\alpha'(A)B\rangle = \langle A\alpha(B)\rangle = \mathrm{Tr}[\alpha^{\ddagger}(\rho A)B] = \left\langle\rho^{-1}\alpha^{\ddagger}(\rho A)B\right\rangle$, and therefore

$$\alpha'(A) = \rho^{-1}\alpha^{\ddagger}(\rho A).$$

Furthermore,

$$\begin{aligned}\langle A\alpha(B)\rangle &= \langle\alpha'(A)B\rangle = \mathrm{Tr}[B\rho\alpha'(A)] = \mathrm{Tr}[\alpha'^{\ddagger}(B\rho)A] = \mathrm{Tr}[\rho A\alpha'^{\ddagger}(B\rho)\rho^{-1}] \\ &= \left\langle A\alpha'^{\ddagger}(B\rho)\rho^{-1}\right\rangle\end{aligned}$$

so $\alpha(B) = \alpha'^{\ddagger}(B\rho)\rho^{-1}$, i.e. $\alpha'^{\ddagger}(B\rho) = \alpha(B)\rho$. Assuming that α and α' are Hermitian, it follows that $\alpha^{\ddagger} = \alpha^{\dagger}$ and $\alpha'^{\ddagger} = \alpha'^{\dagger}$ are also Hermitian, therefore we also have $\alpha'^{\dagger}(\rho B) = \rho\alpha(B)$. Therefore

$$\begin{aligned}\langle A\alpha(B)\rangle &= \mathrm{Tr}[\rho A\rho^{-1}\alpha'^{\dagger}(\rho B)] = \mathrm{Tr}[\alpha'(\rho A\rho^{-1})\rho B] = \left\langle\alpha'(\rho A\rho^{-1})\rho B\rho^{-1}\right\rangle \\ &= \left\langle A\rho^{-1}\alpha(\rho B\rho^{-1})\rho\right\rangle\end{aligned}$$

from which it follows that $\alpha(B) = \rho^{-1}\alpha(\rho B\rho^{-1})\rho$.

I.e. we have shown that

$$\alpha\Delta = \Delta\alpha \tag{12}$$

if both α and α' are Hermitian maps. Since τ_t and τ_t' are both Hermitian maps as established above, (12) holds for $\alpha = \tau_t$ and all t.

Furthermore, Eq. (11) holds, since $\langle\tau_t(A)\rangle = \langle\tau_t'(I)A\rangle = \langle A\rangle$ directly from the definition of τ_t' and detailed balance II.

Now we prove the converse. First note that for a linear map $\alpha : M_n \to M_n$ we have that α is completely positive if and only if $\alpha^{\dagger}$ is completely positive. This follows immediately from the definition of $\alpha^{\dagger}$ and the fact [**39, 50**] that a linear map $\varphi : M_n \to M_n$ is completely positive if and only if it can be written in the form

$$\varphi(A) = \sum_{j=1}^{n^2} V_j A V_j^{\dagger}$$

for all A, for some set of matrices $V_j \in M_n$. (It can also be shown by a slightly longer argument that α is positive if and only if $\alpha^\dagger$ is positive.)

Assuming Eq. (10) and Eq. (11), we define $\varphi_t : M_n \to M_n$ by

$$\varphi_t(A) = \rho^{-1/2}\tau_t^\dagger(\rho^{1/2}A\rho^{1/2})\rho^{-1/2},$$

from which follows that

$$\begin{aligned}\langle \varphi_t(A)B \rangle &= \mathrm{Tr}[\rho\rho^{-1/2}\tau_t^\dagger(\rho^{1/2}A\rho^{1/2})\rho^{-1/2}B] = \mathrm{Tr}[\rho^{1/2}A\rho^{1/2}\tau_t(\rho^{-1/2}B\rho^{1/2})] \\ &= \left\langle A\rho^{1/2}\tau_t(\rho^{-1/2}B\rho^{1/2})\rho^{-1/2} \right\rangle = \langle A\tau_t(B) \rangle\end{aligned}$$

where in the last step we applied $\tau_t(\rho^z A\rho^{-z}) = \rho^z\tau_t(A)\rho^{-z}$ which follows from Lemma 1.3.1 and our assumption Eq. (10). This shows that $\tau_t' = \varphi_t$, i.e.

$$\tau_t'(A) = \rho^{-1/2}\tau_t^\dagger(\rho^{1/2}A\rho^{1/2})\rho^{-1/2}, \tag{13}$$

from which we conclude that τ_t' is completely positive, since τ_t and therefore $\tau_t^\dagger$ are. (Similarly, τ_t' is positive if we only assume that τ_t is positive.) Furthermore

$$\langle \tau_t'(I)A \rangle = \langle \tau_t(A) \rangle = \langle A \rangle ,$$

implying that τ_t' is unital. This shows that τ_t satisfies detailed balance II with respect to ρ as required. □

We will now use this characterization of detailed balance II to derive the characterization of detailed balance II in terms of the entangled state ω.

For any linear map $\alpha : M_n \to M_n$ we can define another linear map $\hat{\alpha} : M_n \to M_n$ by

$$\hat{\alpha}(A) = \alpha'(A^\intercal)^\intercal$$

where α' is as defined in Section 1.2. In order to formulate the characterization of detailed balance II in terms of ω, we apply this to the dynamics τ_t, i.e. we consider $\hat{\tau}_t$ given by

$$\hat{\tau}_t(A) = \tau_t'(A^\intercal)^\intercal \tag{14}$$

for all $n \times n$ matrices A and every t. Keep in mind that τ_t' and therefore $\hat{\tau}_t$ are mathematically well-defined operators for every t. However, it is only under the condition of detailed balance II that τ_t' becomes dynamics, i.e. that it is unital and completely positive. When this is the case, $\hat{\tau}_t$ similarly becomes dynamics.

Note that since the transpose appears in Eq. (14), the definition of $\hat{\tau}_t$ is basis dependent, so we have made a specific choice to fit in with ω from Section 1.1.

Now we can characterize detailed balance II in terms of entanglement.

THEOREM 1.3.3. *The dynamics τ_t satisfies detailed balance II with respect to ρ if and only if*

$$\omega[A \otimes \hat{\tau}_t(B)] = \omega[\tau_t(A) \otimes B] \tag{15}$$

for all $n \times n$ matrices A and B, and

$$\hat{\tau}_t(I) = I, \tag{16}$$

for every t. Alternatively Eq. (15) can be expressed as

$$\omega \circ (\mathrm{id}_{M_n} \otimes \hat{\tau}_t) = \omega \circ (\tau_t \otimes \mathrm{id}_{M_n}),$$

i.e. evolving the 2-system by $\mathrm{id}_{M_n} \otimes \hat{\tau}_t$ has the same effect on the entangled pure state ω as $\tau_t \otimes \mathrm{id}_{M_n}$, where id_{M_n} denotes the identity map on the algebra M_n.

PROOF. Assume that τ_t satisfies detailed balance II with respect to ρ. Then Eq. (13) holds, so by also using Eq. (6) and Eq. (14) it follows that

$$\begin{aligned}\omega[A \otimes \hat{\tau}_t(B)] &= \mathrm{Tr}[\rho^{1/2} A \rho^{1/2} \tau_t'(B^\intercal)] \\ &= \mathrm{Tr}[\rho^{1/2} A \rho^{1/2} \rho^{-1/2} \tau_t^\dagger(\rho^{1/2} B^\intercal \rho^{1/2}) \rho^{-1/2}] \\ &= \mathrm{Tr}[\tau_t(A) \rho^{1/2} B^\intercal \rho^{1/2}] = \omega[\tau_t(A) \otimes B],\end{aligned}$$

i.e. Eq. (15) holds. Since $\tau_t'(I) = I$ because of detailed balance II, we also have Eq. (16) by Eq. (14).

Conversely, assuming Eqs. (15) and (16), we are going to use Theorem 1.3.2 to show that τ_t satisfies detailed balance II with respect to ρ. Since

$$\omega(A \otimes B) = \mathrm{Tr}(\rho B^\intercal \rho^{1/2} A \rho^{-1/2}) = \left\langle B^\intercal \Delta^{1/2}(A) \right\rangle,$$

we have by our assumption Eq. (15) that

$$\begin{aligned}\left\langle B^\intercal \tau_t[\Delta^{1/2}(A)] \right\rangle &= \left\langle \tau_t'(B^\intercal) \Delta^{1/2}(A) \right\rangle = \omega[A \otimes \tau_t'(B^\intercal)^\intercal] = \omega[A \otimes \hat{\tau}_t(B)] \\ &= \omega[\tau_t(A) \otimes B] = \left\langle B^\intercal \Delta^{1/2}[\tau_t(A)] \right\rangle\end{aligned}$$

which means that $\tau_t \Delta^{1/2} = \Delta^{1/2} \tau_t$, hence $\tau_t \Delta = \Delta \tau_t$. Furthermore,

$$\langle \tau_t(A) \rangle = \omega[\tau_t(A) \otimes I] = \omega[A \otimes \hat{\tau}_t(I)] = \langle A \rangle,$$

since we assumed that $\hat{\tau}_t(I) = I$. The conditions in Theorem 1.3.2 are therefore satisfied, implying that τ_t satisfies detailed balance II with respect to ρ. $\square$

Next we consider a similar characterization of Θ-sqdb. The definition of Θ-sqdb is indeed already in a form that is aligned with ω. We simply define $\alpha^\Theta : M_n \to M_n$ by

$$\alpha^\Theta(A) = (\Theta \circ \alpha \circ \Theta(A^\intercal))^\intercal$$

for any linear $\alpha : M_n \to M_n$. Then one can immediately reformulate the definition of Θ-sqdb to obtain the following characterization which is inherent to the work in [**12**, **29**, **30**]:

PROPOSITION 1.3.4. *The dynamics τ_t satisfies Θ-sqdb with respect to ρ if and only if*

$$\omega[A \otimes \tau_t^{\Theta}(B)] = \omega[\tau_t(A) \otimes B]$$

for all $n \times n$ matrices A and B and every t.

A typical choice of Θ is $\Theta(A) = A^{\intercal}$. In this case $\tau_t^{\Theta} = \tau_t$ and the above condition simplifies to

$$\omega[A \otimes \tau_t(B)] = \omega[\tau_t(A) \otimes B].$$

So this choice of Θ seems to fit in naturally with our choice of ω.

It is straightforward to construct examples of Θ-sqdb in M_2 where τ_t does not commute with Δ, unlike the case of detailed balance II. This aspect of standard quantum detailed balance was emphasized in for example [**27**].

On the other hand, should we assume that τ_t does commute with Δ, one can show that Θ-sqdb implies detailed balance II.

We also mention that all of the results in this section still hold if we work in terms of positivity instead of complete positivity, as discussed in the previous section.

Theorem 1.3.3 and Proposition 1.3.4 gives us a hint as to how we may explore a more abstract version of detailed balance. In both results the dynamics of the system, and a kind of dual mechanics, is "balanced" relative to a state defined on a composition of the system with itself, and derived from the original system's state. So by considering a state on a composition of two different systems, derived from or related to both systems' states in some way, and then requiring a condition such as Eq. (15) to hold where $\hat{\tau}_t$ is the dynamics on the other system, one could potentially determine a more abstract and far more versatile "balance" condition.

In Chapter 3 we will show how this can be done in the context of W^*-dynamical systems in such a way that, one can not only easily characterize a quantum detailed balance condition, but one can easily generalize the condition in a number of natural ways. Moreover, we'll see how properties not directly related to detailed balance can be characterized and generalized as well, as we'll illustrate with an ergodic property in Section 3.3

CHAPTER 2

Mathematical Background

In this chapter we cover some of the main mathematical tools that we will be working with in the next two chapters. We also introduce notations and explain some conventions that we will adopt in the remainder of the thesis.

In Section 2.1 we briefly discuss and define cyclic representations, which will form the backbone of a great deal of analysis in the next chapter. We then prove the existence of a cyclic representation given by a Hilbert space in a special case that will be of importance.

In Section 2.2 we briefly discuss von Neumann algebras, in terms of which W^*-systems are defined in Chapter 3. We also derive a useful characterization for continuity on a von Neumann algebra relative to the topology that we will be mainly interested in, the σ-weak topology. We then provide a brief summary of the operators of Tomita Takesaki theory, and state one of the theory's main results.

In Section 2.3 we define completely positive maps between C^*-algebras and list two results that we will use multiple times: a general characterization of completely positive maps by Stinespring, known as the Stinespring dilation, and Kadison's inequality, which is an inequality that holds for completely positive maps that are also contractions.

Finally, in Section 2.4 we prove that, for a Hilbert space H with a countable total orthonormal basis, $(\mathscr{L}(H) \otimes 1)' = 1 \otimes \mathscr{L}(H)$ and $(1 \otimes \mathscr{L}(H))' = \mathscr{L}(H) \otimes 1$ in $\mathscr{L}(H \otimes H)$.

2.1. Cyclic representations

The majority of our analysis of systems, as will be defined in the next chapter, will be done on Hilbert space level through cyclic representations. In this section we briefly relate the definition of a cyclic representation, discuss some of the conventions we will adopt and then prove the existence of such a representation in a special case that will be important to us.

Definition 2.1.1. A *cyclic representation* for a unital $*$-algebra and state pair (A, μ) is a triple (X, π, Ω) with X an inner-product space, $\Omega \in X$ and $\pi : A \to L(X)$ a linear operator that satisfies

$$\overline{\pi(A)\Omega} = \overline{X}$$
$$\mu(a) = \langle \Omega, \pi(a)\Omega \rangle$$

for all $a \in A$, where $\overline{X}$ is the completion of X relative to the norm determined by its inner product.
A *cyclic representation* for a unital C^*-algebra and state pair (A, μ) is a triple (H, π, Ω) with H a Hilbert space, $\Omega \in H$ and $\pi : A \to \mathscr{L}(H)$ a linear $*$-morphism that satisfies

$$\overline{\pi(A)\Omega} = H$$
$$\mu(a) = \langle \Omega, \pi(a)\Omega \rangle$$

for all $a \in A$.

By a state μ on the unital $*$-algebra A we mean that $\mu : A \to \mathbb{C}$ is linear, $\mu(1) = 1$ and $\mu(a^*a) \geq 0$ for all $a \in A$. With this definition of a state the existence of a cyclic representation in both cases of Definition 2.1.1 is established through the GNS construction, and in the case of a C^*-algebra it follows that the cyclic representation is unique up to unitary transformations (see [**14**, Section 2.3.3] for details). Moreover, if a state μ on a C^*-algebra A is faithful then (A, μ) has a faithful cyclic representation (H, π, Ω). That is, π is a $*$-isomorphism onto its range. In that case it is sometimes convenient to assume, without loss of generality, that (A, μ) is in its cyclic representation from the start. In other words, it has cyclic representation $(H, \mathrm{id}_A, \Omega)$ with $\mathrm{id}_A : A \to \mathscr{L}(H)$ the identity operator.

The special case we will be interested is a state ω on the algebraic tensor product $A \odot B$ where A, B are C^*-algebras. This is a unital $*$-algebra and state pair, however because A and B are C^*-algebras a Hilbert space cyclic representation can be shown to exist.

PROPOSITION 2.1.2. *Let A and B be C^*-algebras, and ω a state on their algebraic tensor product $A \odot B$. Then ω is bounded with respect to the maximal C^*-norm on $A \odot B$.*

A proof of this known proposition can be found in [**22**, Proposition 4.1]

THEOREM 2.1.3. *$(A \odot B, \omega)$ in Proposition 2.1.2 has a cyclic representation (H, π, Ω) where H is a Hilbert space, $\Omega \in H$ is a unit vector and $\pi : A \odot B \to \mathscr{L}(H)$ is a $*$-morphism satisfying*

$$\overline{\pi(A \odot B)\Omega} = H$$
$$\omega(z) = \langle \Omega, \pi(z)\Omega \rangle$$

for all $z \in A \odot B$.

PROOF. Equip $A \odot B$ with the maximal C^*-norm $\|\cdot\|_m$. Then $A \odot B$ is a normed space and by Proposition 2.1.2 ω is bounded relative to $\|\cdot\|_m$. That is, $\omega_m \in (A \odot B)^*$, so ω has a unique bounded linear extension to $A \otimes_m B$, the maximal tensor product of A and B. We denote this extension by ω as well and assume without loss of generality that $A \odot B \subset A \otimes_m B$, so $\overline{A \odot B} = A \otimes_m B$.

It is easily verified that ω is a state on the C^*-algebra $A \otimes_m B$, thus $(A \otimes_m B, \omega)$ has a cyclic representation (G, φ, Ω) in the sense of Definition 2.1.1. It follows that (H, π, Ω) is a cyclic representation with the required properties where $H = \overline{\varphi(A \odot B)\Omega}$, and $\pi : A \odot B \to \mathscr{L}(H) : z \mapsto \varphi(z)|_H$. □

2.2. von Neumann algebra tools

There are a couple of different equivalent definitions of von Neumann algebras, depending on the source text, but arguably the simplest one is in terms of the double commutant:

Definition 2.2.1. If H is a Hilbert space and M is a $*$-subalgebra of $\mathscr{L}(H)$ then M is a *von Neumann algebra* if $M'' = M$.

A trivial example of a von Neumann algebra therefore is $\mathscr{L}(H)$ itself, since $\mathscr{L}(H)' = \mathbb{C}1$ and $(\mathbb{C}1)' = \mathscr{L}(H)$. Also note that a von Neumman algebra in $\mathscr{L}(H)$ necessarily contains the identity operator $1 : H \to H$.

The topology that we will be most concerned with on a von Neumann algebra is the σ-weak topology on $\mathscr{L}(H)$, i.e. the locally convex topology generated by the family of seminorms $\{|\operatorname{Tr}(\cdot u)| : u \in L^1(H)\}$ on $\mathscr{L}(H)$. We will also use the term *normal* to refer to σ-weakly continuous maps. This not only keeps the terminology simple, but reflects the fact that a $*$-morphism between von Neumann algebras is normal if and only if it is σ-weakly continuous, where a normal $*$-morphism $\eta : M \to N$ is by definition a $*$-morphism which satisfies

$$\phi(\sup\{x_i\}) = \sup\{\phi(x_i)\}$$

for all bounded increasing nets $(x_i) \subset M$. See [**10**, III.2.2] for further details.

It's a fundamental fact that the σ-weak topology is a w^* topology, which can make it easier to work with. For example, the σ-weak topology inherits useful w^*-topology properties such as:

Proposition 2.2.2. *Let X and Y be normed spaces, and V a subset of X^*. A map $\eta : V \to Y^*$ is w^*-continuous if and only if, for all w^*-continuous $\theta : Y^* \to \mathbb{C}$, the map $\theta \circ \eta : V \to \mathbb{C}$ is w^*-continuous.*

Proof. The reverse implication is obvious and the forward implication follows directly from the fact [**44**, A.2. Theorem] that $\{\theta_y : Y^* \to \mathbb{C} : g \mapsto g(y)\}$ represents all w^*-continuous functions $Y^* \mapsto \mathbb{C}$. That is, if (f_λ) is a net in X^* converging to some $f \in X^*$ then

$$\theta_y \circ \eta(f_\lambda) = \eta(f_\lambda)(y) \longrightarrow \eta(f)(y) = \theta_y \circ \eta(f)$$

for all $y \in Y$. Thus $\eta(f_\lambda) \longrightarrow \eta(f)$ by definition of the w^*-topology. □

Since the σ-weak topology is a w^*-topology, Proposition 2.2.2 can be used to establish the following criteria for normal maps:

THEOREM 2.2.3. *Let H, K be Hilbert spaces and $M \subset \mathscr{L}(H)$ an arbitrary subset. A map $\eta : M \to \mathscr{L}(K)$ is normal if and only if $\theta \circ \eta$ is normal for all normal positive linear functionals $\theta : \mathscr{L}(K) \to \mathbb{C}$.*

PROOF. By [44, Thm 4.2.3], $\Gamma : \mathscr{L}(H) \to L^1(H)^* : u \mapsto \operatorname{Tr}(\cdot u)$ is an isometric linear isomorphism. Since the σ-weak topology on $\mathscr{L}(H)$ is generated by semi-norms of the form

$$\mathscr{L}(H) \to \mathbb{R}^+ : u \mapsto |\operatorname{Tr}(uv)|, v \in L^1(H)$$

and the w^*-topology on $L^1(H)$ by semi-norms of the form

$$L^1(H)^* \to \mathbb{R}^+ : \operatorname{Tr}(\cdot u) \mapsto |\operatorname{Tr}(uv)|, v \in L^1(H)$$

it is clear that Γ defines a homeomorphism between $\mathscr{L}(H)$ equipped with the σ-weak topology, and $L^1(H)^*$ equipped with the w^*-topology. Therefore the proposition will follow from Proposition 2.2.2 if we can show that $\theta \circ \eta$ is normal for all normal linear functionals $\theta : \mathscr{L}(K) \to \mathbb{C}$. Hence what remains to be shown is that if $\theta \circ \eta$ is normal for all *positive* normal linear functionals $\theta : \mathscr{L}(K) \to \mathbb{C}$, then it is normal for all normal linear functionals $\theta : \mathscr{L}(K) \to \mathbb{C}$.

Consider any normal linear functional $\theta : \mathscr{L}(K) \to \mathbb{C}$. By [44, Theorem 4.2.10] there is a $v \in L^1(K)$ such that

$$\theta(u) = \operatorname{Tr}(uv) \tag{17}$$

for all $u \in \mathscr{L}(K)$, and it is easy to see that $\theta \geq 0$ if $v \geq 0$. For any $a \in \mathscr{L}(K)$ and any hermitian $b \in \mathscr{L}(K)$,

$$a_0 = \frac{1}{2}(a + a^*); a_1 = \frac{1}{2i}(a - a^*); b^+ = \frac{1}{2}(|b| + b); b^- = \frac{1}{2}(|b| - b)$$

are all trace-class if a, b are. See [44, Theorem 2.4.15]. Setting $v = v_0^+ - v_0^- + iv_1^+ - iv_1^-$ in (17) we see that

$$\theta(u) = \operatorname{Tr}(uv_0^+) - \operatorname{Tr}(uv_0^-) + i\operatorname{Tr}(uv_1^+) - i\operatorname{Tr}(uv_1^-).$$

Thus θ can be expressed as a linear combination of normal positive functionals since $v_0^+, v_0^-, v_1^+, v_1^- \geq 0$.

Hence the results follows since vector operations are continuous in the σ-weak topology. $\square$

We proceed by giving a brief summary of the relevant terminology and results from Tomita-Takesaki theory that we will use. Let A be a von Neumann algebra on a Hilbert space H, with a cyclic and separating vector Ω. That is, $\overline{A\Omega} = H$ and if $a \in A$, then $a\Omega = 0$ only if $a = 0$. Define a conjugate linear operator S_0 on $A\Omega$ by

$$S_0 a\Omega = a^*\Omega$$

for all $a \in A$. Then S_0 is a closable operator, and we denote its closure by S (see [49, Chapter 1] for a detailed development of closed operators). That is, if $x_n \to x$ *and* $S_0 x_n \to y$ in H, then $\overline{S}x := y$. $\overline{S}$ is a densely defined closed operator, and has a polar decomposition

$$S = J\Delta^{\frac{1}{2}} \tag{18}$$

(see [**37**, Section 9.2] and [**14**, Definition 2.5.10.]) where Δ is called the modular operator associated with (A, Ω) and J is called the modular conjugation associated with (A, Ω).

We will only use the J operator in Eq. (18), which is an anti-unitary operator satisfying $J^* = J$, $J\Omega = \Omega$ and, since it is anti-unitary, $J^2 = J^*J = 1$.

A principal result from Tomita-Takesaki theory is as follows:

Theorem 2.2.4. *Let A be a von Neumann algebra with associated modular conjugation J. Then*

$$JAJ = A'.$$

2.3. Completely positive maps

In Chapter 1 we considered dynamics given by completely positive maps on finite dimensional matrices, however in Chapter 3 we will consider dynamics given by completely positive maps on general von Neumann algebras (equipped with a faithful state). In this section we briefly discuss some results and go over the definition of completely positive maps, which is in terms of matrix algebras with C^*-algebra entries:

It is easy to show that for a Hilbert space H, $\mathscr{L}(H^n)$ is $*$-isomorphic to $M_n(\mathscr{L}(H))$, where if $[a_{ij}] \in M_n(\mathscr{L}(H))$ is the matrix representation of $T \in \mathscr{L}(H^n)$, then $[b_{ij}]$ with $b_{ij} = a_{ji}^*$ is the matrix representation of $T^* \in \mathscr{L}(H^n)$. It follows that if $A \subset \mathscr{L}(H)$ is a C^*-algebra then $M_n(A)$ acts as a C^*-algebra of operators on H^n. More generally, in this sense we can consider $M_n(A)$ for any C^*-algebra A by first taking a faithful representation π of A on some Hilbert space H.

Definition 2.3.1. Let A and B be C^*-algebras and $\phi : A \to B$ a linear map. Then ϕ is said to be *completely positive* if for all $n \in \mathbb{N}$,

$$\phi^{(n)} : M_n(A) \to M_n(B) : [a_{ij}] \mapsto [\phi(a_{ij})]$$

is positive.

Note that a completely positive map is necessarily positive, and hence $*$-linear in particular. That is, if $\phi : A \to B$ is a positive (linear) map between C^*-algebras A and B, and for any $a \in A$

$$a = \sum_i \alpha_i a_i$$

is a expressed as a finite linear combination of the positive elements $a_i \in A_+$, then it follows that

$$\phi(a^*) = \phi\left(\sum_i \overline{\alpha_i} a_i\right) = \sum_i \overline{\alpha_i}\phi(a_i) = \left(\sum_i \alpha_i \phi(a_i)\right)^* = (\phi(a))^*$$

A useful characterization of completely positive maps was derived by Stinespring in his original 1955 paper and is known as the *Stinespring dilation*:

THEOREM 2.3.2. *Let A be a unital C^*-algebra, H a Hilbert space and $\phi : A \to \mathscr{L}(H)$ a linear map. Then ϕ is completely positive if and only if there is a Hilbert space K, an operator $V \in \mathscr{L}(H, K)$ and a representation π of A on K such that*

$$\phi(a) = V^*\pi(a)V.$$

PROOF. For the original proof by Stinespring, refer to [50]. □

Most completely positive maps we will consider will also be unital, which are necessarily contractions by [10, Proposition II.6.9.4], and completely positive contractions satisfy *Kadison's inequality* which we will encounter several times in the next chapter:

PROPOSITION 2.3.3. *Let A, B be C^*-algebras and $\phi : A \to B$ a completely positive contraction. Then*

$$\phi(a^*a) \geq \phi(a)^*\phi(a) \tag{19}$$

for all $a \in A$.

PROOF. Follows easily from the Stinespring dilation. See [10, II.6.9.14 Proposition] □

2.4. The commutant of $\mathscr{L}(H) \otimes 1$ in $\mathscr{L}(H \otimes H)$

In Chapter 4 we will construct an example to illustrate some of the ideas from Chapter 3, and we will use the fact that $\mathscr{L}(H) \otimes \mathscr{L}(H) \subset \mathscr{L}(H \otimes H)$, relative to the spatial tensor product, and that $(\mathscr{L}(H) \otimes 1)' = 1 \otimes \mathscr{L}(H)$ in $\mathscr{L}(H \otimes H)$. The former is well known (see for example [36, Proposition 2.6.12]), and the latter can be proved using theory from the tensor products of von Neumann algebras (see [52, IV. Theorem 5.9.]). However, we won't consider a general Hilbert space H in Chapter 4, instead we'll assume it has a countable orthonormal basis. For curiosity's sake we'll directly prove the latter result for this special case.

Let H be a Hilbert space with a total orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$. Then $\{e_i \otimes e_j : i, j \in \mathbb{N}\}$ is a countable orthonormal basis for $H \otimes H$. We can thus consider an infinite matrix representation of any $a \in \mathscr{L}(H \otimes H)$

given by matrix entries $\{\alpha_{j,k,l,m} : j,k,l,m \in \mathbb{N}\}$, relative to the basis $\{e_i \otimes e_j : i,j \in \mathbb{N}\}$. That is, analogous to the finite dimensional case,

$$a = \sum_{j,k,l,m\in\mathbb{N}} \alpha_{j,k,l,m}\ e_j \otimes e_l \bowtie e_k \otimes e_m. \tag{20}$$

Note however that this is only a way to express the matrix representation of a in the above basis. That is, Eq. (20) does not imply the convergence of an infinite series. In fact the series in Eq. (20) will not converge in general. Instead the interpretation of Eq. (20) is that a is the operator in $\mathscr{L}(H \otimes H)$ uniquely determined by:

$$a(e_k \otimes e_m) = \sum_{j,l\in\mathbb{N}} \alpha_{j,k,l,m}\ e_j \otimes e_l \tag{21}$$

for all $e_k \otimes e_m \in H \otimes H$. We will not directly utilize this matrix representation, but we include it as it is occasionally useful to "visualize" an operator $a \in \mathscr{L}(H \otimes H)$ as an infinite matrix.

THEOREM 2.4.1. *$(\mathscr{L}(H) \otimes 1)' = 1 \otimes \mathscr{L}(H)$ and $(1 \otimes \mathscr{L}(H))' = \mathscr{L}(H) \otimes 1$ in $\mathscr{L}(H \otimes H)$.*

PROOF. It is clear that $1 \otimes \mathscr{L}(H) \subset (\mathscr{L}(H) \otimes 1)'$, so we only have to show that $(\mathscr{L}(H) \otimes 1)' \subset 1 \otimes \mathscr{L}(H)$.

Consider any $a \in \mathscr{L}(H \otimes H)$ defined by

$$a(e_k \otimes e_m) = \sum_{j,l\in\mathbb{N}} \alpha_{j,k,l,m}\ e_j \otimes e_l \tag{22}$$

for all $k, m \in \mathbb{N}$ as explained above. For any $p, q \in \mathbb{N}$, let us denote the operator $e_p \bowtie e_q \in \mathscr{L}(H)$ simply by e_{pq}.

Assume that $a \in (\mathscr{L}(H) \otimes 1)'$. Thus, in particular,

$$a(e_{pq} \otimes 1) = (e_{pq} \otimes 1)a$$

for all $p, q \in \mathbb{N}$. Consider any $e_s \otimes e_t \in H \otimes H$. It follows that

$$\begin{aligned}
(e_{pq} \otimes 1)a(e_s \otimes e_t) &= (e_{pq} \otimes 1) \sum_{j,l\in\mathbb{N}} \alpha_{j,s,l,t}\ e_j \otimes e_l \\
&= \sum_{j,l\in\mathbb{N}} \alpha_{j,s,l,t}\ e_{pq}e_j \otimes e_l \\
&= \sum_{l=1}^{\infty} \alpha_{q,s,l,t}\ e_p \otimes e_l
\end{aligned}$$

where we used the continuity of $e_{pq} \otimes 1 \in \mathscr{L}(H) \otimes \mathscr{L}(H) \subset \mathscr{L}(H \otimes H)$. It also follows that

$$\begin{aligned} a(e_{pq} \otimes 1) e_s \otimes e_t &= a(e_{pq} e_s \otimes e_t) \\ &= \delta_{q,s} a(e_p \otimes e_t) \\ &= \delta_{q,s} \sum_{j,l \in \mathbb{N}} \alpha_{j,p,l,t}\ e_j \otimes e_l \end{aligned}$$

where δ is the Kronecker delta function. That is, $\delta_{q,s} = 0$ if $q \neq s$, and $\delta_{q,q} = 1$ for all $q \in \mathbb{N}$.

Hence, since a commutes with e_{pq} we have that

$$\sum_{l=1}^{\infty} \alpha_{q,s,l,t}\ e_p \otimes e_l = \delta_{q,s} \sum_{j,l \in \mathbb{N}} \alpha_{j,p,l,t}\ e_j \otimes e_l \tag{23}$$

for all $p, q, s, t \in \mathbb{N}$. It is clear that we must have $\alpha_{j,k,l,t} = 0$ if $j \neq k$, so Eq. (23) simplifies to

$$\sum_{l=1}^{\infty} \alpha_{q,q,l,t}\ e_p \otimes e_l = \sum_{l=1}^{\infty} \alpha_{p,p,l,t}\ e_p \otimes e_l$$

for all $p, q, t \in \mathbb{N}$, so it also follows that, for any $l, t \in \mathbb{N}$, $\alpha_{q,q,l,t} = \alpha_{p,p,l,t}$ for all $p, q \in \mathbb{N}$. We can thus set $\varphi_{l,t} := \alpha_{p,p,l,t}$ for any $p, l, t \in \mathbb{N}$.

Applying this to Eq. (22) it follows for any $e_k \otimes e_m \in H \otimes H$ that

$$\begin{aligned} a(e_k \otimes e_m) &= \sum_{j,l \in \mathbb{N}} \alpha_{j,k,l,m}\ e_j \otimes e_l \\ &= \sum_{l=1}^{\infty} \alpha_{k,k,l,m}\ e_k \otimes e_l \\ &= e_k \otimes \sum_{l=1}^{\infty} \varphi_{l,m}\ e_l \\ &= (1 \otimes d) e_k \otimes e_m \end{aligned}$$

where $d \in \mathscr{L}(H)$ is the operator uniquely determined by

$$d e_m = \sum_{l=1}^{\infty} \varphi_{l,m} e_l.$$

That d is well defined, specifically that it is bounded linear, follows from the fact that a is bounded linear. That is, for any $x \in \operatorname{span}\{e_i : i \in \mathbb{N}\} \subset H$ it is easy to show that

$$\|a(e_k \otimes x)\| = \|e_k \otimes d(x)\| = \|d(x)\|$$

and since a is bounded linear, $\|a(e_k \otimes x)\| \leq \|a\| \|e_k \otimes x\| = \|a\| \|x\|$.

Since $\{e_i \otimes e_j : (j, l) \in \mathbb{N}^2\}$ is an orthonormal basis for $H \otimes H$, it follows that $a = 1 \otimes d$. Thus $(\mathscr{L}(H) \otimes 1)' = 1 \otimes \mathscr{L}(H)$, with $(1 \otimes \mathscr{L}(H))' = \mathscr{L}(H) \otimes 1$ following similarly. $\square$

It is clear from Theorem 2.4.1 that $\mathscr{L}(H) \otimes 1$ and $1 \otimes \mathscr{L}(H)$ are each equal to their double commutants in $\mathscr{L}(H \otimes H)$, so we have the following simple corollary by definition of a von Neumann algebra.

COROLLARY 2.4.2. *$\mathscr{L}(H) \otimes 1$ and $1 \otimes \mathscr{L}(H)$ are von Neumann algebras in $\mathscr{L}(H \otimes H)$.*

CHAPTER 3

Balance

In this chapter we turn to our main goal, to define and study the notion of balance between two W^*-dynamical systems, as an extension of the conditions in Chapter 1 for quantum detailed balance. In particular the characterizations derived in terms of the entangled state (Theorem 1.3.3 and Proposition 1.3.4).

In Section 3.1 we give the definition of balance, along with relevant mathematical background. In particular we define the dual of a W^*-dynamical system, in terms of which balance is defined, and prove that it exists. Couplings of states on two von Neumann algebras are also defined here, and two elementary couplings, the diagonal and product couplings, are discussed.

In Section 3.2 we show how couplings lead to unital completely positive (u.c.p.) maps from one von Neumann algebra to another. Of central importance in this regard, is the diagonal coupling defined in Section 3.1. In certain standard special cases of states on the algebra $\mathscr{L}(\mathfrak{H})$, with $\mathfrak{H}$ a finite dimensional or separable Hilbert space, the diagonal coupling is the maximally entangled bipartite state compatible with the single system states (see Section 4.3), indicating a close connection between these u.c.p. maps and entanglement.

Section 3.3 gives a characterization of balance in terms of intertwinement with the u.c.p. maps defined in Section 3.2. The role of KMS-duals and the special case of KMS-symmetry are also briefly discussed in the context of symmetry of balance. A simple application of balance is then given in the form of an ergodic result, by characterizing an ergodicity condition in a way analogous to the theory of joinings (Proposition 3.3.4).

The development of the theory of balance continues in Section 3.4, where balance is shown to be transitive using the composition of couplings, as an easy result of the characterization of balance in Section 3.3. The definition and properties of such compositions are treated in some detail. The connection to correspondences in the sense of Connes is also discussed.

In Section 3.5 we discuss a quantum detailed balance condition (namely standard quantum detailed balance with respect to a reversing operation, from [26] and [29]) in terms of balance.

3.1. Definition of balance

This section gives the definition of balance, but for convenience and completeness also collects some related known results that we need in the formulation of this definition as well as later on in the chapter. Some of the notation used in the rest of the thesis is also introduced.

DEFINITION 3.1.1. A *W^*-dynamical system* $\mathbf{A} = (A, \alpha, \mu)$ consists of a faithful normal state μ on a (necessarily σ-finite) von Neumann algebra A, and a unital completely positive (u.c.p.) map $\alpha : A \to A$, such that $\mu \circ \alpha = \mu$.

Recall that by normal we mean that μ is σ-weakly continuous (see Section 2.2). For simplicity we will refer to W^*-dynamical systems simply as *systems* from now on.

Note that we only consider a single u.c.p. map, since throughout the chapter we can develop the theory at a single point in time. This can then be applied to a semigroup of u.c.p. maps by applying the definitions and results to each element of the semigroup separately.

In the rest of the chapter the symbols $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ will denote systems (A, α, μ), (B, β, ν) and (C, γ, ξ) respectively. The unit of a von Neumann algebra will be denoted by 1. When we want to emphasize it is the unit of, say, A, the notation 1_A will be used.

Without loss of generality, we will always assume that these von Neumann algebras are in the cyclic representations associated with the given states, i.e. the cyclic representation of (A, μ) is of the form $(G_\mu, \mathrm{id}_A, \Lambda_\mu)$, where G_μ is the Hilbert space, id_A denotes the identity map of A into $\mathscr{L}(G_\mu)$, and Λ_μ is the cyclic and separating vector such that $\mu(a) = \langle \Lambda_\mu, a\Lambda_\mu \rangle$. We know that Λ_μ is separating for A since μ is faithful. Hence Λ_μ is cyclic and separating for A' as well by [**14**, Proposition 2.5.3].

The dynamics α of a system $\mathbf{A}$ is necessarily a contraction, since it is positive and unital (see again [**10**, Proposition II.6.9.4]). Furthermore, α is automatically normal. This is due to the following result:

THEOREM 3.1.2. *Let M and N be von Neumann algebras on the Hilbert spaces H and K respectively, and consider states on them respectively given by $\mu(a) = \langle \Omega, a\Omega \rangle$ and $\nu(b) = \langle \Lambda, b\Lambda \rangle$, with $\Omega \in H$ and $\Lambda \in K$ cyclic vectors, i.e. $\overline{M\Omega} = H$ and $\overline{N\Lambda} = K$. Assume that ν is faithful and consider a positive linear (but not necessarily unital) $\eta : M \to N$ such that*

$$\nu(\eta(a)^*\eta(a)) \leq \mu(a^*a) \tag{24}$$

for all $a \in M$. Then it follows that η is normal, i.e. σ-weakly continuous.

PROOF. By Theorem 2.2.3 we only need to show that η composed with an arbitrary normal positive functional $\mathscr{L}(K) \to \mathbb{C}$ is normal.

For any $x, y \in \mathscr{K}$, define the mapping

$$\omega_{x,y} : \mathscr{L}(K) \to \mathbb{C} : b \mapsto \langle x, by \rangle.$$

By [14, Theorem 2.5.31], for any positive normal function $\varphi : \mathscr{L}(K) \to \mathbb{C}$ there is an $x \in K$ such that $\varphi = \omega_{x,x} =: \omega_x$. So let $x \in K$ be arbitrary and consider the map $\omega_x \circ \eta$. We proceed to show that $\omega \circ \eta$ may be approximated in norm by a weak operator continuous (wo-continuous) linear function $M \to \mathbb{C}$.

Since ν is faithful, Λ is separating for N and thus cyclic for N' by [14, Proposition 2.5.3]. Therefore $\|x - b'\Lambda\|$ can be made arbitrarily small for an appropriate choice of $b' \in N'$. Hence it follows that for any $\epsilon > 0$ there is a $b' \in N'$ such that

$$\begin{aligned}
|\omega_x \circ \eta(a) - \omega_{x,b'\Lambda} \circ \eta(a)| &= |\langle x, \eta(a)x \rangle - \langle x, \eta(a)b'\Lambda \rangle| \\
&= |\langle x, \eta(a)x - \eta(a)b'\Lambda \rangle| \\
&\leq \|x\|\|\eta\|\|a\|\|x - b'\Lambda\| \\
&< \|a\|\epsilon
\end{aligned}$$

for all $a \in M$, where we used the fact that η is necessarily bounded by [10, II.6.9.2]. We now show that $\omega_{x,b'\Lambda} \circ \eta$ is continuous in the weak operator topology.

From (24) it follows that

$$a\Omega \mapsto \eta(a)\Omega$$

uniquely determines an operator $T \in \mathscr{L}(H, K)$. Hence, for any $a \in M$

$$\begin{aligned}
\omega_{x,b'\Lambda} \circ \eta(a) &= \langle x, \eta(a)b'\Lambda \rangle \\
&= \langle x, b'\eta(a)\Lambda \rangle \\
&= \langle x, b'Ta\Lambda \rangle \\
&= \langle T^*b^*x, a\Lambda \rangle
\end{aligned}$$

from which it easy to see that $\omega_{x,b'\Lambda} \circ \eta$ is wo-continuous.

Since $\epsilon > 0$ above was arbitrary we have thus shown that $\omega_x \circ \eta$ lies in the norm-closure of the space of all wo-continuous linear functions on M. Therefore $\omega_x \circ \eta$ is normal by [51, 1.10 Theorem. (iii)].

□

To apply this result to the dynamics of a system **A** we note that because α is a completely positive contraction it follows from Kadison's inequality (Proposition 2.3.3) that

$$\mu(\alpha(a)^*\alpha(a)) \leq \mu(\alpha(a^*a))$$

for all $a \in A$. Hence α is normal as a special case of Theorem 3.1.2.

A central notion going forward will be the dual of a system, defined as follows:

DEFINITION 3.1.3. The *dual* of the system $\mathbf{A}$, is the system $\mathbf{A}' = (A', \alpha', \mu')$ where A' is the commutant of A (in $\mathscr{L}(G_\mu)$), μ' is the state on A' given by $\mu'(a') = \langle \Lambda_\mu, a'\Lambda_\mu \rangle$ for all $a' \in A'$, and $\alpha' : A' \to A'$ is the unique map such that

$$\langle \Lambda_\mu, a\alpha'(a')\Lambda_\mu \rangle = \langle \Lambda_\mu, \alpha(a)a'\Lambda_\mu \rangle$$

for all $a \in A$ and all $a' \in A'$.

Let J_μ be the modular conjugation associated with (A, Λ_μ), as defined in Section 2.2, or just the modular conjugation associated with μ for short. Define

$$j_\mu := J_\mu(\cdot)^* J_\mu \tag{25}$$

Then $j_\mu(A) = A'$ by Theorem 2.2.4 and

$$\langle \Lambda_\mu, j_\mu(a)\Lambda_\mu \rangle = \langle \Lambda_\mu, Ja^*J\Lambda_\mu \rangle = \overline{\langle J^*\Lambda_\mu, a^*\Lambda_\mu \rangle} = \langle a^*\Lambda_\mu, \Lambda_\mu \rangle$$

That is,

$$\mu' = \mu \circ j_\mu$$

in Definition 3.1.3. We will return to these operators in Section 3.3.

Before we proceed to use the dual to define balance between two systems we have to establish that it is well defined. Specifically, that $\alpha' : A' \to A'$ is well-defined and satisfies the required properties for $\mathbf{A}'$ to be a system follows from the following result, which is a more general version of [**1**, Proposition 3.1] and [**6**, Theorem 2.1]:

THEOREM 3.1.4. *Let H and K be Hilbert spaces, M a (not necessarily unital) $*$-subalgebra of $\mathscr{L}(H)$, and N a (not necessarily unital) C^*-subalgebra of $\mathscr{L}(K)$. Let $\Omega \in H$ with $\|\Omega\| = 1$ be cyclic for M, i.e. $M\Omega$ is dense in H, and let $\Lambda \in K$ be any unit vector. Set*

$$\mu : M \to \mathbb{C} : a \mapsto \langle \Omega, a\Omega \rangle$$

and

$$\nu : N \to \mathbb{C} : b \mapsto \langle \Lambda, b\Lambda \rangle .$$

Consider any positive linear $\eta : M \to N$, i.e. for a positive operator $a \in M$, we have that $\eta(a)$ is a positive operator. Assume furthermore that

$$\nu \circ \eta = \mu.$$

Then there exists a unique map, called the dual *of η,*

$$\eta' : N' \to M'$$

such that

$$\langle \Omega, a\eta'(b')\Omega \rangle = \langle \Lambda, \eta(a)b'\Lambda \rangle \tag{26}$$

for all $a \in M$ and $b' \in N'$. The map η' is necessarily linear, positive and unital, i.e. $\eta'(1) = 1$, and $\|\eta'\| = 1$. Furthermore the following two results hold:

(a) If M and N are von Neumann algebras then, if η is n-positive, then η' is n-positive as well. In particular, if η is completely positive, then η' is as well.

(b) If M and N contain the identity operators on H and K respectively, and η is unital (i.e. $\eta(1) = 1$), then it follows that

$$\mu' \circ \eta' = \nu',$$

where $\mu'(a') := \langle \Omega, a'\Omega \rangle$ and $\nu'(b') := \langle \Lambda, b'\Lambda \rangle$ for all $a' \in M'$ and $b' \in N'$. If in addition Λ is separating for N', then η' is faithful in the sense that when $\eta'(b'^*b') = 0$, it follows that $b' = 0$.

PROOF. Fix any $b' \in (N')_+$ and define

$$\psi : M \to \mathbb{C} : a \mapsto \langle \Lambda, \eta(a)b'\Lambda \rangle$$

It is easy to see that ψ is a positive linear functional since if $a \geq 0$ then

$$\psi(a) = \langle \Lambda, (\eta(a))^{\frac{1}{2}}(\eta(a))^{\frac{1}{2}}(b')^{\frac{1}{2}}(b')^{\frac{1}{2}}\Lambda] \rangle = \|(b')^{\frac{1}{2}}(\eta(a))^{\frac{1}{2}}\Lambda\|^2$$

where we used the fact that, since N and N' are both C^*-algebras (see [14, p. 71]), $\eta(a)$ and b' have positive square roots in N and N' respectively.

Furthermore, if $a \geq 0$ then

$$\begin{aligned}
\psi(a) &= \langle \Lambda, (\eta(a))^{\frac{1}{2}}(\eta(a))^{\frac{1}{2}}b'\Lambda \rangle \\
&= \langle \Lambda, (\eta(a))^{\frac{1}{2}}b'(\eta(a))^{\frac{1}{2}}\Lambda \rangle \\
&\leq \langle \Lambda, \eta(a)\Lambda \rangle \|b'\| \\
&= \nu \circ \eta(a)\|b'\| \\
&= \mu(a)\|b'\| \\
&= \left\langle \frac{\Omega}{\sqrt{\|b'\|}}, a\frac{\Omega}{\sqrt{\|b'\|}} \right\rangle
\end{aligned}$$

where we used [14, Proposition 2.3.11(3)] to obtain the inequality. It now follows by [20, Lemma 1 on p. 53] that there is a $d' \in M'$ such that

$$\begin{aligned}
\psi(a) &= \left\langle d'\frac{\Omega}{\sqrt{\|b'\|}}, ad'\frac{\Omega}{\sqrt{\|b'\|}} \right\rangle \\
&= \left\langle \Omega, a\frac{(d')^*d'}{\|b'\|}\Omega \right\rangle
\end{aligned}$$

for all $a \in M$.

Thus, if $b' \in N'$ is positive then there exists an element $\eta'(b') \in (M')_+$ such that (26) holds for all $a \in M$, namely $\eta'(b') = ((d')^*d')/\|b'\|$. Moreover, this element is necessarily unique since $\langle \Omega, ac'\Omega \rangle = \langle \Omega, ad'\Omega \rangle$ for some $c', d' \in (M')_+$ and all $a \in M$ implies that $c'\Omega = d'\Omega$ and thus that $c' = d'$. This follows due to Ω being cyclic for M, and thus also separating for M' as can be easily checked.

In other words, by defining $\eta(b')$ as the unique element in $(M')_+$ satisfying (26) we obtain a well-defined mapping $\eta' : (N')_+ \to (M')_+$.

Consider any $b' \in N'$ and let

$$b' = \sum_i \alpha_i c_i' = \sum_i \gamma_i d_i'$$

be two expressions of b' as finite linear combinations of positive elements in N'. Such an expression always exists since N' is a C^*-algebra (for example, as in the proof of Theorem 2.2.3). It follows that

$$\begin{aligned}
\langle \Omega, a \sum_i \alpha_i \eta'(c_i')\Omega\rangle &= \sum_i \alpha_i \langle \Omega, a\eta'(c_i')\Omega\rangle \\
&= \sum_i \alpha_i \langle \Omega, a\eta'(c_i')\Omega\rangle \\
&= \sum_i \alpha_i \langle \Omega, \eta(a) c_i'\Omega\rangle \\
&= \langle \Omega, \eta(a) \sum_i \alpha_i c_i'\Omega\rangle \\
&= \langle \Omega, \eta(a) \sum_i \gamma_i d_i'\Omega\rangle \\
&= \langle \Omega, a \sum_i \gamma_i \eta'(d_i')\Omega\rangle
\end{aligned}$$

for all $a \in M$. Thus $\sum_i \alpha_i \eta'(c_i') = \sum_i \gamma_i \eta'(d_i')$ for the same reason that $c' = d'$ above. This shows that, firstly, η' has a well defined linear extension $\eta' : N' \to M'$ and, secondly, that the linear extension satisfies (26) for all $a \in M$ and $b' \in N'$.
Furthermore, substituting $b' = 1$ in (26) shows that η' is unital:

$$\langle \Omega, a\eta'(1)\Omega\rangle = \langle \Omega, \eta(a)\Omega\rangle = \mu \circ \eta(a) = \nu(a) = \langle \Omega, a\Omega\rangle = \langle \Omega, a1\Omega\rangle.$$

Also, η' is clearly positive by our construction, so it is a contraction by [**10**, II.6.9.4 Proposition]. Hence, as $\eta'(1) = 1$ it follows that $\|\eta'\| = 1$.

(a) Assume that $\eta : M \to N$ is n-positive. Thus

$$\eta^{(n)} : M_n(M) \to M_n(N)$$

defined pointwise is positive, or equivalently,

$$M \odot M_n \to N \odot M_n : a \otimes x_n \mapsto \eta(a) \otimes x_n \tag{27}$$

is positive. To prove that η' is n-positive we will show that $\eta^{(n)}$ has a positive dual $(\eta^{(n)})'$, and reason that this is equivalent to n' being n-positive, in the same way that the n-positivity of η is equivalent to the positivity of $\eta^{(n)}$. However, to this end the algebra M_n is too "small", so let $\pi : M_n \to M_{n^2}$ be the $*$-morphism defined by

$$\pi : x_n \mapsto x_n \otimes I_n = \begin{bmatrix} [x_n] & & 0 \\ & \ddots & \\ 0 & & [x_n] \end{bmatrix}$$

for all $x_n \in M_n$ where I_n is the $n \times n$ identity matrix. For simplicity we'll use the shorthand notation $x_n \in \mathscr{C}_n (= \pi(M_n))$ to refer to the element $\pi(x_n) = x_n \otimes 1_n$ where $x_n \in M_n$. Thus $\pi : M_n \to \mathscr{C}_n$ is a $*$-isomorphism and we have that the following linear operator is positive:

$$\eta^{(n)} : \eta \odot \iota_n : M \odot \mathscr{C}_n \to N \odot \mathscr{C}_n : a \otimes x_n \mapsto \eta(a) \otimes x_n.$$

For any $i = 1, ..., n$ let γ_i be the $n \times 1$ column vector with a $1/\sqrt{n}$ in position i and zeros elsewhere, and define $\Omega_n \in \mathbb{C}^{n^2}$ to be a stacking of these vectors:

$$\Omega_n = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}.$$

Set $\overline{\Omega} = \Omega \otimes \Omega_n$ and $\overline{\Lambda} = \Lambda \otimes \Omega_n$, both of which are clearly unit vectors in $H \odot \mathbb{C}^{n^2}$. With a little thought it can also be seen that $(M \odot \mathscr{C}_n)\overline{\Omega}$ is dense in $H \otimes \mathbb{C}^{n^2}$. That is, $\overline{\Omega}$ is cyclic for $M \odot \mathscr{C}_n \subset \mathscr{L}(H \otimes \mathbb{C}^{n^2})$.

Furthermore, for any $a \in M$ and $x_n \in \mathscr{C}_n$ it follows that

$$\begin{aligned} \langle \overline{\Lambda}, \eta \odot \iota_n(a \otimes x_n)\overline{\Lambda} \rangle &= \langle \Lambda \otimes \Omega_n, \eta(a) \otimes x_n \Lambda \otimes \Omega_n \rangle \\ &= \langle \Lambda, \eta(a)\Lambda \rangle \langle \Omega_n, x_n \Omega_n \rangle \\ &= \langle \Omega, a\Omega \rangle \langle \Omega_n, x_n \Omega_n \rangle \\ &= \langle \overline{\Omega}, a \otimes x_n \overline{\Omega} \rangle \end{aligned}$$

Thus $\langle \overline{\Lambda}, \eta \odot \iota_n(z)\overline{\Lambda} \rangle = \langle \overline{\Omega}, z\overline{\Omega} \rangle$ for all $z \in M \odot \mathscr{C}_n$.

Finally, $\mathscr{C}_n$ is a finite-dimensional von Neumann algebra, so $M \odot \mathscr{C}_n = M \overline{\otimes} \mathscr{C}_n$ and $N \odot \mathscr{C}_n = N \overline{\otimes} \mathscr{C}_n$. Hence it follows from [**52**, IV. Theorem 5.9.] that

$$M' \odot \mathscr{C}_n' = M' \overline{\otimes} \mathscr{C}_n' = (M \overline{\odot} \mathscr{C}_n)' = (M \odot \mathscr{C}_n)'.$$

Hence it now follows from the first part of the theorem that $\eta^{(n)}$ has a unique dual

$$(\eta^{(n)})' : N' \odot \mathscr{C}_n' \to M' \odot \mathscr{C}_n'$$

which is necessarily positive.

Now, let $\iota_n' : \mathscr{C}_n' \to \mathscr{C}_n'$ be the identity mapping on $\mathscr{C}_n' \subset M_{n^2}$. It follows, for any $b \in N', a \in M, b_n' \in \mathscr{C}_n'$ and $a_n \in \mathscr{C}_n$, that

$$\begin{aligned}
\left\langle \overline{\Omega}, \eta' \odot \iota_n'(b' \otimes b_n')a \otimes a_n\overline{\Omega} \right\rangle &= \langle \Omega \otimes \Omega_n, \eta'(b')a\Omega \otimes b_n'a_n\Omega_n \rangle \\
&= \langle \Omega, \eta'(b')a\Omega \rangle \langle \Omega_n, b_n'a_n\Omega_n \rangle \\
&= \langle \Omega, b'\eta(a)\Omega \rangle \langle \Omega_n, b_n'a_n\Omega_n \rangle \\
&= \langle \Omega \otimes \Omega_n, b'\eta(a)\Omega \otimes b_n'a_n\Omega_n \rangle \\
&= \left\langle \overline{\Omega}, b' \otimes b_n'(\eta \odot \iota_n)(a \otimes a_n)\overline{\Omega} \right\rangle .
\end{aligned}$$

Thus $\langle \overline{\Omega}, \eta' \odot \iota_n'(b')a\overline{\Omega} \rangle = \langle \overline{\Omega}, b'\eta \odot \iota_n(a)\overline{\Omega} \rangle = \langle \overline{\Omega}, b'\eta^{(n)}(a)\overline{\Omega} \rangle$, so it follows by uniqueness of the dual that $\eta' \odot \iota_n' = (\eta^{(n)})'$.
The proof of (a) is concluded by noting that n-positivity of η' is equivalent to $\eta' \odot \iota_n$ being positive, which in turn is equivalent to $\eta' \odot \iota_n'$ being positive. This follows since $\mathscr{C}_n$ and $\mathscr{C}_n'$ are isomorphic in M_{n^2}. Specifically, $\mathscr{C}_n' = (M_n \otimes I_n)' = I_n \otimes M_n$. This follows from a modification of the arguments used to prove Theorem 2.4.1, however in the finite dimensional case this can be proven more directly and with a much shorter argument using the fact that $M_n' = \mathbb{C}1_n$.

(b) Assume $1_{\mathscr{L}(H)} \in M$ and $1_{\mathscr{L}(K)} \in N$. If η is unital then it follows that

$$\begin{aligned}
\mu' \circ \eta'(b') &= \langle \Omega, \eta'(b')\Omega \rangle \\
&= \langle \Lambda, \eta(1)b'\Lambda \rangle \\
&= \nu'(b')
\end{aligned}$$

for all $b' \in N'$.
If $\eta'(b'^*b') = 0$ for some $b' \in N'$, then since $\mu' \circ \eta' = \nu'$ it follows that $\nu'(b'^*b') = 0$, or $\langle \Lambda, b'^*b'\Lambda \rangle = \|b'\Lambda\|^2 = 0$. Thus $b' = 0$ follows if Λ is separating for N'. $\square$

Strictly speaking one should say that η' is the dual of η with respect to μ and ν. However the states will always be implicitly clear.

In particular, with $M = N = A$ and $\Omega = \Lambda = \Lambda_\mu$, we see from this theorem that the dual of the system $\mathbf{A}$ is well-defined.

If instead of the single map α we have a semigroup of u.c.p. maps $(\alpha_t)_{t\geq 0}$ leaving μ invariant, then $\alpha_t' \equiv (\alpha_t)'$ also gives a semigroup of u.c.p. maps leaving μ' invariant. The continuity or measurability properties of this dual semigroup (as function of t) will depend on those of α_t. Consider for example the standard assumption made for (continuous time) quantum Markov semigroups, namely that $t \mapsto \alpha_t(a)$ is σ-weakly continuous for every $a \in A$. Then it can be shown that $t \mapsto \varphi(\alpha_t'(a'))$ is continuous for every $a' \in A'$ and every normal state φ on A', so $t \mapsto \alpha_t'(a')$ is σ-weakly continuous for every $a' \in A'$. I.e. $(\alpha_t')_{t\geq 0}$ is also a quantum Markov semigroup (with the same type of continuity property). If we were to include these assumptions in our definition of a system, then the dual of such a system would therefore still be a system. Our example in Chapter 4 will indeed be for semigroups indexed by $t \geq 0$, with even stronger continuity properties.

It is helpful to keep the following fact about duals in mind:

COROLLARY 3.1.5. *If in addition to the assumptions in Theorem 3.1.4 (prior to parts (a) and (b)), we have that M and N are von Neumann algebras, and Λ is cyclic for N', then we have*

$$\eta'' = \eta.$$

PROOF. This follows directly from the theorem itself, since $\eta'' : M \to N$ is then the unique map such that $\langle \Lambda, b'\eta''(a)\Lambda \rangle = \langle \Omega, \eta'(b')a\Omega \rangle$ for all $a \in M$ and $b' \in N'$, while we know (again from Theorem 3.1.4) that $\langle \Lambda, b'\eta(a)\Lambda \rangle = \langle \Omega, \eta'(b')a\Omega \rangle$ for all $a \in M$ and $b' \in N'$. □

We also record the following simple result:

PROPOSITION 3.1.6. *If in Theorem 3.1.4 we assume in addition that μ and ν are faithful normal states on von Neumann algebras M and N (so Ω and Λ are the corresponding cyclic and separating vectors), then*

$$(j_\nu \circ \eta \circ j_\mu)' = j_\mu \circ \eta' \circ j_\nu$$

for the map $j_\nu \circ \eta \circ j_\mu : M' \to N'$ obtained in terms of Eq. (25).

PROOF. It is a straightforward calculation to show that

$$\langle \Omega, a' j_\mu \circ \eta' \circ j_\nu(b)\Omega \rangle = \langle \Lambda, j_\nu \circ \eta \circ j_\mu(a')b\Lambda \rangle$$

for all $a' \in M'$ and $b \in N$. □

We can now state the main definition for our work ahead:

DEFINITION 3.1.7. Let μ and ν be faithful normal states on the von Neumann algebras A and B respectively. A *coupling* of (A, μ) and (B, ν), is a state ω on the algebraic tensor product $A \odot B'$ such that

$$\omega(a \otimes 1) = \mu(a) \text{ and } \omega(1 \otimes b') = \nu'(b')$$

for all $a \in A$ and $b \in B'$. We also call such an ω a coupling of μ and ν. Let $\mathbf{A}$ and $\mathbf{B}$ be systems. We say that $\mathbf{A}$ and $\mathbf{B}$ (in this order) are in *balance* with respect to a coupling ω of μ and ν, expressed in symbols as

$$\mathbf{A}\omega\mathbf{B},$$

if

$$\omega(\alpha(a) \otimes b') = \omega(a \otimes \beta'(b'))$$

for all $a \in A$ and $b' \in B'$.

By a state on the algebraic tensor product $A \odot B'$ we mean that ω is linear, $\omega(1) = 1$ and ω is positive in the sense that $\omega(z^*z) \geq 0$ for all $z \in A \odot B'$ (See Definition 2.1.1 and the discussion immediately after).

Notice that Definition 3.1.7 is in terms of the dual $\mathbf{B}'$ rather than in terms of $\mathbf{B}$ itself. To define balance in terms of $\omega(\alpha(a) \otimes b) =$

$\omega(a \otimes \beta(b))$, for $a \in A$ and $b \in B$, turns out to be a less natural convention, in particular with regards to transitivity (see Section 3.4). Also, strictly speaking, saying that **A** and **B** are in balance, implies a direction, say from **A** to **B**. These points will become more apparent in subsequent sections.

For systems given by quantum Markov semigroups $(\alpha_t)_{t\geq 0}$ and $(\beta_t)_{t\geq 0}$, instead of a single map for each system, we note that balance is defined by requiring $\omega(\alpha_t(a) \otimes b') = \omega(a \otimes \beta_t'(b'))$ at every $t \geq 0$.

For comparison to the theory of joinings [**21**, **22**, **23**], note that a joining of systems **A** and **B**, with α and β $*$-automorphisms, is a state ω on $A \odot B$ such that $\omega(a \otimes 1) = \mu(a)$, $\omega(1 \otimes b) = \nu(b)$ and $\omega \circ (\alpha \odot \beta) = \omega$. In addition [**9**] also assumes that $\omega \circ (\sigma_t^\mu \odot \sigma_t^\nu) = \omega$, where σ_t^μ and σ_t^ν are the modular groups associated with μ and ν. In [**9**], however, it is formulated in terms of the opposite algebra of B, which is in that sense somewhat closer to the conventions used above for balance.

We now define two simple couplings that always exist, and in terms of which several important characterizations will be derived in the subsequent sections.

Consider two von Neumann algebras and faithful normal state pairs (A, μ) and (B, ν), and assume without loss of generality that both are in their cyclic representations $(G_\mu, \mathrm{id}_A, \Lambda_\mu)$ and $(G_\nu, \mathrm{id}_A, \Lambda_\nu)$.

Let $\varphi : A \odot A' \to \mathscr{L}(G_\mu)$ be the linear extension of the bilinear map $A \times A' \to \mathscr{L}(G_\mu) : (a, a') \mapsto aa'$, by the universal property of tensor products. The *diagonal coupling* of (A, μ) with itself, or simply of μ with itself, is defined:

$$\delta_\mu(z) = \langle \Lambda_\mu, \varphi(z)\Lambda_\mu \rangle \tag{28}$$

for all $z \in A \odot A'$. That is,

$$\delta_\mu(a \otimes a') = \langle \Lambda_\mu, aa'\Lambda \rangle$$

which would've sufficed as a definition for δ_μ, but it is more readily seen that δ_μ is positive by observing that $\varphi : A \otimes A' \to \mathscr{L}(G_\mu)$ is a $*$-morphism:

$$\begin{aligned}
\varphi((a \otimes a')(b \otimes b')) &= aba'b' \\
&= aa'bb' \\
&= \varphi(a \otimes a')\varphi(b \otimes b') \\
\varphi((a \otimes a')^*) &= \varphi(a^* \otimes a'^*) \\
&= (aa')^* \\
&= \varphi(a \otimes a')^*
\end{aligned}$$

for all $a, b \in A$ and $a', b' \in A'$, from which it readily follows for any $z = \sum_i^n a_i \otimes a_i' \in A \odot A'$ that $\varphi(z^*z) \geq 0$:

$$\begin{aligned}\varphi(z^*z) &= \varphi\left(\sum_{i=1}^n a_i^* \otimes a_i'^* \sum_{j=1}^n a_j \otimes a_j'\right) \\ &= \sum_{i,j=1}^n \varphi(a_i^* \otimes a_i'^*)\varphi(a_j \otimes a_j') \\ &= \sum_{i=1}^n \varphi(a_i^* \otimes a_i'^*) \sum_{j=1} \varphi(a_j \otimes a_j') \\ &= \varphi(z^*)\varphi(z)\end{aligned}$$

Thus δ_μ in (28) clearly defines a state and a coupling:

$$\delta_\mu(a \otimes 1) = \langle \Lambda_\mu, \varphi(a \otimes 1)\Lambda \rangle = \langle \Lambda, a\Lambda \rangle = \mu(a)$$

$$\delta_\mu(1 \otimes a') = \langle \Lambda_\mu, \varphi(1 \otimes a')\Lambda \rangle = \langle \Lambda, a'\Lambda \rangle = \mu'(a)$$

Moreover, for any system $\mathbf{A}$ it easy to see that $\mathbf{A}\delta_\mu\mathbf{A}$.

The *trivial/product coupling* $\mu \odot \nu'$ of (A, μ) and (B, ν), or simply of μ and ν, is defined by

$$\mu \odot \nu'(a \otimes b') = \mu(a)\nu'(b) \tag{29}$$

for all $a \otimes b' \in A \odot B'$, which is a well-defined functional by the universal property of the tensor products. It is clear that $\mu \odot \nu'(1 \otimes 1) = 1$, and to see that $\mu \odot \nu'$ is positive consider the inner product space representation of (29):

$$\begin{aligned}\mu \odot \nu'(a \otimes b') &= \langle \Lambda_\mu, a\Lambda_\mu \rangle \langle \Lambda_\nu, b'\Lambda_\nu \rangle \\ &= \langle \Lambda_\mu \otimes \Lambda_\nu, a \otimes b'\Lambda_\mu \otimes \Lambda_\nu \rangle\end{aligned}$$

for all $a \otimes b' \in A \odot B'$, so it follows that

$$\mu \odot \nu'(t) = \langle \Lambda_\mu \otimes \Lambda_\nu, t(\Lambda_\mu \otimes \Lambda_\nu) \rangle \tag{30}$$

for all $t \in A \odot B'$, where the latter inner product is the inner product on $G_\mu \odot G_\nu$ uniquely determined by

$$\langle x \otimes y, p \otimes q \rangle = \langle x, p \rangle \langle y, q \rangle$$

for all $x \otimes y, p \otimes q \in \mathscr{G}_\mu \odot G_\nu$. Hence $\langle x \otimes y, a \otimes b' x \otimes y \rangle = \langle a^* \otimes b'^* x \otimes y, x \otimes y \rangle = \langle (a \otimes b')^* x \otimes y, x \otimes y \rangle$ for all $x \otimes y \in G_\mu \odot G_\nu$ and $a \otimes b' \in A \odot B'$, from which it follows that $\langle a \otimes y, t(x \otimes y) \rangle = \langle t^*(a \otimes y), x \otimes y \rangle$ for all $t \in A \odot B'$. Thus $\mu \odot \nu'(t^*t) \geq 0$ for all $t \in A \odot B'$ in (30).

3.2. Couplings and u.c.p. maps

Here we define and study a map E_ω associated with a coupling ω. This map is of fundamental importance in the theory of balance, as will be seen the next two sections. We do not consider systems in this section, only couplings. Some aspects of this section and the next are closely related to [9, Section 4] regarding joinings.

Let ω be a coupling of (A,μ) and (B,ν) as in Definition 3.1.7. Assume without loss of generality that (B,ν) is in its cyclic representation, denoted here by $(G_\nu, \mathrm{id}_B, \Lambda_\nu)$, which means that $(G_\nu, \mathrm{id}_{B'}, \Lambda_\nu)$ is a cyclic representation of (B',ν') (see Section 2.1). Similarly, we assume that (A,μ) is in the cyclic representation $(G_\mu, \mathrm{id}_A, \Lambda_\mu)$.

Denoting the cyclic representation of $(A \odot B', \omega)$ by $(H_\omega, \pi_\omega, \Omega_\omega)$ (see Theorem 2.1.3), we obtain a second cyclic representation $(H_\mu, \pi_\mu, \Omega_\mu)$ of (A,μ) by setting

$$H_\mu := \overline{\pi_\omega(A \otimes 1)\Omega_\omega},\ \pi_\mu(a) := \pi_\omega(a \otimes 1)|_{H_\mu} \text{ and } \Omega_\mu := \Omega_\omega \tag{31}$$

for all $a \in A$, since

$$\langle \Omega_\mu, \pi_\mu(a)\Omega_\mu \rangle = \langle \Omega_\omega, \pi_\omega(a \otimes 1)\Omega_\omega \rangle = \omega(a \otimes 1) = \mu(a).$$

Similarly

$$H_\nu := \overline{\pi_\omega(1 \otimes B')\Omega_\omega},\ \pi_{\nu'}(b') := \pi_\omega(1 \otimes b')|_{H_\nu} \text{ and } \Omega_\nu := \Omega_\omega \tag{32}$$

for all $b' \in B'$, gives a second cyclic representation $(H_\nu, \pi_{\nu'}, \Omega_\nu)$ of (B',ν'). In particular H_μ and H_ν are subspaces of H_ω.

By the unitary equivalence of $(G_\nu, \mathrm{id}_B, \Lambda_\nu)$ and $(H_\nu, \pi_{\nu'}, \Omega_\nu)$ there is a unitary operator

$$u_\nu : G_\nu \to H_\nu \tag{33}$$

defined by

$$u_\nu b' \Lambda_\nu := \pi_{\nu'}(b')\Omega_\nu$$

for all $b' \in B' \subset \mathscr{L}(H)$. More generally, since G_ν and H_ν are the "same" through u_ν, any $t \in \mathscr{L}(G_\nu)$ can be viewed as an operator on G_ν, as t, or on H_ν, as $u_\nu t u_\nu^*$, and likewise for any $t \in \mathscr{L}(H_\nu)$. In particular, by setting

$$\pi_\nu(b) := u_\nu b u_\nu^* \tag{34}$$

for all $b \in B$ we obtain a second cyclic representation $(H_\nu, \pi_\nu, \Omega_\nu)$ of (B,ν) with the property

$$\pi_\nu(B)' = \pi_{\nu'}(B)$$

and

$$\pi_{\nu'}(b') = u_\nu b' u_\nu^* \tag{35}$$

as is easily verified.

Now, let

$$P_\nu \in \mathscr{L}(H_\omega)$$

be the projection operator of H_ω onto H_ν. Then, for any $a \in A$,

$$\pi_\omega(a \otimes 1) \in \mathscr{L}(H_\omega)$$
$$P_\nu \pi_\omega(a \otimes 1) P_\nu^* \in \mathscr{L}(H_\nu)$$
$$u_\nu^* P_\nu \pi_\omega(a \otimes 1) P_\nu^* u_\nu \in \mathscr{L}(G_\nu)$$

That is, we can directly compare the elements of A to B, B' as operators on the same Hilbert space. Doing so leads to our first main result:

PROPOSITION 3.2.1. *In terms of the notation above, we have*

$$u_\nu^* \iota_{H_\nu}^* \pi_\omega(a \otimes 1) \iota_{H_\nu} u_\nu = u_\nu^* P_\nu \pi_\omega(a \otimes 1) u_\nu \in B$$

for all $a \in A$, where $\iota_{H_\nu} : H_\nu \to H_\omega$ is the inclusion map, and $\iota_{H_\nu}^* : H_\omega \to H_\nu$ its adjoint.

PROOF. Note that $P_\nu = \iota_{H_\nu}^*$, so indeed $u_\nu^* \iota_{H_\nu}^* \pi_\omega(a \otimes 1) \iota_{H_\nu} u_\nu = u_\nu^* P_\nu \pi_\omega(a \otimes 1) u_\nu$. We now show that this is in B.

For any $b' \in B'$ we have $\pi_\omega(1 \otimes b') H_\nu^\perp \subset H_\nu^\perp$, since $\pi_\omega(1 \otimes b'^*) H_\nu \subset H_\nu$. It follows that $P_\nu \pi_\omega(1 \otimes b') = \pi_\omega(1 \otimes b') P_\nu = \pi_{\nu'}(b') P_\nu$. Therefore,

$$\begin{aligned}
u_\nu^* P_\nu \pi_\omega(a \otimes 1) u_\nu b' &= u_\nu^* P_\nu \pi_\omega(a \otimes 1) u_\nu b' u_\nu^* u_\nu \\
&= u_\nu^* P_\nu \pi_\omega(a \otimes 1) \pi_{\nu'}(b') u_\nu \\
&= u_\nu^* P_\nu \pi_\omega(a \otimes 1) \pi_\omega(1 \otimes b') u_\nu \\
&= u_\nu^* P_\nu \pi_\omega(1 \otimes b') \pi_\omega(a \otimes 1) u_\nu \\
&= u_\nu^* \pi_\omega(1 \otimes b') P_\nu \pi_\omega(a \otimes 1) u_\nu \\
&= u_\nu^* \pi_{\nu'}(b') P_\nu \pi_\omega(a \otimes 1) u_\nu \\
&= b' u_\nu^* P_\nu \pi_\omega(a \otimes 1) u_\nu.
\end{aligned}$$

Hence, since $b' \in B'$ is arbitrary, it follows that $u_\nu^* P_\nu \pi_\omega(a \otimes 1) u_\nu \in B'' = B$. □

This proposition proves part of the following result, which defines the central object of this section, namely the map $E_\omega : A \to B$

THEOREM 3.2.2. *In terms of the notation above we have the following well-defined linear map*

$$E_\omega : A \to B : a \mapsto u_\nu^* \iota_{H_\nu}^* \pi_\omega(a \otimes 1) \iota_{H_\nu} u_\nu \tag{36}$$

which is normal and completely positive. It has the following properties:

$$E_\omega(1) = 1$$
$$\|E_\omega\| = 1$$
$$\nu \circ E_\omega = \mu. \tag{37}$$

PROOF. E_ω is precisely the form of a completely positive map according to the Stinespring dilation (see Theorem 2.3.2), with $\iota_\nu u_\nu \in \mathscr{L}(G_\nu, H_\omega)$ and $\pi_\omega((\cdot) \otimes 1)$ a representation of A on H_ω.

From Eq. (36) we have $E_\omega(1) = u_\nu^* \iota_{H_\nu}^* \iota_{H_\nu} u_\nu = 1$ as well as $\|E_\omega\| \leq 1$, since $\|u_\nu\| = \|P_\nu\| = 1$ and $\|\pi(a \otimes 1)\| \leq \|a\|$ ([14, Proposition 2.3.1]). Thus $\|E_\omega\| = 1$.

Furthermore,

$$\nu \circ E_\omega(a) = \langle \Lambda_\nu, E_\omega(a)\Lambda_\nu \rangle = \langle \Omega_\omega, \pi_\omega(a \otimes 1)\Omega_\omega \rangle = \omega(a \otimes 1) = \mu(a)$$

for all $a \in A$.

Lastly, Kadison's inequality (Proposition 2.3.3), $E_\omega(a)^* E_\omega(a) \leq E_\omega(a^*a)$, holds, since E_ω is a completely positive contraction. So $\nu(E_\omega(a)^* E_\omega(a)) \leq \nu(E_\omega(a^*a)) = \mu(a^*a)$, for all $a \in A$. Hence, E_ω is normal, due to Theorem 3.1.2. $\square$

We proceed by discussing some further general properties of E_ω which will be useful for us later. These results will be in terms of the diagonal and product couplings defined in the previous section. In terms of the diagonal coupling we have the following characterization of E_ω which will often be used:

Proposition 3.2.3. *The map E_ω is the unique operator from A to B such that*

$$\omega(a \otimes b') = \delta_\nu(E_\omega(a) \otimes b')$$

for all $a \in A$ and $b' \in B'$.

Proof. We simply calculate:

$$\begin{aligned}
\delta_\nu(E_\omega(a) \otimes b') &= \langle \Lambda_\nu, E_\omega(a) b' \Lambda_\nu \rangle \\
&= \langle \Lambda_\nu, u_\nu^* P_\nu \pi_\omega(a \otimes 1) u_\nu b' \Lambda_\nu \rangle \\
&= \langle P_\nu \Omega_\nu, \pi_\omega(a \otimes 1) \pi_{\nu'}(b') \Omega_\nu \rangle \\
&= \langle \Omega_\nu, \pi_\omega(a \otimes b') \Omega_\nu \rangle \\
&= \omega(a \otimes b')
\end{aligned}$$

for all $a \in A$ and $b' \in B'$. Secondly, suppose that for some $b_1, b_2 \in B$ we have $\delta_\nu(b_1 \otimes b') = \delta_\nu(b_2 \otimes b')$ for all $b' \in B'$. Then $\langle b_1^* \Lambda_\nu, b' \Lambda_\nu \rangle = \langle b_2^* \Lambda_\nu, b' \Lambda_\nu \rangle$ for all $b' \in B'$, so $b_1^* \Lambda_\nu = b_2^* \Lambda_\nu$, since $B' \Lambda_\nu$ is dense in G_ν. But Λ_ν is separating for B, hence $b_1 = b_2$. Therefore E_ω is indeed the unique function as stated. $\square$

This has four simple corollaries:

Corollary 3.2.4. *If ω_1 and ω_2 are both couplings of μ and ν, then $\omega_1 = \omega_2$ if and only if $E_{\omega_1} = E_{\omega_2}$.*

Corollary 3.2.5. *The map E_ω is faithful in the sense that if $E_\omega(a^*a) = 0$, then $a = 0$.*

Proof. If $E_\omega(a^*a) = 0$, then $\mu(a^*a) = \omega((a^*a) \otimes 1) = \delta_\nu(E_\omega(a^*a) \otimes 1) = 0$. But μ is faithful, and hence $a = 0$. $\square$

The latter also follows from Theorem 3.1.4 and $E''_\omega = E_\omega$.

The next corollary is relevant when we consider cases of trivial balance, i.e. balance with respect to the product coupling $\mu \odot \nu'$, and will be applied toward the end of the next section, in relation to ergodicity:

Corollary 3.2.6. *Let ω be a coupling of (A, μ) and (B, ν). If $\omega = \mu \odot \nu'$, then $E_\omega(a) = \mu(a)1_B$ for all $a \in A$. Conversely, if $E_\omega(A) = \mathbb{C}1_B$, then $\omega = \mu \odot \nu'$.*

Proof. If $\omega = \mu \odot \nu'$, then $E_\omega(a) = \mu(a)1_B$ follows from Proposition 3.2.3. Conversely, again using Proposition 3.2.3, if $E_\omega(A) = \mathbb{C}1_B$, then $\omega(a \otimes b')1_B = \delta_\nu(E_\omega(a) \otimes b')1_B = E_\omega(a)\delta_\nu(1 \otimes b') = E_\omega(a)\nu'(b')$. In particular, setting $b' = 1$, $E_\omega(a) = \mu(a)1_B$, so $\omega = \mu \odot \nu'$. □

Corollary 3.2.7. *We have $\omega = \delta_\nu$ if and only if $E_\omega = \mathrm{id}_B$.*

A natural question to ask in light of Proposition 3.2.3 is, given a linear map $E : A \to B$, when does $\omega_E(a \otimes b') = \delta_\nu(E(a) \otimes b')$ define a coupling of μ and ν? If it does define a coupling then $E_{\omega_E} = E$ by Proposition 3.2.3. Hence, by Theorem 3.2.2 a necessary condition for E_ω to be a coupling is that E is a normal, completely positive, unital contraction satisfying $\nu \circ E = \mu$. Or, equivalently, a completely positive unital map satisfying $\nu \circ E = \mu$ (see proof of Theorem 3.2.2). The following proposition establishes that this is a sufficient condition as well:

Proposition 3.2.8. *Let μ and ν be faithful normal states on the von Neumann algebras A and B respectively. Consider a linear map $E : A \to B$ and define a linear functional $\omega_E : A \odot B' \to \mathbb{C}$ by*

$$\omega_E := \delta_\nu \circ (E \odot \mathrm{id}_{B'}),$$

i.e.

$$\omega_E(a \otimes b') = \delta_\nu(E(a) \otimes b')$$

for all $a \in A$ and $b \in B'$. Then ω_E is a coupling of μ and ν if and only if E is completely positive, unital and $\nu \circ E = \mu$. In this case $E = E_{\omega_E}$.

Proof. Consider a completely positive linear map $E : A \to B$. Then $E \odot \mathrm{id}_{B'}$ is positive (see e.g. [II.9.7][10]), so ω_E is positive, since δ_ν is. If we furthermore assume that E is unital, then $\omega_E(1 \otimes 1) = 1$, so ω_E is a state. Assuming in addition that $\nu \circ E = \mu$, we conclude that $\omega_E(a \otimes 1) = \nu(E(a)) = \mu(a)$ and $\omega_E(1 \otimes b') = \nu'(b')$, so ω_E is indeed a coupling of μ and ν. Because of Proposition 3.2.3 we necessarily have $E = E_{\omega_E}$. The converse is covered by Theorem 3.2.2 and Proposition 3.2.3. □

So in effect we can define couplings as maps E of the form described in this proposition.

Lastly we study the dual E'_ω of E_ω, given by Theorem 3.1.4. Given a coupling ω of μ and ν, we define

$$\omega' := \delta'_\mu \circ (E'_\omega \odot \mathrm{id}_A) : B' \odot A \to \mathbb{C}$$

where $\delta'_\mu : A \odot A' \to \mathbb{C}$ is the state defined by $\delta_{\mu'}(a' \otimes a) = \langle \Lambda_\mu, a'a\Lambda_\mu \rangle$ for all $a \in A, a' \in A'$ (So δ'_μ is the diagonal coupling of μ' with itself). Since E'_ω is a u.c.p. map, it then follows, using Theorem 3.1.4, Proposition 3.2.8 and Proposition 3.2.3, that ω' is a coupling of ν' and μ' such that

$$\omega'(b' \otimes a) = \omega(a \otimes b') \tag{38}$$

for all $a \in A$ and $b' \in B'$.

Proposition 3.2.9. *In terms of the above notation we have*

$$E'_\omega = E_{\omega'} : B' \to A'$$

and

$$E_{\omega'}(b') = u^*_\mu \iota^*_{H_\mu} \pi_\omega(1 \otimes b')\iota_{H_\mu} u_\mu$$

for all $b' \in B'$, where $u_\mu : G_\mu \to H_\mu$ is the unitary operator defined by

$$u_\mu a\Lambda_\mu := \pi_\mu(a)\Omega_\mu$$

for all $a \in A$, $\iota_{H_\mu} : H_\mu \to H_\omega$ is the inclusion map, and $\iota^*_{H_\mu} : H_\omega \to H_\mu$ its adjoint.

Proof. That $E'_\omega = E_{\omega'}$, follows from the definition of ω' and Proposition 3.2.3 applied to ω' and $\delta_{\mu'}$ instead of ω and δ_ν.

Note that u_μ is defined in perfect analogy to u_ν in Eq. (33): As the cyclic representation of $(B' \odot A, \omega')$ we can use $(H_\omega, \pi_{\omega'}, \Omega_\omega)$ with $\pi_{\omega'}$ defined via

$$\pi_{\omega'}(b' \otimes a) := \pi_\omega(a \otimes b')$$

(and the universal property of tensor products) for all $b' \in B'$ and $a \in A$. Then, referring to the form of Eq. (32), we see that in the place of $(H_\nu, \pi_{\nu'}, \Omega_\nu)$ we have $(H_\mu, \pi_\mu, \Omega_\mu)$, as we would expect, since $\overline{\pi_{\omega'}(1 \otimes A)\Omega_\omega} = \overline{\pi_\omega(A \otimes 1)\Omega_\omega} = H_\mu$, $\pi_{\omega'}(1 \otimes a)|_{H_\mu} = \pi_\omega(a \otimes 1)|_{H_\mu} = \pi_\mu(a)$ and $\Omega_\mu = \Omega_\omega$ for all $a \in A$.

So u_μ plays the same role for E'_ω as u_ν does for E_ω, i.e. by definition (see Theorem 3.2.2)

$$E_{\omega'}(b') = u^*_\mu \iota^*_{H_\mu} \pi_{\omega'}(b' \otimes 1)\iota_{H_\mu} u_\mu = u^*_\mu \iota^*_{H_\mu} \pi_\omega(1 \otimes b')\iota_{H_\mu} u_\mu$$

for all $b' \in B'$. □

We are now in a position to apply E_ω to balance in subsequent sections.

3.3. A Characterization of balance

In this section we derive a characterization of balance in terms of the map E_ω from the previous section, and consider some of its consequences, including a condition for symmetry of balance. This gives insight into the meaning and possible applications of balance. We continue with the notation from Section 3.2.

The dynamics α of a system **A** can be represented by a contraction U on H_μ defined as the unique extension of

$$U\pi_\mu(a)\Omega_\mu := \pi_\mu(\alpha(a))\Omega_\mu$$

for $a \in A$. Note that U is indeed a contraction, as α satisfies Kadison's inequality (see Proposition 2.3.3). That is, $\mu(\alpha(a)^*\alpha(a)) \leq \mu(a^*a)$. (It is also simple to check from the definition of the dual system that U^* is the corresponding representation of α' on H_μ.) Similarly

$$V\pi_\nu(b)\Omega_\nu := \pi_\nu(\beta(b))\Omega_\nu$$

for all $b \in B$, to represent β on H_ν by the contraction V.

Also set

$$P_\omega := P_\nu|_{H_\mu} : H_\mu \to H_\nu, \tag{39}$$

where P_ν is again the projection of H_ω onto H_ν. Then it follows from from Eqs. (34), (36) and $\pi_\omega(a \otimes 1)H_\mu \subset H_\mu$ that

$$\begin{aligned} P_\omega\pi_\mu(a)\Omega_\mu &= P_\nu|_{H_\mu}\pi_\omega(a\otimes 1)|_{H_\mu}\Omega_\mu \\ &= P_\nu\pi_\omega(a\otimes 1)\Omega_\mu \\ &= u_\nu u_\nu^* P_\nu \pi_\omega(a\otimes 1)u_\nu u_\nu^*\Omega_\mu \\ &= u_\nu E_\omega(a)u_\nu^*\Omega_\mu \\ &= \pi_\nu(E_\omega(a))\Omega_\nu \end{aligned} \tag{40}$$

for all $a \in A$ since $\Omega_\mu = \Omega_\omega = \Omega_\nu$. Thus, P_ω is a Hilbert space representation of E_ω.

The characterization of balance in terms of E_ω is the following:

Theorem 3.3.1. *For systems* **A** *and* **B**, *let* ω *be a coupling of* μ *and* ν. *Then* **A**ω**B**, *i.e.* **A** *and* **B** *are in balance with respect to* ω, *if and only if*

$$E_\omega \circ \alpha = \beta \circ E_\omega$$

holds, or equivalently, if and only if $P_\omega U = VP_\omega$.

Proof. We prove it on Hilbert space level. Note that P_ω as defined in Eq. (39) is the unique function $H_\mu \to H_\nu$ such that $\langle P_\omega x, y\rangle = \langle x, y\rangle$ for all $x \in H_\mu$ and $y \in H_\nu$. (This is a Hilbert space version of Proposition 3.2.3, but it follows directly from the definition of P_ω.)

Assume that $\mathbf{A}$ and $\mathbf{B}$ are in balance with respect to ω. Then, for $x = \pi_\mu(a)\Omega_\mu \in H_\mu$ and $y = \pi_{\nu'}(b')\Omega_\nu \in H_\nu$, where $a \in A$ and $b' \in B'$,

$$\begin{aligned}
\langle P_\omega U x, y\rangle &= \langle Ux, y\rangle \\
&= \langle \pi_\omega(\alpha(a) \otimes 1)\Omega_\omega, \pi_\omega(1 \otimes b')\Omega_\omega\rangle \\
&= \langle \Omega_\omega, \pi_\omega(\alpha(a^*) \otimes b')\Omega_\omega\rangle = \omega(\alpha(a^*) \otimes b') \\
&= \omega(a^* \otimes \beta'(b')) \\
&= \langle \pi_\omega(a \otimes 1)\Omega_\omega, \pi_\omega(1 \otimes \beta'(b'))\Omega_\omega\rangle \\
&= \langle x, V^* y\rangle = \langle P_\omega x, V^* y\rangle = \langle V P_\omega x, y\rangle
\end{aligned}$$

which implies that $P_\omega U = V P_\omega$. Therefore, using Eqs. (31), (34), (36) and $u_\nu \Lambda_\nu = \Omega_\omega$,

$$\begin{aligned}
E_\omega \circ \alpha(a)\Lambda_\nu &= u_\nu^* P_\omega \pi_\mu(\alpha(a))\Omega_\omega = u_\nu^* P_\omega U \pi_\mu(a)\Omega_\omega \\
&= u_\nu^* V P_\omega \pi_\mu(a)\Omega_\omega = u_\nu^* V u_\nu E_\omega(a) u_\nu^* \Omega_\omega \\
&= u_\nu^* V \pi_\nu(E_\omega(a))\Omega_\omega = u_\nu^* \pi_\nu(\beta \circ E_\omega(a))\Omega_\omega \\
&= \beta \circ E_\omega(a)\Lambda_\nu.
\end{aligned}$$

But since Λ_ν is separating for B, this means that $E_\omega \circ \alpha(a) = \beta \circ E_\omega(a)$.

Conversely, if $E_\omega \circ \alpha = \beta \circ E_\omega$, then by Eq. (40),

$$\begin{aligned}
P_\omega U \pi_\mu(a)\Omega_\mu &= P_\omega \pi_\mu(\alpha(a))\Omega_\omega = \pi_\nu(E_\omega(\alpha(a)))\Omega_\omega \\
&= \pi_\nu(\beta \circ E_\omega(a))\Omega_\omega = V \pi_\nu(E_\omega(a))\Omega_\omega \\
&= V P_\omega \pi_\mu(a)\Omega_\mu
\end{aligned}$$

and so $P_\omega U = V P_\omega$. Therefore, similar to the beginning of this proof,

$$\omega(\alpha(a^*) \otimes b') = \langle P_\omega U x, y\rangle = \langle V P_\omega x, y\rangle = \omega(a^* \otimes \beta'(b'))$$

for all $a \in A$ and $b' \in B'$, as required. □

This theorem can be compared to the case of joinings in [**9**, Theorems 4.1 and 4.3]. Keep in mind that in [**9**] the dynamics of systems are given by $*$-automorphisms, and secondly an additional assumption is made involving the modular groups. The u.c.p. map obtained in [**9**] from a joining then also intertwines the modular groups, not just the dynamics.

From Theorem 3.3.1 one starts to see some aspects of the meaning of balance. In particular it can be seen from $E_\omega \circ \alpha = \beta \circ E_\omega$ that part of the dynamics of $\mathbf{B}$, more precisely the restriction $\beta|_{E_\omega(A)} : E_\omega(A) \to E_\omega(A)$ to the space $E_\omega(A)$, is given by the dynamics of $\mathbf{A}$, via E_ω.

A natural question is whether or not balance is symmetric. I.e., are $\mathbf{A}$ and $\mathbf{B}$ in balance with respect to ω if and only if $\mathbf{B}$ and $\mathbf{A}$ are in balance with respect to some coupling (related in some way to ω)? Below we derive balance conditions equivalent to $\mathbf{A}\omega\mathbf{B}$, but where (duals of) the systems **A** and **B** appear in the opposite order. This is then used to find conditions under which balance is symmetric.

By Proposition 3.2.8 and Theorem 3.3.1 the question is equivalent to asking if there is a u.c.p. map $\tilde{E} : B \to A$ such that

$$\mu \circ \tilde{E} = \nu \text{ and } \tilde{E} \circ \beta = \alpha \circ \tilde{E} \ ?$$

By Proposition 3.2.9 $E'_\omega : B' \to A'$, so let

$$j_\mu : \mathscr{L}(G_\mu) \to \mathscr{L}(G_\mu) : a \mapsto J_\mu a^* J_\mu,$$

where as in the previous section we assume that (A, μ) is in the cyclic representation $(G_\mu, \mathrm{id}_A, \Lambda_\mu)$ and J_μ is the corresponding modular conjugation. We Similarly define j_ν.

So, given a coupling ω of μ and ν, this allows us to define

$$E^\sigma_\omega := j_\mu \circ E'_\omega \circ j_\nu : B \to A,$$

Since j_μ is an anti-$*$-automorphism, the conjugate linear map j^*_μ : $\mathscr{L}(G_\mu) \to \mathscr{L}(G_\mu)$ obtained by composing j_μ with the involution, i.e.

$$j^*_\mu(a) := j_\mu(a^*)$$

for all $a \in \mathscr{L}(G_\mu)$, is completely positive in the sense that if it is applied entry-wise to elements of the matrix algebra $M_n(A)$, then it maps positive elements to positive elements for every n, just like complete positivity of linear maps (see Section 2.3). It follows that $E^\sigma_\omega = j^*_\mu \circ E'_\omega \circ j^*_\nu$ is a u.c.p. map, since E'_ω is. Consequently, since $\mu \circ E^\sigma_\omega = \mu' \circ E'_\omega \circ j_\nu = \nu' \circ j_\nu = \nu$, it follows from Proposition 3.2.8 that

$$\omega^\sigma := \delta_\mu \circ (E^\sigma_\omega \odot \mathrm{id}_{A'}) : B \odot A' \to \mathbb{C}$$

is a coupling of ν and μ. It is then also clear that

$$E_{\omega^\sigma} = E^\sigma_\omega \tag{41}$$

by applying Proposition 3.2.3.

By the definition of a dual in Theorem 3.1.4 it can be easily seen that $(E_\omega \circ \alpha)' = \alpha' \circ E'_\omega$ and $(\beta \circ E_\omega)' = E'_\omega \circ \beta'$. So if ω is a coupling of μ and ν then by Theorem 3.3.1 and the definition of E^σ_ω:

$$\begin{gathered}
\alpha' \circ E'_\omega = E'_\omega \circ \beta' \\
j_\mu \circ \alpha' \circ j_\mu \circ j_\mu \circ E'_\omega \circ j_\nu = j_\mu \circ E'_\omega \circ j_\nu \circ j_\nu \circ \beta' \circ j_\nu \\
j_\mu \circ \alpha' \circ j_\mu \circ E^\sigma_\omega = E^\sigma_\omega \circ j_\nu \circ \beta' \circ j_\nu \\
\alpha^\sigma \circ E^\sigma_\omega = E^\sigma_\omega \circ \beta^\sigma
\end{gathered}$$

if we define

$$\alpha^\sigma = j_\mu \circ \alpha' \circ j_\mu \text{ and } \beta^\sigma = j_\mu \circ \alpha' \circ j_\mu \tag{42}$$

The operator α^σ in (42) is known as the KMS-dual of α (see [**29**]), and similarly for β. This means that

$$\langle \Lambda_\mu, a_1 j_\mu(\alpha^\sigma(a_2))\Lambda_\mu \rangle = \langle \Lambda_\mu, \alpha(a_1) j_\mu(a_2)\Lambda_\mu \rangle$$

for all $a_1, a_2 \in A$, which corresponds to the definition of the KMS-dual given in [**29**, Section 2], in connection with quantum detailed balance. (In [**29**], however, the KMS-dual is indicated by a prime rather than the symbol σ.) However as mentioned in Chapter 1 when we defined Θ-sqdb, we will not explore KMS-theory in this thesis.

Finally, note that α^σ is also u.c.p. map, by the same argument as for E_ω^σ above, and that $\nu \circ \alpha_\sigma = \mu' \circ \alpha' \circ j_\mu = \mu' \circ j_\mu = \mu$.

We summarize the above findings in a single proposition:

Proposition 3.3.2. *In terms of the above notation, if ω is a coupling of μ and ν, then*

$$\mathbf{A}^\sigma := (A, \alpha^\sigma, \mu) \text{ and } \mathbf{B}^\sigma := (B, \beta^\sigma, \nu)$$

are systems and

$$\mathbf{A}\omega\mathbf{B} \Leftrightarrow \mathbf{B}'\omega'\mathbf{A}' \Leftrightarrow \mathbf{B}^\sigma\omega^\sigma\mathbf{A}^\sigma.$$

For a QMS $(\alpha_t)_{t\geq 0}$ with the σ-weak continuity property, we again have that the same σ-weak continuity property holds for $(\alpha_t^\sigma)_{t\geq 0}$ as well, where $\alpha_t^\sigma := (\alpha_t)^\sigma$ for every t. This follows from the corresponding property of $(\alpha_t')_{t\geq 0}$.

Proposition 3.3.2 is not quite symmetry of balance. However, if

$$\alpha^\sigma = \alpha \text{ and } \beta^\sigma = \beta \tag{43}$$

then it follows that

$$\mathbf{A}\omega\mathbf{B} \Leftrightarrow \mathbf{B}\omega^\sigma\mathbf{A},$$

which expresses symmetry of balance in this special case.

The condition $\alpha^\sigma = \alpha$ is known as KMS-symmetry and was studied in [**16**], [**26**], [**33**] and [**34**].

We have however not excluded the possibility that there is some coupling other than ω^σ that could be used to show symmetry of balance more generally. This possibility seems unlikely, given how natural the foregoing arguments and constructions are.

We end this section by studying a simple application of balance that follow from Theorem 3.3.1 and the facts derived in the previous section.

We consider *ergodicity* of a system $\mathbf{B}$, which we define to mean

$$B^\beta := \{b \in B : \beta(b) = b\} = \mathbb{C}1_B \tag{44}$$

in analogy to the case for $*$-automorphisms instead of u.c.p. maps. This is certainly not the only notion of ergodicity available; see for example [**6**] for an alternative definition which implies Eq. (44), because of [**6**, Lemma 2.1]. The definition we give here is however convenient to illustrate how balance can be applied: this form of ergodicity can be characterized in terms of balance, similar to how it is done in the theory of joinings (see [**21**, Theorem 3.3], [**22**, Theorem 2.1] and [**9**, Theorem 6.2]), as we now explain.

DEFINITION 3.3.3. A system $\mathbf{B}$ is said to be *disjoint* from a system $\mathbf{A}$ if the only coupling ω with respect to which $\mathbf{A}$ and $\mathbf{B}$ (in this order) are in balance, is the trivial coupling $\omega = \mu \odot \nu'$.

In the next result, an *identity system* is a system $\mathbf{A}$ with $\alpha = \mathrm{id}_A$.

PROPOSITION 3.3.4. *A system is ergodic if and only if it is disjoint from all identity systems.*

PROOF. Suppose $\mathbf{B}$ is ergodic and $\mathbf{A}$ an identity system. If $\mathbf{A}\omega\mathbf{B}$ for some coupling ω, then $\beta \circ E_\omega = E_\omega$ by Theorem 3.3.1. So $E_\omega(A) = \mathbb{C}1_B$, since $\mathbf{B}$ is ergodic. By Corollary 3.2.6 we conclude that $\omega = \mu \odot \nu'$.

Conversely, suppose that $\mathbf{B}$ is disjoint from all identity systems. Recall that $A := B^\beta$ is a von Neumann algebra (see for example [**9**, Lemma 6.4] for a proof). Therefore $\mathbf{A} := (A, \mathrm{id}_A, \mu)$ is an identity system, where $\mu := \nu|_A$. Define a coupling of μ and ν by $\omega := \delta_\nu|_{A \odot B'}$ (see Eq. (28)), then from Proposition 3.2.3 we have $E_\omega = \mathrm{id}_A$. So $E_\omega \circ \alpha = \mathrm{id}_A = \beta \circ E_\omega$, implying that $\mathbf{A}$ and $\mathbf{B}$ are in balance with respect to ω by Theorem 3.3.1. Hence, by our supposition and Corollary 3.2.6, $B^\beta = E_\omega(A) = \mathbb{C}1_B$, which means that $\mathbf{B}$ is ergodic. □

It seems plausible that some other ergodic properties can be similarly characterized in terms of balance. Given two systems, one can also ask whether balance with respect to some coupling means that an ergodic property on the one system must necessarily hold on the other system, possibly in a weaker form.

Conversely, one can in principle use balance as a way to impose less stringent alternative versions of a given property, not necessarily ergodicity, by requiring a system to be in balance with another system having the property in question. We expect that such conditions need not be directly comparable (and strictly weaker) than the property in question. This idea will be discussed further in relation to detailed balance in Section 3.5.

3.4. Transitivity of balance

Here we show transitivity of balance: if $\mathbf{A}$ and $\mathbf{B}$ are in balance with respect to ω, and $\mathbf{B}$ and $\mathbf{C}$ are in balance with respect to ψ, then $\mathbf{A}$ and $\mathbf{C}$ are in balance with respect to a certain coupling obtained from ω and ψ, and denoted by $\omega \circ \psi$. The coupling $\omega \circ \psi$ is the composition of ω and ψ, as defined and discussed in detail below. Furthermore, we discuss the connection between couplings and correspondences in the sense of Connes.

Let ω be a coupling of (A, μ) and (B, ν), and let ψ be a coupling of (B, ν) and (C, ξ). Note that $E_\psi \circ E_\omega : A \to C$ is a u.c.p. map such that $\xi \circ E_\psi \circ E_\omega = \mu$ by Theorem 3.2.2. Therefore, by Proposition 3.2.8, setting

$$\omega \circ \psi := \delta_\xi \circ ((E_\psi \circ E_\omega) \odot \mathrm{id}_{C'}), \tag{45}$$

i.e.

$$\omega \circ \psi(a \otimes c') = \delta_\xi(E_\psi(E_\omega(a)) \otimes c')$$

for all $a \in A$ and $c \in C'$, we obtain a coupling $\omega \circ \psi$ of μ and ξ such that

$$E_{\omega\circ\psi} = E_\psi \circ E_\omega. \tag{46}$$

This construction forms the foundation for the rest of this section.

We call the coupling $\omega \circ \psi$ the *composition* of the couplings ω and ψ. We can view it as an analogue of a construction appearing in the theory of joinings in classical ergodic theory; see for example [**32**, Definition 6.9].

We can immediately give the main result of this section, namely that we have transitivity of balance in the following sense:

THEOREM 3.4.1. *If* $\mathbf{A}\omega\mathbf{B}$ *and* $\mathbf{B}\psi\mathbf{C}$*, then* $\mathbf{A}(\omega \circ \psi)\mathbf{C}$.

PROOF. By Theorem 3.3.1 we have $E_\omega \circ \alpha = \beta \circ E_\omega$ and $E_\psi \circ \beta = \gamma \circ E_\psi$, so

$$E_{\omega\circ\psi} \circ \alpha = E_\psi \circ \beta \circ E_\omega = \gamma \circ E_{\omega\circ\psi},$$

which again by Theorem 3.3.1 means that $\mathbf{A}(\omega \circ \psi)\mathbf{C}$. □

In order to gain a deeper understanding of the transitivity of balance, we now study properties of the composition of couplings.

PROPOSITION 3.4.2. *The diagonal coupling* δ_ν *in Eq. (28) is the identity for composition of couplings in the sense that* $\delta_\nu \circ \psi = \psi$ *and* $\omega \circ \delta_\nu = \omega$.

PROOF. By Corollary 3.2.7, $E_{\delta_\nu} = \mathrm{id}_B$. Hence, from Eq. (46), we obtain $E_{\delta_\nu\circ\psi} = E_\psi \circ E_{\delta_\nu} = E_\psi$ and $E_{\omega\circ\delta_\nu} = E_{\delta_\nu} \circ E_\omega = E_\omega$, which concludes the proof by Corollary 3.2.4. □

In order to treat further properties of $\omega \circ \psi$ and the connection with the theory of correspondences, we need to set up the relevant notation:

Continuing with the notation in the previous two sections, also assuming (C, ξ) to be in its cyclic representation $(G_\xi, \mathrm{id}_C, \Lambda_\xi)$, and denoting the cyclic representation of $(B \odot C', \psi)$ by $(K_\psi, \varphi_\psi, \Psi_\psi)$, it follows that

$$K_\nu := \overline{\pi_\psi(B \otimes 1)\Psi_\psi},\ \varphi_\nu(b) := \varphi_\psi(b \otimes 1)|_{K_\nu} \text{ and } \Psi_\nu := \Psi_\psi$$

gives a third cyclic representation $(K_\nu, \varphi_\nu, \Lambda_\nu)$ of (B, ν), and that

$$K_\xi := \overline{\pi_\psi(1 \otimes C')\Psi_\psi},\ \varphi_{\xi'}(c') := \varphi_\psi(1 \otimes c')|_{K_\xi} \text{ and } \Psi_\xi := \Psi_\psi \tag{47}$$

gives a cyclic representation $(K_\xi, \varphi_{\xi'}, \Psi_\xi)$ of (C', ξ'). Note that to help keep track of where we are, we use the symbol K instead of H for the Hilbert spaces originating from ψ (as opposed to ω), and similarly we use φ instead of π, and Ψ instead of Ω.

We can define a unitary equivalence

$$v_\nu : G_\nu \to K_\nu \tag{48}$$

from $(G_\nu, \mathrm{id}_B, \Lambda_\nu)$ to $(K_\nu, \varphi_\nu, \Psi_\nu)$ by

$$v_\nu b\Lambda_\nu := \varphi_\nu(b)\Psi_\nu$$

for all $b \in B$. Then

$$\varphi_\nu(b) := v_\nu b v_\nu^*$$

for all $b \in B$.

By Theorem 3.2.2 we can then define the normal u.c.p. map $E_{\psi'} : C' \to B'$. By Proposition 3.2.9 this map is the dual E'_ψ of E_ψ, and we can write it as

$$E'_\psi : C' \to B' : c' \mapsto v_\nu^* \iota_{K_\nu}^* \varphi_\psi(1 \otimes c')\iota_{K_\nu} v_\nu = v_\nu^* Q_\nu \varphi_\psi(1 \otimes c')v_\nu \tag{49}$$

where Q_ν is the projection of K_ψ onto K_ν, and $Q_\nu = \iota_{K_\nu}^*$ with $\iota_{K_\nu} : K_\nu \to K_\psi$ the inclusion map, in analogy to $P_\nu = \iota_{H_\nu}^*$ in Proposition 3.2.1.

The coupling $\omega \circ \psi$ can now be expressed in various ways:

Proposition 3.4.3. *The coupling $\omega \circ \psi$ is given by the following formulas:*

$$\omega \circ \psi = \delta_\nu \circ (E_\omega \odot E'_\psi) \tag{50}$$

and

$$\omega \circ \psi = \delta_\mu \circ (\mathrm{id}_A \odot (E'_\omega \circ E'_\psi))$$

in terms of Eq. (28), as well as

$$\omega \circ \psi(a \otimes c') = \psi(E_\omega(a) \otimes c') = \omega(a \otimes E'_\psi(c')) \tag{51}$$

and

$$\omega \circ \psi(a \otimes c') = \langle u_\nu^* P_\nu \pi_\mu(a^*)\Omega_\omega, v_\nu^* Q_\nu \varphi_{\xi'}(c')\Psi_\psi\rangle \tag{52}$$

(in the inner product of the Hilbert space G_ν) for all $a \in A$ and $c' \in C'$.

Proof. From Eqs. (45) and (28), and Theorem 3.1.4, we have

$$\begin{aligned}\omega \circ \psi(a \otimes c') &= \langle \Lambda_\xi, E_\psi(E_\omega(a))c'\Lambda_\xi\rangle \\ &= \left\langle \Lambda_\nu, E_\omega(a)E'_\psi(c')\Lambda_\nu \right\rangle \end{aligned} \tag{53}$$

from which Eq. (50) follows. Continuing with the last expression above, we respectively have by Theorem 3.1.4 that

$$\begin{aligned}\omega \circ \psi(a \otimes c') &= \left\langle \Lambda_\mu, aE'_\omega(E'_\psi(c'))\Lambda_\mu \right\rangle \\ &= \delta_\mu \circ (\mathrm{id}_A \odot (E'_\omega \circ E'_\psi))(a \otimes c'),\end{aligned}$$

by Proposition 3.2.3 that

$$\omega \circ \psi(a \otimes c') = \omega(a \otimes E'_\psi(c'))$$

and by Proposition 3.2.9 that

$$\begin{aligned}\omega \circ \psi(a \otimes c') &= \langle \Lambda_\nu, E_{\psi'}(c') E_\omega(a) \Lambda_\nu \rangle \\ &= \psi'(c' \otimes E_\omega(a)) \\ &= \psi(E_\omega(a) \otimes c'),\end{aligned}$$

where in the second line we again applied Proposition 3.2.3, while the last line follows from the definition of ψ', as in Eq. (38).

On Hilbert space level we again have from Eq. (53) that

$$\begin{aligned}\omega \circ \psi(a \otimes c') &= \left\langle E_\omega(a^*)\Lambda_\nu, E'_\psi(c')\Lambda_\nu \right\rangle \\ &= \langle u_\nu^* P_\nu \pi_\omega(a^* \otimes 1) u_\nu \Lambda_\nu, v_\nu^* Q_\nu \varphi_\psi(1 \otimes c') v_\nu \Lambda_\nu \rangle \\ &= \langle u_\nu^* P_\nu \pi_\mu(a^*)\Omega_\omega, v_\nu^* Q_\nu \varphi_{\xi'}(c')\Psi_\psi \rangle\end{aligned}$$

for all $a \in A$ and $c' \in C'$, using Theorem 3.2.2 (and Proposition 3.2.1) as well as Eqs. (49), (31) and (47). □

At the end of this section $\omega \circ \psi$ will also be expressed in terms of the theory of relative tensor products of bimodules; see Corollary 3.4.7.

Next we consider triviality of transitivity, namely when $\omega \circ \psi = \mu \odot \xi'$, in which case we also say that the couplings ω and ψ are *orthogonal*, in analogy to the case of classical joinings [**32**, Definition 6.9]. We first note the following:

Proposition 3.4.4. *If either $\omega = \mu \odot \nu'$ or $\psi = \nu \odot \xi'$, then $\omega \circ \psi = \mu \odot \xi'$.*

Proof. By Proposition 3.2.3, $E_{\mu\odot\nu'} = \mu(\cdot)1_B$ and $E_{\nu\odot\xi'} = \nu(\cdot)1_C$, so $(\mu \odot \nu') \circ \psi(a \otimes c') = \delta_\xi(\mu(a)1_C \otimes c') = \mu(a)\xi'(c')$ and $\omega \circ (\nu \odot \xi')(a \otimes c') = \delta_\xi(\nu(E_\omega(a))1_C \otimes c') = \mu(a)\xi'(c')$ according to Eq. (45) and Theorem 3.2.2. □

However, as will be seen by example in Section 4.4, in general it is possible that $\omega \circ \psi = \mu \odot \xi'$ even when $\omega \neq \mu \odot \nu'$ and $\psi \neq \nu \odot \xi'$. In order for $\omega \circ \psi \neq \mu \odot \xi'$ to hold, there has to be sufficient "overlap" between ω and ψ. The following makes this precise on Hilbert space level and also explains the use of the term "orthogonal" above:

Proposition 3.4.5. *We have $\omega \circ \psi = \mu \odot \xi'$ if and only if*

$$u_\nu^*[P_\nu H_\mu \ominus \mathbb{C}\Omega_\omega] \perp v_\nu^*[Q_\nu K_\xi \ominus \mathbb{C}\Psi_\psi]$$

in the Hilbert space G_ν (see Section 3.2), where P_ν and Q_ν are the projections of H_ω onto H_ν and K_ψ onto K_ν respectively, and u_ν and v_ν are the unitaries defined above (see Eqs. (33) and (48)).

Proof. In terms of the projections P_{Ω_ω} and Q_{Ψ_ψ} of H_ω and K_ψ onto $\mathbb{C}\Omega_\omega$ and $\mathbb{C}\Psi_\psi$ respectively, we have

$$\begin{aligned}
&\left\langle u_\nu^* P_{\Omega_\omega}\pi_\mu(a^*)\Omega_\omega, v_\nu^* Q_{\Psi_\psi}\varphi_{\xi'}(c')\Psi_\psi\right\rangle \\
&= \left\langle \langle\Omega_\omega, \pi_\mu(a^*)\Omega_\omega\rangle u_\nu^*\Omega_\omega, \langle\Psi_\psi, \varphi_{\xi'}(c')\Psi_\psi\rangle v_\nu^*\Psi_\psi\right\rangle \\
&= \mu(a)\xi'(c')\langle\Lambda_\nu, \Lambda_\nu\rangle \\
&= \mu\odot\xi'(a\otimes c')
\end{aligned}$$

for all $a \in A$ and $c' \in C'$. In terms of $P := P_\nu - P_{\Omega_\omega}$ and $Q := Q_\nu - Q_{\Psi_\psi}$, it then follows from Eq. (52) that

$$\begin{aligned}
\omega\circ\psi(a\otimes c') - \mu\odot\xi'(a\otimes c') &= \langle u_\nu^* P\pi_\mu(a^*)\Omega_\omega, v_\nu^* Q\varphi_{\xi'}(c')\Psi_\psi\rangle \\
&\quad + \left\langle u_\nu^* P\pi_\mu(a^*)\Omega_\omega, v_\nu^* Q_{\Psi_\psi}\varphi_{\xi'}(c')\Psi_\psi\right\rangle \\
&\quad + \langle u_\nu^* P_{\Omega_\omega}\pi_\mu(a^*)\Omega_\omega, v_\nu^* Q\varphi_{\xi'}(c')\Psi_\psi\rangle \\
&= \langle u_\nu^* P\pi_\mu(a^*)\Omega_\omega, v_\nu^* Q\varphi_{\xi'}(c')\Psi_\psi\rangle .
\end{aligned}$$

For the last line we used $u_\nu^* P H_\omega = G_\nu \ominus \mathbb{C}\Lambda_\nu$ and $v_\nu^* Q_{\Psi_\psi} K_\psi = \mathbb{C}\Lambda_\nu$ to obtain the one term as zero, while the other term is zero, since $v_\nu^* Q K_\psi = G_\nu \ominus \mathbb{C}\Lambda_\nu$ and $u_\nu^* P_{\Omega_\omega} H_\omega = \mathbb{C}\Lambda_\nu$. Therefore $\omega\circ\psi(a\otimes c') - \mu\odot\xi'(a\otimes c')$ is zero for all $a \in A$ and $c' \in C'$ if and only if $u_\nu^*[P_\nu H_\mu \ominus \mathbb{C}\Omega_\omega] \perp v_\nu^*[Q_\nu K_\xi \ominus \mathbb{C}\Psi_\psi]$. □

To conclude this section, we discuss bimodules and correspondences, the main goal being to show how $\omega\circ\psi$ can be expressed in terms of the relative tensor product of bimodules obtained from ω and ψ. Along the way we get an indication of the connection between couplings and correspondences. Also see [**9**] for a related discussion of correspondences in the context of joinings.

The theory of correspondences was originally developed by Connes, but never published in full, although it is discussed briefly in his book [**17**, Appendix V.B]. In short, a *correspondence* from one von Neumann algebra, M, to another, N, is an M-N-bimodule (where the direction *from M to N*, is the convention used in the thesis).

For details on the relative tensor product, see for example [**52**, Section IX.3] and [**31**], but also [**48**] for some of the early work on this topic. We only outline the most pertinent aspects of relative tensor products, and the reader is referred to these sources, in particular [**52**, Section IX.3], for a more systematic exposition.

As before, let

$$j_\nu(b) := J_\nu b^* J_\nu$$

for all $b \in \mathscr{L}(G_\nu)$, with $J_\nu : G_\nu \to G_\nu$ the modular conjugation associated with (B, Λ_ν). Similarly, with (C, ξ) in its cyclic representation $(G_\xi, \mathrm{id}_C, \Lambda_\xi)$, let

$$j_\xi(c) := J_\xi c^* J_\xi$$

for all $c \in \mathscr{L}(G_\xi)$, with $J_\xi : G_\xi \to G_\xi$ the modular conjugation associated with (C, Λ_ξ).

Given a coupling ω of (A, μ) and (B, ν) as at the beginning of this section, we can view $H = H_\omega$ as an A-B-bimodule by setting

$$\pi_H(a) := \pi_\omega(a \otimes 1)$$

and

$$\pi'_H(b) := \pi_\omega(1 \otimes j_\nu(b)),$$

and writing

$$axb := \pi_H(a)\pi'_H(b)x$$

for all $a \in A$, $b \in B$, and $x \in H$. The map π_H can be shown to be normal (see for example the proof of [**9**, Theorem 3.3]), as required for it to give a left A-module, and similarly π'_H gives a normal right action of B on H; again see [**9**, Theorem 3.3]. When viewing H as the A-B-bimodule thus defined, we also denote it by ${}_AH_B$. This module is therefore an example of a correspondence from A to B.

With ψ a coupling of (B, ν) and (C, ξ) as at the beginning of this section, and $(K_\psi, \varphi_\psi, \Psi_\psi)$ the corresponding cyclic representation as before, but now using the notation $K = K_\psi$, we analogously obtain the B-C-bimodule ${}_BK_C$ via π_K and π'_K given by

$$\pi_K(b) := \varphi_\psi(b \otimes 1)$$

and

$$\pi'_K(c) := \varphi_\psi(1 \otimes j_\xi(c))$$

which enables us to write

$$byc := \pi_K(b)\pi'_K(c)y$$

for all $b \in B$, $c \in C$, and $y \in K$.

Now we form the relative tensor product (see [**52**, Definition IX.3.16])

$${}_AX_C := H \otimes_\nu K$$

with respect to the faithful normal state ν. This is also a Hilbert space (its inner product will be discussed below) and, as the notation on the left suggests, the relative tensor product is itself a A-C-bimodule. This is a special case of [**52**, Corollary IX.3.18]. The reason it works is that since H is a A-B-bimodule, any element of $\pi_H(A)$ can be viewed as an element of $\mathcal{L}(H_B)$, the space of all bounded (in the usual sense of linear operators on Hilbert spaces) right B-module maps. Similarly for the right action of C. So ${}_AX_C$ is a correspondence from A to C, which can be viewed as the composition of the correspondences ${}_AH_B$ and ${}_BK_C$.

As one may expect, the actions of A and C on $H \otimes_\nu K$ are given by

$$a(x \otimes_\nu y)c = (ax) \otimes_\nu (yc)$$

for all $a \in A$ and $c \in C$. However, in general this does not hold for all $x \in H$ and $y \in K$. In fact the elementary tensor $x \otimes_\nu y$ does not exist for all $x \in H$ and $y \in K$. However, it does work if we restrict either x or y to a certain dense subspace, say $x \in \mathfrak{D}(H, \nu) \subset H$ and $y \in K$. (See below for further details on the space $\mathfrak{D}(H,\nu)$.) We correspondingly use $x \in H$ and $y \in \mathfrak{D}'(K,\nu) \subset K$ if we rather want to restrict y to a dense subspace of K.

In particular we have $\Omega_\omega \in \mathfrak{D}(H,\nu)$ and $\Psi_\psi \in \mathfrak{D}'(K,\nu)$, and so we set

$$\Omega := \Omega_\omega \otimes_\nu \Psi_\psi \in H \otimes_\nu K,$$

which we use to define a state, denoted by $\omega \diamond \psi$, on $A \odot C'$ as follows:

$$\omega \diamond \psi(d) := \langle \Omega, \pi_X(d)\Omega \rangle \tag{54}$$

for all $d \in A \odot C'$, where π_X is the representation of $A \odot C'$ on ${}_A X_C$ given in terms of its bimodule structure by

$$\pi_X(a \otimes c')x := axj_\xi(c')$$

for all $x \in {}_A X_C$. Below we show that $\omega \diamond \psi = \omega \circ \psi$, so we have the composition of couplings expressed in terms of the relative tensor product of bimodules, i.e. in terms of the composition of correspondences.

We first review the inner product of the relative tensor product in more detail, in order to clarify its use below. Write

$$\eta'_\nu(b) := j_\nu(b)\Lambda_\nu = J_\nu b^* \Lambda_\nu \tag{55}$$

for all $b \in B$.

For every $x \in \mathfrak{D}(H,\nu)$, define the bounded linear operator $L_\nu(x) : G_\nu \to H$ by setting

$$L_\nu(x)\eta'_\nu(b) = xb \equiv \pi'_H(b)x$$

for all $b \in B$, and uniquely extending to G_ν. We note that the space $\mathfrak{D}(H,\nu)$ is defined to ensure that $L_\nu(x)$ is indeed bounded:

$$\mathfrak{D}(H,\nu) = \{x \in H : \|xb\| \leq k_x \|\eta'_\nu(b)\| \text{ for all } b \in B, \text{ for some } k_x \geq 0\}$$

It then follows that $L_\nu(x_1)^* L_\nu(x_2) \in B$ for all $x_1, x_2 \in \mathfrak{D}(H,\nu)$. The space $H \otimes_\nu K$ and its inner product is obtained from a quotient construction such that we have

$$\langle x_1 \otimes_\nu y_1, x_2 \otimes_\nu y_2 \rangle = \langle y_1, \pi_K(L_\nu(x_1)^* L_\nu(x_2)) y_2 \rangle_K \tag{56}$$

for $x_1, x_2 \in \mathfrak{D}(H,\nu)$ and $y_1, y_2 \in K$, where for emphasis we have denoted the inner product of K by $\langle \cdot, \cdot \rangle_K$. This is the "left" version, but there is also a corresponding "right" version of this formula for the inner product (see [**52**, Section IX.3]). It can be shown from the definition of $\mathfrak{D}(H,\nu)$, that $\pi_H(a)\pi_\nu(b)\Omega_\omega \in D(H,\nu)$ for all $a \in A$ and $b \in B$, from which in turn it follows that $\mathfrak{D}(H,\nu)$ is dense in H, and that $\Omega_\omega \in \mathfrak{D}(H,\nu)$. Similarly $D'(K,\nu)$, which is defined analogously, is dense in K.

From this short review of the inner product, we can show that it has the following property:

PROPOSITION 3.4.6. *In $H \otimes_\nu K$,*

$$\langle a_1 \Omega c_1, a_2 \Omega c_2 \rangle = \psi(E_\omega(a_1^* a_2) \otimes j_\xi(c_2 c_1^*)) \tag{57}$$

for $a_1, a_2 \in A$ and $c_1, c_2 \in C$.

PROOF. Firstly, we obtain a formula for $L_\nu(x)$ for elements of the form $x = \pi_H(a)\pi_\nu(b)\Omega_\omega \in D(H, \nu)$, where $a \in A$ and b. For all $b_1 \in B$ we have

$$\begin{aligned} L_\nu(x)\eta'_\nu(b_1) &= \pi'_H(b_1)\pi_H(a)\pi_\nu(b)\Omega_\omega \\ &= \pi_H(a)\pi_\nu(b)\pi_{\nu'}(j_\nu(b_1))\Omega_\omega \\ &= \pi_H(a)\pi_\nu(b)u_\nu\eta'_\nu(b_1), \end{aligned}$$

by Eqs. (35) and (55), which means that

$$L_\nu(\pi_H(a)\pi_\nu(b)\Omega_\omega) = \pi_H(a)\pi_\nu(b)u_\nu. \tag{58}$$

Applying the special case $L_\nu(\pi_H(a)\Omega_\omega) = \pi_H(a)u_\nu$ of this formula, for $a_1, a_2 \in A$ we have

$$\begin{aligned} L_\nu(\pi_H(a_1)\Omega_\omega)^* L_\nu(\pi_H(a_2)\Omega_\omega) &= u_\nu^* P_\nu \pi_H(a_1^* a_2) u_\nu \\ &= E_\omega(a_1^* a_2). \end{aligned}$$

by Theorem 3.2.2 and Proposition 3.2.1. From Eq. (56) we therefore have

$$\begin{aligned} \langle a_1 \Omega c_1, a_2 \Omega c_2 \rangle &= \langle \pi'_K(c_1)\Psi_\psi, \pi_K(E_\omega(a_1^* a_2))\pi'_K(c_2)\Psi_\psi \rangle_K \\ &= \langle \Psi_\psi, \pi_K(E_\omega(a_1^* a_2))\pi'_K(c_2 c_1^*)\Psi_\psi \rangle_K \\ &= \langle \Psi_\psi, \varphi_\psi(E_\omega(a_1^* a_2) \otimes j_\xi(c_2 c_1^*))\Psi_\psi \rangle_K \\ &= \psi(E_\omega(a_1^* a_2) \otimes j_\xi(c_2 c_1^*)). \end{aligned}$$

□

Now we can confirm that Eq. (54) is indeed equivalent to the original definition Eq. (45):

COROLLARY 3.4.7. *We have*

$$\omega \diamond \psi = \omega \circ \psi$$

in terms of the definitions Eq. (54) and Eq. (45).

PROOF. From Eq. (54)

$$\begin{aligned} \omega \diamond \psi(a \otimes c') &= \langle \Omega, \pi_X(a \otimes c')\Omega \rangle = \langle \Omega, a\Omega j_\xi(c') \rangle \\ &= \psi(E_\omega(a) \otimes c')) \end{aligned}$$

by Eq. (57), for all $a \in A$ and $c' \in C'$. By Eq. (51), $\omega \diamond \psi = \omega \circ \psi$. □

So we have $\omega \circ \psi$ expressed in terms of the vector $\Omega \in H \otimes_\nu K$. Note, however, that in general $H \otimes_\nu K$ is not the GNS Hilbert space for the state $\omega \circ \psi$, although the former contains the latter. Consider for example the simple case where $\omega = \mu \odot \nu'$ and $\psi = \nu \odot \xi'$. Then, by

Proposition 3.4.4, $\omega \circ \psi = \mu \odot \xi'$, and the GNS Hilbert space obtained from this state is $G_\mu \otimes G_\xi$, whereas $H \otimes_\nu K = G_\mu \otimes G_\nu \otimes G_\xi$.

When $(A, \mu) = (B, \nu)$ and ω is the diagonal coupling δ_ν in Eq. (28), then by [**52**, Proposition IX.3.19], ${}_A X_C$ is isomorphic to ${}_B K_C$, so in this case the correspondence ${}_A H_B$ acts as an identity from the left. Similarly from the right when ψ is the diagonal coupling. This is the correspondence version of Proposition 3.4.2.

Lastly, by Eq. (58) we have $L_\nu(\Omega_\omega) = \iota_{H_\nu} u_\nu$, therefore $L_\nu(\Omega_\omega)^* = u_\nu^* P_\nu$, which by Theorem 3.2.2 means that

$$E_\omega(a) = L_\nu(\Omega_\omega)^* \pi_H(a) L_\nu(\Omega_\omega)$$

for all $a \in A$. This is the form in which E_ω has appeared in the theory of correspondences, as a special case of maps of the form $a \mapsto L_\nu(x)^* \pi_H(a) L_\nu(x)$ for arbitrary $x \in \mathfrak{D}(H, \nu)$; see for example [**47**, Section 1.2].

3.5. Detailed balance in terms of balance

Our main goal in this section is to suggest how balance can be used to define conditions that generalize detailed balance. In order to motivate these generalized conditions, we present a specific instance of how detailed balance can be expressed in terms of balance. We focus on one form of detailed balance, namely standard quantum detailed balance with respect to a reversing operation, as defined in [**26**, Definition 3 and Lemma 1] and [**29**, Definition 1]. This form of detailed balance has only appeared in the literature relatively recently.

The basic idea of this section should also apply to properties other than detailed balance conditions, as will be explained.

We begin by noting the following simple fact in terms of the diagonal coupling δ_μ (see Eq. (28)):

PROPOSITION 3.5.1. *A system* $\mathbf{A}$ *is in balance with itself with respect to the diagonal coupling* δ_μ*, i.e.* $\delta_\mu(\alpha(a) \otimes a') = \delta_\mu(a \otimes \alpha'(a'))$ *for all* $a \in A$ *and* $a' \in A'$*. Conversely, if two systems* $\mathbf{A}$ *and* $\mathbf{B}$*, with* $(A, \mu) = (B, \nu)$*, are in balance with respect to the diagonal coupling* δ_μ*, then* $\mathbf{A} = \mathbf{B}$*, i.e.* $\alpha = \beta$*.*

PROOF. The first part is simply the definition of the dual (see Definition 3.1.3 and Theorem 3.1.4):

$$\langle \Lambda_\mu, \alpha(a) a' \Lambda_\mu \rangle = \langle \Lambda_\mu, a \alpha'(a') \Lambda_\mu \rangle$$
$$\delta_\mu(\alpha(a) \otimes a') = \delta_\mu(a \otimes \alpha'(a'))$$

for all $a \in A$ and $a' \in A'$.

And the second part simply follows from the uniqueness of the dual. That is, if $\mathbf{A}\delta_\mu\mathbf{B}$ with $A = B$ and $\mu = \nu$ then the above equations hold with α' replaced with β'. Hence $\beta' = \alpha'$ by Theorem 3.1.4, so $\beta = \alpha$

by Corollary 3.1.5. Alternatively one can also use Theorem 3.3.1 and Corollary 3.2.7. □

So, if **A** and **B** are in balance with respect to the diagonal coupling and one of the systems has some property, then the other system has it as well, since the systems are necessarily the same.

One avenue of investigation is therefore to define generalized versions of a given property by demanding only that a system is in balance with another system with the given property, with respect to a coupling (or set of couplings) other than the diagonal coupling. In particular we then do not need to assume that the two systems have the same algebra and state.

We demonstrate this idea below for a specific property, namely standard quantum detailed balance with respect to a reversing operation. In order to do so, we discuss this form of detailed balance along with Θ-KMS-duals:

Definition 3.5.2. Consider a system **A**. A *reversing operation* for **A** (or for (A, μ)), is a $*$-antihomorphism $\Theta : A \to A$ (i.e. Θ is linear, $\Theta(a^*) = \Theta(a)^*$, and $\Theta(a_1 a_2) = \Theta(a_2)\Theta(a_1)$) such that $\Theta^2 = \mathrm{id}_A$ and $\mu \circ \Theta = \mu$. Furthermore we define

$$\alpha^\Theta := \Theta \circ \alpha^\sigma \circ \Theta$$

of α in terms of the $\alpha^\sigma = j_\mu \circ \alpha' \circ j_\mu$ in Eq. (42).

In [**12**] α^Θ was introduced in the context of systems on $\mathscr{L}(\mathfrak{H})$, with $\mathfrak{H}$ a separable Hilbert space, and called a Θ-*KMS-dual*. There may be a scarcity of examples of reversing operations for general von Neumann algebras, but a standard example for $\mathscr{L}(\mathfrak{H})$ is mentioned in Section 4.5.

Using the Θ-KMS-dual, we can define the above mentioned form of detailed balance:

Definition 3.5.3. A system **A** satisfies *standard quantum detailed balance with respect to the reversing operation* Θ for (A, μ), or Θ-*sqdb*, when $\alpha^\Theta = \alpha$.

To complete the picture, we state some straightforward properties related to reversing operations Θ and the Θ-KMS-dual:

Proposition 3.5.4. *Given a reversing operation* Θ *for* **A** *as in Definition 3.5.2, we define an anti-unitary operator* $\theta : G_\mu \to G_\mu$ *by extending*

$$\theta a \Lambda_\mu := \Theta(a^*)\Lambda_\mu \tag{59}$$

which in particular gives $\theta^2 = 1$ *and* $\theta\Lambda_\mu = \Lambda_\mu$. *Then*

$$\Theta(a) = \theta a^* \theta$$

for all $a \in A$, *and consequently* Θ *is normal. This allows us to define*

$$\Theta' : A' \to A' : a' \mapsto \theta a'^* \theta$$

which is the dual of Θ *in the sense that*

$$\langle \Lambda_\mu, a\Theta'(a')\Lambda_\mu \rangle = \langle \Lambda_\mu, \Theta(a)a'\Lambda_\mu \rangle$$

for all $a \in A$ *and* $a' \in A'$. *We also have*

$$\theta J_\mu = J_\mu \theta$$

from which

$$\alpha^\Theta = (\Theta \circ \alpha \circ \Theta)^\sigma$$

and

$$(\alpha^\Theta)^\Theta = \alpha$$

follow.

Proof. Even though θ is defined similarly to the conjugate linear operator S_0 from Tomita Takesaki theory, (59) does have a bounded linear extension in $\mathscr{L}(G_\mu)$:

$$\begin{aligned}\|\Theta(a^*)\Lambda_\mu\|^2 = \langle \Theta(a^*)\Lambda_\mu, \Theta(a^*)\Lambda_\mu \rangle &= \langle \Lambda_\mu, \Theta(a)\Theta(a^*)\Lambda_\mu \rangle \\ &= \langle \Lambda_\mu, \Theta(a^*a)\Lambda_\mu \rangle \\ &= \mu \circ \Theta(a^*a) \\ &= \mu(a^*a) \\ &= \|a\Lambda_\mu\|^2\end{aligned}$$

From the definition of θ, the properties of Θ and $\theta\Lambda_\mu = \Lambda_\mu$ it follows that

$$\theta a^* \theta b \Lambda_\mu = \Theta((a^*\Theta(b^*))^*)\Lambda_\mu = \Theta(a)b\Lambda_\mu$$

for all $a, b \in A$, so $\Theta(a) = \theta a^* \theta$. Normality (i.e. σ-weak continuity) follows from this and the definition of the σ-weak topology. For $a \in A$ and $a' \in A'$ we now have $a\theta a'\theta = \theta\Theta(a^*)a'\theta = \theta a'\Theta(a^*)\theta = \theta a'\theta a$, hence $\theta a'\theta \in A'$. So Θ' is well-defined, and that it is the dual of Θ follows easily.

Denoting the closure of the operator

$$A\Lambda_\mu \to A\Lambda_\mu : a\Lambda_\mu \mapsto a^*\Lambda_\mu$$

by $S_\mu = J_\mu \Delta_\mu^{1/2}$, as usual in Tomita-Takesaki theory, we obtain $S_\mu = \theta S_\mu \theta = \theta J_\mu \theta \theta \Delta_\mu^{1/2} \theta$, hence $\theta J_\mu \theta = J_\mu$ by the uniqueness of polar decomposition, proving $\theta J_\mu = J_\mu \theta$.

Then by definition

$$\begin{aligned}\alpha^\Theta &= \Theta \circ j_\mu \circ \alpha' \circ j_\mu \circ \Theta = j_\mu \circ \Theta' \circ \alpha' \circ \Theta' \circ j_\mu = j_\mu \circ (\Theta \circ \alpha \circ \Theta)' \circ j_\mu \\ &= (\Theta \circ \alpha \circ \Theta)^\sigma\end{aligned}$$

follows. So $(\alpha^\Theta)^\Theta = \Theta \circ \Theta \circ \alpha \circ \Theta \circ \Theta = \alpha$ by Corollary 3.1.5 and Proposition 3.1.6. □

Returning now to the main goal of this section, it will be convenient for us to express the Θ-KMS dual as a system:

Proposition 3.5.5. *For a reversing operation* Θ *as in Definition 3.5.2,*

$$\mathbf{A}^{\Theta} := (A, \alpha^{\Theta}, \mu)$$

is a system, called the Θ-KMS-dual *of* $\mathbf{A}$*.*

Proof. Recall from Proposition 3.3.2 that $\mathbf{A}^{\sigma}$ is a system. Since α^{σ} is u.c.p., it can be checked as in Proposition 3.3.2 from $\alpha^{\Theta} = \Theta^{*} \circ \alpha^{\sigma} \circ \Theta^{*}$, where $\Theta^{*}(a) := \Theta(a^{*})$ for all $a \in A$, that α^{Θ} is u.c.p. as well. From $\mu \circ \Theta = \mu$, we obtain $\mu \circ \alpha^{\Theta} = \mu$. □

Similar to before, for a QMS $(\alpha_t)_{t\geq 0}$ with the σ-weak continuity, we have that this continuity property also holds for $(\alpha_t^{\Theta})_{t\geq 0}$, where $\alpha_t^{\Theta} := (\alpha_t)^{\Theta}$ for every t. This follows from the continuity of $(\alpha_t^{\sigma})_{t\geq 0}$, and the fact that Θ is normal (Proposition 3.5.4).

As a simple corollary of Proposition 3.5.1 we have:

Corollary 3.5.6. *Let* $\mathbf{A}$ *be a system and let* Θ *be a reversing operation for* $\mathbf{A}$*. Then the following are equivalent:*

(a) $\mathbf{A}$ *satisfies* Θ*-sqdb.*

(b) $\mathbf{A}$ *and* $\mathbf{A}^{\Theta}$ *are in balance with respect to* δ_{μ}*.*

(c) $\mathbf{A}^{\Theta}$ *and* $\mathbf{A}$ *are in balance with respect to* δ_{μ}*.*

When two systems are in balance, we expect the one system to partially inherit properties of the other, so for any given property that a system may have, we can in principle consider generalized forms of the property via balance. In particular for Θ-sqdb:

- We can consider systems $\mathbf{A}$ and $\mathbf{B}$ which are in balance with respect to a coupling ω (or a set of couplings) other than $\mu \odot \nu'$, but not necessarily with respect to δ_{μ}. Assuming that either $\mathbf{A}$ or $\mathbf{B}$ satisfies Θ-sqdb, for some reversing operation Θ for $\mathbf{A}$ or $\mathbf{B}$ respectively, the other system can then be viewed as satisfying a generalized version of Θ-sqdb.

A second possible way of obtaining conditions generalizing Θ-sqdb for a system $\mathbf{A}$, is simply to adapt Corollary 3.5.6 more directly:

- We can require $\mathbf{A}$ and $\mathbf{A}^{\Theta}$ to be in balance with respect to some coupling ω (or a set of couplings) other than $\mu \odot \mu'$, but not necessarily with respect to δ_{μ}. Or $\mathbf{A}^{\Theta}$ and $\mathbf{A}$ to be in balance with respect to some coupling ω (or a set of couplings) other than $\mu \odot \mu'$, but not necessarily with respect to δ_{μ}.

Under KMS-symmetry (see Eq. (43)), the two options in the second condition, namely $\mathbf{A}$ and $\mathbf{A}^{\Theta}$ in balance, versus $\mathbf{A}^{\Theta}$ and $\mathbf{A}$ in balance, are equivalent:

Proposition 3.5.7. *If the system* $\mathbf{A}$ *is KMS-symmetric, then* $\mathbf{A}\omega\mathbf{A}^{\Theta}$ *if and only if* $\mathbf{A}^{\Theta}\omega_E\mathbf{A}$*, where* $E := \Theta \circ E_{\omega} \circ \Theta$*. (See Proposition 3.2.8 for* ω_E*.)*

PROOF. By KMS-symmetry $\alpha^{\Theta} = \Theta \circ \alpha \circ \Theta$. Note that for any coupling ω we have that $E = \Theta \circ E_{\omega} \circ \Theta$ is u.c.p. like α^{Θ} in the proof of Proposition 3.5.5, and $\mu \circ E = \mu$ by Theorem 3.2.2 and $\mu \circ \Theta = \mu$. Then ω_E is a coupling by Proposition 3.2.8. From Theorem 3.3.1 we have

$$\mathbf{A}\omega\mathbf{A}^{\Theta} \Leftrightarrow E_{\omega} \circ \alpha = \Theta \circ \alpha \circ \Theta \circ E_{\omega} \Leftrightarrow E \circ \alpha^{\Theta} = \alpha \circ E \Leftrightarrow \mathbf{A}^{\Theta}\omega_E\mathbf{A}.$$

□

The two types of conditions suggested above will be illustrated by a simple example in the next chapter, where the conditions obtained will in fact be weaker than Θ-sqdb.

CHAPTER 4

A balance example

In this chapter we use a simple example based on the examples in [2, Section 6], [11], [28, Section 5] and [29, Subsection 7.1] to illustrate some of the ideas discussed in this thesis. Our main reason for considering this example is that it is comparatively easy to manipulate mathematically.

4.1. The algebra and state

Let $\mathfrak{H}$ be a separable Hilbert space with total orthonormal set $e_1, e_2, e_3, \ldots$. We are going to consider systems on the von Neumann algebra $\mathscr{L}(\mathfrak{H})$. These systems will all have the same faithful normal state ζ on $\mathscr{L}(\mathfrak{H})$ given by the diagonal (in the mentioned basis) density matrix

$$\rho = \begin{bmatrix} \rho_1 & & \\ & \rho_2 & \\ & & \ddots \end{bmatrix}$$

where $\rho_1, \rho_2, \rho_3, \ldots > 0$ satisfy $\sum_{n=1}^{\infty} \rho_n = 1$. That is, $\rho \in \mathscr{L}(\mathfrak{H})$, $\rho e_n = \rho_n e_n$ for all n, and

$$\zeta(a) = \mathrm{Tr}(\rho a) = \sum_{n=1}^{\infty} \langle e_n, \rho a e_n \rangle$$

for all $a \in \mathscr{L}(\mathfrak{H})$. That ζ is faithful can easily be checked and follows from $0 < \rho_n < 1$ for all n, and that ζ is normal follows from [44, 4.2.10. Theorem] since clearly $\rho \in L^1(\mathfrak{H})$.

We now briefly explain what the cyclic representation looks like for the state ζ:

A (faithful) cyclic representation of $(\mathscr{L}(\mathfrak{H}), \zeta)$ can be written as (H, π, Ω) where $H = \mathfrak{H} \otimes \mathfrak{H}$,

$$\pi(a) = a \otimes 1$$

for all $a \in \mathscr{L}(\mathfrak{H})$, and the maximally entangled state (reducing to ρ)

$$\Omega = \sum_{n=1}^{\infty} \sqrt{\rho_n} e_n \otimes e_n$$

is the cyclic vector. For any $i, j \in \mathbb{N}$ it's easy to see that $\pi(e_i \bowtie e_j)\Omega = \sqrt{\rho_j} e_i \otimes e_j$. That is, $\pi(\mathscr{L}(\mathfrak{H}))\Omega$ contains all the elementary tensors of and is therefore dense in H. It can also easily be checked that $\langle \Omega, \pi(a)\Omega \rangle = \zeta(a)$ for all $a \in \mathscr{L}(\mathfrak{H})$. Lastly, since ζ is faithful the cyclic representation is necessarily faithful.

Our von Neumann algebra is therefore represented as

$$A = \pi(\mathscr{L}(\mathfrak{H})) = \mathscr{L}(\mathfrak{H}) \otimes 1,$$

and the state ζ is represented by the state μ on A given by

$$\mu(\pi(a)) = \zeta(a)$$

for all $a \in A$. That is, (A, μ) is a von Neumann algebra and faithful normal state pair in a cyclic representation (H, id_A, Ω). Moreover, by Theorem 2.4.1, $A' = 1 \otimes \mathscr{L}(H)$. Hence if we consider a second representation π' given by

$$\pi'(a) = 1 \otimes a$$

for all $a \in \mathscr{L}(\mathfrak{H})$, then (H, π', Ω) is similarly a second (faithful) cyclic representation of $(\mathscr{L}(\mathfrak{H}), \zeta)$. So it follows that $A' = \pi'(\mathscr{L}(\mathfrak{H}))$ and the state μ' on A' is given by

$$\mu'(\pi'(a)) = \langle \Omega, \pi'(a)\Omega \rangle = \zeta(a)$$

for all $a \in A$.

Regarding notation: Recall that for any $x, y \in \mathfrak{H}$ we denote by $x \bowtie y$ the operator defined

$$(x \bowtie y)z := x \langle y, z \rangle$$

for all $z \in \mathfrak{H}$.

4.2. The couplings

We consider couplings of ζ with itself. A coupling of ζ with itself corresponds to a coupling of μ with itself in the cyclic representation, which is a state ω on $A \odot A' = \pi(\mathscr{L}(\mathfrak{H})) \odot \pi'(\mathscr{L}(\mathfrak{H})) \cong \mathscr{L}(\mathfrak{H}) \odot \mathscr{L}(\mathfrak{H})$ such that

$$\omega(\pi(a) \otimes 1) = \mu(\pi(a)) \quad \text{and} \quad \omega(1 \otimes \pi'(a)) = \mu'(\pi'(a))$$

for all $a \in \mathscr{L}(\mathfrak{H})$. However, in this concrete example it is clearly equivalent, and simpler in terms of notation, to view ω directly as a state on $\mathscr{L}(\mathfrak{H}) \odot \mathscr{L}(\mathfrak{H})$ such that

$$\omega(a \otimes 1) = \zeta(a) \quad \text{and} \quad \omega(1 \otimes a) = \zeta(a) \tag{60}$$

for all $a \in \mathscr{L}(\mathfrak{H})$, rather than to work via the cyclic representation.

Consider any disjoint subsets $Y_1, Y_2, Y_3, \ldots$ of $\mathbb{N} := \{1, 2, 3, 4, \ldots\}$ such that $\cup_{n=1}^{\infty} Y_n = \mathbb{N}$. We construct a coupling ω which is given by a density matrix $\kappa \in \mathscr{L}(\mathfrak{H} \otimes \mathfrak{H})$, i.e.

$$\omega(c) = \mathrm{Tr}(\kappa c) = \mathrm{Tr}(c\kappa) = \sum_{i,j=1}^{\infty} \langle e_i \otimes e_j, (c\kappa)e_i \otimes e_j \rangle$$

for all $c \in \mathscr{L}(\mathfrak{H}) \odot \mathscr{L}(\mathfrak{H})$. Therefore we may as well allow $c \in \mathscr{L}(\mathfrak{H} \otimes \mathfrak{H})$, and define ω on the latter algebra, even though our theory only needs it to be defined on the algebraic tensor product $\mathscr{L}(\mathfrak{H}) \odot \mathscr{L}(\mathfrak{H}) \cong A \odot A'$.

We begin by obtaining a positive trace-class operator κ_n corresponding to the set Y_n for every n. Each κ_n will be one of three types, namely a (maximally) entangled type, a mixed type, or a product type, each of which we now discuss in turn for any n.

First, the *entangled type* (corresponding to an entangled pure state): Set

$$\Omega_n = \sum_{q \in Y_n} \sqrt{\rho_q} e_q \otimes e_q$$

and

$$\kappa_n = \Omega_n \bowtie \Omega_n = \sum_{p \in Y_n} \sum_{q \in Y_n} \sqrt{\rho_p \rho_q} (e_p \bowtie e_q) \otimes (e_p \bowtie e_q).$$

Secondly, the *mixed type* (corresponding to a mixture of pure states): Set

$$\kappa_n = \sum_{q \in Y_n} \rho_q (e_q \otimes e_q) \bowtie (e_q \otimes e_q) = \sum_{q \in Y_n} \rho_q (e_q \bowtie e_q) \otimes (e_q \bowtie e_q).$$

Thirdly, the *product type*: Set

$$\kappa_n = d_n \otimes d_n$$

where

$$d_n := \left(\sum_{p \in Y_n} \rho_p \right)^{-1/2} \sum_{q \in Y_n} \rho_q (e_q \bowtie e_q).$$

For each type we take

$$\kappa_n = 0$$

if Y_n is empty (this allows for a partition of $\mathbb{N}$ into a finite number of non-empty subsets).

For each $n \in \mathbb{N}$, define

$$\omega_n : \mathscr{L}(\mathfrak{H}) \odot \mathscr{L}(\mathfrak{H}) : c \mapsto \operatorname{Tr}(\kappa_n c).$$

Keep in mind that ω_n depends on the particular partition of $\mathbb{N}$ and on what the *type* of each κ_n is.

For any $a \in \mathscr{L}(\mathfrak{H})$, if κ_n is of the entangled type then it is straightforward to check that

$$\begin{aligned}
\omega_n(a \otimes 1) &= \sum_{i,j=1}^{\infty} \left\langle e_i \otimes e_j, (a \otimes 1)\Omega_n \bowtie \left(\sum_{q \in Y_n} \sqrt{\rho_q} e_q \otimes e_q \right) e_i \otimes e_j \right\rangle \\
&= \sum_{q \in Y_n} \left\langle e_q \otimes e_q, \sum_{p \in Y_n} \sqrt{\rho_q} \sqrt{\rho_p}\, a e_p \otimes e_p \right\rangle \\
&= \sum_{q \in Y_n} \rho_q \langle e_q, a e_q \rangle,
\end{aligned}$$

if κ_n is of the mixed type, that

$$\begin{aligned}
\omega_n(a\otimes 1) &= \sum_{i,j=1}^{\infty}\left\langle e_i\otimes e_j, a\otimes 1\left(\sum_{q\in Y_n}\rho_q(e_q\otimes e_q)\bowtie(e_q\otimes e_q)\right)e_i\otimes e_j\right\rangle\\
&= \sum_{q\in Y_n}\langle e_q\otimes e_q, a\otimes 1\,(\rho_q e_q\otimes e_q)\rangle\\
&= \sum_{q\in Y_n}\rho_q\langle e_q, ae_q\rangle
\end{aligned}$$

and if κ_n is of the product type that

$$\begin{aligned}
\omega_n(a\otimes 1) &= \sum_{i,j=1}^{\infty}\left\langle e_i\otimes e_j, a\otimes 1\left(\frac{1}{\sqrt{\rho_{Y_N}}}\sum_{q\in Y_n}(e_q\bowtie e_q)\otimes\frac{1}{\sqrt{\rho_{Y_N}}}\sum_{q\in Y_n}(e_q\bowtie e_q)\right)e_i\otimes e_j\right\rangle\\
&= \sum_{p,q\in Y_n}\left\langle e_p\otimes e_q, a\otimes 1\frac{1}{\sqrt{\rho_{Y_N}}}\rho_p e_p\otimes\frac{1}{\sqrt{\rho_{Y_N}}}\rho_q e_q\right\rangle\\
&= \frac{1}{p_{Y_n}}\sum_{p\in Y_n}\sum_{q\in Y_n}\rho_p\langle e_p, ae_p\rangle\rho_q\langle e_q, e_q\rangle\\
&= \sum_{q\in Y_n}\rho_q\langle e_q, ae_q\rangle
\end{aligned}$$

where $\rho_{Y_n} := \sum_{n\in Y_n}\rho_n$. The same equalities similarly hold for $1\otimes a$, that is

$$\omega_n(a\otimes 1) = \omega_n(1\otimes a) = \sum_{q\in Y_n}\rho_q\,\langle e_q, ae_q\rangle \tag{61}$$

for all $a\in B(\mathfrak{H})$. It is also straightforward to verify that

$$\mathrm{Tr}(\kappa_n) = \sum_{q\in Y_n}\rho_q. \tag{62}$$

For each n, let κ_n be any of the three types above. Then κ_n is indeed trace-class and positive, and so setting

$$\omega_n(c) = \mathrm{Tr}(\kappa_n c)$$

for all $c\in\mathscr{L}(\mathfrak{H}\otimes\mathfrak{H})$, we obtain a well-defined positive linear functional ω_n on $\mathscr{L}(\mathfrak{H}\otimes\mathfrak{H})$. Then

$$\omega := \sum_{n=1}^{\infty}\omega_n$$

converges in the norm of $\mathscr{L}(\mathfrak{H}\otimes\mathfrak{H})^*$, since $\|\omega_n\| = \omega_n(1) = \mathrm{Tr}(\kappa_n)$, so $\sum_{n=1}^{\infty}\|\omega_n\| = 1$. Correspondingly,

$$\kappa := \sum_{n=1}^{\infty}\kappa_n \tag{63}$$

converges in the trace-class norm $\|\cdot\|_1$, since

$$\sum_{n=1}^{\infty} \|\kappa_n\|_1 = \sum_{n=1}^{\infty} \mathrm{Tr}(\kappa_n) = 1.$$

Then it indeed follows that

$$\omega(c) = \sum_{n=1}^{\infty} \mathrm{Tr}(\kappa_n c) = \mathrm{Tr}(\kappa c),$$

since $|\sum_{n=1}^{m} \mathrm{Tr}(\kappa_n c) - \mathrm{Tr}(\kappa c)| \leq \|\sum_{n=1}^{m} \kappa_n - \kappa\|_1 \|c\|$ (where $\|\cdot\|_1$ denotes the trace-class norm; see [44, Theorem 2.4.16.]).

Furthermore $\omega(1) = \sum_{n=1}^{\infty} \omega_n(1) = \sum_{n=1}^{\infty} \rho_n = 1$, and from Eq. (61) it follows that the conditions in Eq. (60) hold. So ω is a coupling of ζ with itself as required.

For $Y_1 = \mathbb{N}$, i.e. $\kappa = \kappa_1$ and $\Omega_1 = \Omega$, we can get two extremes, namely the diagonal coupling ω if κ_1 is of the entangled type:

$$\begin{aligned}
\omega(a \otimes b) &= \sum_{i,j=1}^{\infty} \langle e_i \otimes e_j, a \otimes b\ \Omega \bowtie \Omega\ e_i \otimes e_j \rangle \\
&= \sum_{i,j=1}^{\infty} \left\langle e_i \otimes e_j, a \otimes b\ \Omega \left(\sum_{n}^{\infty} \sqrt{\rho_n} \langle e_n, e_i \rangle \langle e_n, e_j \rangle \right) \right\rangle \\
&= \sum_{n=1}^{\infty} \langle e_n \otimes e_n, a \otimes b\ \Omega \sqrt{\rho_n} \rangle \\
&= \langle \Omega, a \otimes b\ \Omega \rangle \\
&= \delta_\mu \left(\pi(a) \otimes \pi'(b) \right),
\end{aligned}$$

for all $a, b \in \mathscr{L}(\mathfrak{H})$. Alternatively ω can be the product state $\zeta \otimes \zeta$ on $\mathscr{L}(\mathfrak{H} \otimes \mathfrak{H})$ when κ_1 is of the product type:

$$\begin{aligned}
\omega(a \otimes b) &= \sum_{i,j=1}^{\infty} \langle e_i \otimes e_j, a \otimes b\ d_1 \otimes d_1\ e_i \otimes e_j \rangle \\
&= \sum_{i,j=1}^{\infty} \langle e_i \otimes e_j, a \otimes b\ (\rho_i e_i) \otimes (\rho_j e_j) \rangle \\
&= \sum_{i,j=1}^{\infty} \rho_i \langle e_i, a e_i \rangle \rho_j \langle e_j, b e_j \rangle \\
&= \mu(\pi(a)) \mu'(\pi'(b)) \\
&= \zeta \otimes \zeta (a \otimes b)
\end{aligned}$$

for all $a, b \in \mathscr{L}(\mathfrak{H})$. But the construction above gives many cases other than these two extremes. Then balance with respect to ω is non-trivial, but does not necessarily force two systems $\mathbf{A}$ and $\mathbf{B}$ on the same algebra A to have the same dynamics as in Proposition 3.5.1.

4.3. The dynamics

We now construct dynamics in order to obtain examples of systems on the von Neumann algebra $\mathscr{L}(\mathfrak{H})$. Let $r_j \in \{3, 4, 5, ...\}$ and $0 < k_j < 1$ for $j = 1, 2, 3, ...$, and write $k = (k_1, k_2, k_3, ...)$. In terms of the $n \times n$ matrix

$$O_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix},$$

with the blank spaces all being zero. That is, O_n can be obtained from the $n \times n$ identity matrix by shifting all columns one position to the left, and the first column to the far right. We then define $R_k \in \mathscr{L}(\mathfrak{H})$ by the infinite matrix

$$R_k = \begin{bmatrix} k_1^{1/2} O_{r_1} & & \\ & k_2^{1/2} O_{r_2} & \\ & & \ddots \end{bmatrix}$$

in the basis $e_1, e_2, e_3, ...,$ where again the blank spaces are zero. In other words, $R_k e_1 = k_1^{1/2} e_2$ etc. So R_k consists of a infinite direct sum of finite cycles, each cycle including its own factor $k_n^{1/2}$. Replacing k by $1 - k := (1 - k_1, 1 - k_2, 1 - k_3, ...)$, we similarly obtain R_{1-k}. In the same basis we consider a self-adjoint operator $g \in \mathscr{L}(\mathfrak{H})$ defined by the diagonal matrix

$$g = \begin{bmatrix} g_1 & & \\ & g_2 & \\ & & \ddots \end{bmatrix},$$

with $g_1, g_2, g_3, ...$ a bounded sequence in $\mathbb{R}$. Note that $R_k^* R_k + R_{1-k} R_{1-k}^* = 1$. So by [45, Corollary 30.13] we can define the generator $\mathcal{K}$ of a uniformly continuous semigroup $\mathcal{S} = (\mathcal{S}_t)_{t \geq 0}$ in $\mathscr{L}(\mathfrak{H})$ by

$$\begin{aligned} \mathcal{K}(a) &= i[g, a] - \frac{1}{2}\left[R_k^* R_k a + a R_k^* R_k - 2 R_k^* a R_k\right] \\ &\quad - \frac{1}{2}\left[R_{1-k}^* R_{1-k} a + a R_{1-k}^* R_{1-k} - 2 R_{1-k}^* a R_{1-k}\right] \\ &= R_k^* a R_k + R_{1-k}^* a R_{1-k} - a + i[g, a] \end{aligned}$$

for all $a \in \mathscr{L}(\mathfrak{H})$. That is, $\mathcal{S}_t(a) = e^{t\mathcal{K}}(a)$ for all $a \in \mathscr{L}(\mathfrak{H})$. For the original papers on generators for uniformly continuous semigroups see [35] and [40].

In the same way and still using the same basis, for $l = (l_1, l_2, l_3, ...)$ with $0 < l_j < 1$ we define the generator $\mathcal{L}$ of a second uniformly continuous semigroup $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$ in $\mathfrak{H}$ by

$$\mathcal{L}(b) = R_l^* b R_l + R_{1-l} b R_{1-l}^* - b + i[h, b]$$

for all $b \in \mathscr{L}(\mathfrak{H})$, where the diagonal matrix

$$h = \begin{bmatrix} h_1 & & \\ & h_2 & \\ & & \ddots \end{bmatrix},$$

with $h_1, h_2, h_3, ...$ a bounded sequence in $\mathbb{R}$, defines a self-adjoint operator $h \in \mathscr{L}(\mathfrak{H})$.

In the rest of this chapter, we assume the following:

$$\begin{aligned} \rho_1 &= ... = \rho_{r_1} \\ \rho_{r_1+1} &= ... = \rho_{r_1+r_2} \\ \rho_{r_1+r_2+1} &= ... = \rho_{r_1+r_2+r_3} \\ &\vdots \end{aligned}$$

Then the state ζ is seen to be invariant under both $\mathcal{S}$ and $\mathcal{T}$ by checking that $\zeta \circ \mathcal{K} = 0$ and $\zeta \circ \mathcal{L} = 0$. That is,

$$\frac{d}{dt}\zeta \circ \mathcal{S}_t(a) = \zeta\left(\frac{d}{dt}\mathcal{S}_t(a)\right) = \zeta \circ \mathcal{K} \circ \mathcal{S}_t(a) = 0$$

for all $a \in \mathscr{L}(\mathfrak{H})$. Hence $\zeta \circ \mathcal{S}_t = \zeta \circ \mathcal{S}_0 = \zeta$ and similarly $\zeta \circ \mathcal{T}_t = \zeta \circ \mathcal{T}_0 = \zeta$.

It is going to be simpler (but equivalent) to work directly in terms of $\mathscr{L}(\mathfrak{H})$, rather than its cyclic representation. Nevertheless, since much of the theory of this thesis is expressed in the cyclic representation, it is worth expressing the various objects in this representation as well. In particular we can then see how to obtain duals directly in terms of $\mathscr{L}(\mathfrak{H})$.

Our two systems $\mathbf{A}$ and $\mathbf{B}$, viewed in the cyclic representation, are in terms of $A = B = \pi(\mathscr{L}(\mathfrak{H}))$, with the dynamics given by

$$\alpha_t(\pi(a)) = \pi(\mathcal{S}_t(a))$$

and

$$\beta_t(\pi(b)) = \pi(\mathcal{T}_t(b))$$

and the states μ and ν both given by

$$\mu(\pi(a)) = \nu(\pi(a)) = \zeta(a) = \mathrm{Tr}(\rho a)$$

for all $a, b \in \mathscr{L}(\mathfrak{H})$. The diagonal coupling for μ

$$\delta_\mu : \pi(\mathscr{L}(\mathfrak{H})) \odot \pi'(\mathscr{L}(\mathfrak{H})) \to \mathbb{C}$$

is given by

$$\begin{aligned} \delta_\mu(\pi(a) \odot \pi'(b)) &= \langle \Omega, \pi(a)\pi'(b)\Omega \rangle = \langle \Omega, (a \otimes b)\Omega \rangle \\ &= \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \left\langle e_p, \rho^{1/2} a e_q \right\rangle \left\langle e_q, \rho^{1/2} b^\intercal e_p \right\rangle \\ &= \mathrm{Tr}(\rho^{1/2} a \rho^{1/2} b^\intercal) \end{aligned}$$

where $b^{\intercal} \in \mathscr{L}(\mathfrak{H})$ is obtained as the transpose of the matrix representation of b in terms of the basis $e_1, e_2, e_3, \ldots$. In effect δ_μ is the maximally entangled state $\langle \Omega, (\cdot)\Omega \rangle$ on $\mathscr{L}(\mathfrak{H}) \odot \mathscr{L}(\mathfrak{H})$, reducing to $\mathrm{Tr}(\rho(\cdot))$ on $B(\mathfrak{H})$.

The dual $\beta_t' : \pi'(\mathscr{L}(\mathfrak{H})) \to \pi'(\mathscr{L}(\mathfrak{H}))$ of β_t is given by

$$\langle \Omega, \pi(b)\beta_t'(\pi'(b'))\Omega \rangle = \langle \Omega, \beta_t(\pi(b))\pi'(b')\Omega \rangle$$

for all $b, b' \in \mathscr{L}(\mathfrak{H})$.

We therefore define the dual $\mathcal{L}'$ of $\mathcal{L}$ via the representations by requiring

$$\langle \Omega, \pi(b)\pi'(\mathcal{L}'(b'))\Omega \rangle = \langle \Omega, \pi(\mathcal{L}(b))\pi'(b')\Omega \rangle$$

for all $b, b' \in \mathscr{L}(\mathfrak{H})$, i.e.

$$\mathrm{Tr}(\rho^{1/2} a \rho^{1/2} (\mathcal{L}'(b))^{\intercal}) = Tr(\rho^{1/2} \mathcal{L}(a) \rho^{1/2} b^{\intercal})$$

for all $a, b \in \mathscr{L}(\mathfrak{H})$. Note that $\mathcal{L}'$ is indeed the dual (with respect to ζ) of $\mathcal{L}$ in the sense of Theorem 3.1.4, but represented on $\mathfrak{H}$ instead of on the GNS Hilbert space. It is then straightforward to verify that

$$\mathcal{L}'(b) = R_{1-l}^* b R_{1-l} + R_l b R_l^* - b + i[h, b] \tag{64}$$

for all $b \in \mathscr{L}(\mathfrak{H})$. From this one can see that $\mathcal{L}'$ is also the generator of a uniformly continuous semigroup $\mathcal{T}' = (\mathcal{T}_t')_{t \geq 0}$ in $\mathfrak{H}$, which in addition satisfies

$$\langle \Omega, \pi(b)\pi'(\mathcal{T}_t'(b'))\Omega \rangle = \langle \Omega, \pi(\mathcal{T}_t(b))\pi'(b')\Omega \rangle$$

and therefore

$$\pi'(\mathcal{T}_t'(b')) = \beta_t'(\pi'(b'))$$

for all $b, b' \in \mathscr{L}(\mathfrak{H})$. As with $\mathcal{L}'$ above, $\mathcal{T}_t'$ is the dual of $\mathcal{T}_t$ in the sense of Definition 3.1.3, but represented on $\mathfrak{H}$. So we correspondingly call the semigroup $\mathcal{T}'$ the *dual* of the semigroup $\mathcal{T}$.

We now have a complete description of the systems, as well as their duals.

4.4. Balance

We now show examples of balance between

$$\mathbf{A} := (\mathscr{L}(\mathfrak{H}), \mathcal{S}, \zeta) \quad \text{and} \quad \mathbf{B} := (\mathscr{L}(\mathfrak{H}), \mathcal{T}, \zeta)$$

and illustrate a number of points made in this thesis. Remember that since we now have a continuous time parameter $t \geq 0$, the balance condition in Definition 3.1.7 is required to hold at every t. However, it then follows that $\mathbf{A}$ and $\mathbf{B}$ are in balance with respect to ω if and only if

$$\mathrm{Tr}(\kappa(\mathcal{K}(a) \otimes b)) = \mathrm{Tr}(\kappa(a \otimes \mathcal{L}'(b))$$

for all $a, b \in \mathscr{L}(\mathfrak{H})$. From this one can easily check that **A** and **B** are in balance with respect to ω if and only if

$$\begin{aligned}&(R_k \otimes 1)\kappa(R_k \otimes 1)^* + (R_{1-k} \otimes 1)^*\kappa(R_{1-k} \otimes 1) - i[g \otimes 1, \kappa]\\ &= (1 \otimes R_{1-l})\kappa(1 \otimes R_{1-l})^* + (1 \otimes R_l)^8\kappa(1 \otimes R_l) - i[1 \otimes h, \kappa]\end{aligned}$$

holds. However, equating the real and imaginary parts respectively (keeping in mind that κ as given in Section 4.2 is a real infinite matrix in the basis $e_p \otimes e_q$), we see that this is equivalent to

$$\begin{aligned}&(R_k \otimes 1)\kappa(R_k \otimes 1)^* + (R_{1-k} \otimes 1)^*\kappa(R_{1-k} \otimes 1)\\ &= (1 \otimes R_{1-l})\kappa(1 \otimes R_{1-l})^* + (1 \otimes R_l)^*\kappa(1 \otimes R_l)\end{aligned} \tag{65}$$

and

$$[g \otimes 1, \kappa] = [1 \otimes h, \kappa] \tag{66}$$

both being true.

To proceed, we refine the construction of κ in Section 4.2, by only allowing

$$Y_n = \bigcup_{p \in I_n} Z_p$$

where $Z_1 = \{1, 2, ..., r_1\}$, $Z_2 = \{r_1 + 1, r_1 + 2, ..., r_1 + r_2\}$, etc., and where $I_1, I_2, I_3, ...$ is any sequence of disjoint subsets of $\mathbb{N}_+$ such that $\cup_{n \in \mathbb{N}_+} I_n = \mathbb{N}_+$. Note that an I_n is allowed to be empty (then Y_n is empty), and it is also allowed to be infinite.

It then follows that **A** and **B** are in balance with respect to ω if and only if

$$\begin{aligned}&(R_k \otimes 1)\kappa_n(R_k \otimes 1)^* + (R_{1-k} \otimes 1)^*\kappa_n(R_{1-k} \otimes 1)\\ &= (1 \otimes R_{1-l})\kappa_n(1 \otimes R_{1-l})^* + (1 \otimes R_l)^*\kappa_n(1 \otimes R_l)\end{aligned} \tag{67}$$

and

$$[g \otimes 1, \kappa_n] = [1 \otimes h, \kappa_n] \tag{68}$$

both hold for every n. To see that Eq. (67) and Eq. (68) follow from Eq. (65) and Eq. (66) respectively, place the latter into $\langle e_p \otimes e_q, (\cdot)e_{p'} \otimes e_{q'}\rangle$ for $p, q, p', q' \in Y_n$. The converse holds, since Eq. (63) is convergent in the trace-class norm.

To evaluate these conditions in detail is somewhat tedious, so we just describe it in outline below.

Note that, roughly speaking, in a term like $(R_k \otimes 1)\kappa_n(R_k \otimes 1)^*$, for κ_n of the entangled or mixed type, the first slot in the tensor product structure of κ_n is advanced by one step in each cycle appearing in R_k. In a term like $(1 \otimes R_l)^*\kappa_n(1 \otimes R_l)$, on the other hand, the second slot is rolled back by one step in each cycle, which is equivalent to the first slot being advanced by one step. So, if κ_n is of the entangled or mixed type, and

$$k_p = l_p \tag{69}$$

for each $p \in I_n$, then Eq. (67) holds.

Conversely, for $p \in I_n$, note from the definitions of the entangled and mixed type κ_n that since $r_p > 2$, the terms $(R_k \otimes 1)\kappa_n(R_k \otimes 1)^*$ and $(1 \otimes R_l)^*\kappa_n(1 \otimes R_l)$ have to be equal (hence $k_p = l_p$), for Eq. (67) to hold; the terms $(R_{1-k} \otimes 1)^*\kappa_n(R_{1-k} \otimes 1)$ and $(1 \otimes R_{1-l})\kappa_n(1 \otimes R_{1-l})^*$ involve other basis elements of $\mathfrak{H} \otimes \mathfrak{H}$ and therefore can not ensure Eq. (67) when $(R_k \otimes 1)\kappa_n(R_k \otimes 1)^* \neq (1 \otimes R_l)^*\kappa_n(1 \otimes R_l)$.

For the product type κ_n, Eq. (67) always holds, since κ_n then commutes with $R_k \otimes 1$ and $1 \otimes R_l$.

When κ_n is of the entangled type, one can verify by direct calculation that Eq. (68) holds if and only if

$$g_p - g_q = h_p - h_q \tag{70}$$

for all $p, q \in Y_n$. For the other two types of κ_n, Eq. (68) always holds, since then κ_n, $g \otimes 1$ and $1 \otimes h$ are diagonal, and so the commutators are zero.

We conclude that $\mathbf{A}$ and $\mathbf{B}$ are in balance with respect to ω if and only if the following is true: Eq. (69) holds for all $p \in I_n$ for every n for which κ_n is either of the entangled or mixed type, and Eq. (70) holds for all $p \in I_n$ for every n for which κ_n is of the entangled type.

We now also have an example where the transitivity in Theorem 3.4.1 is trivial, meaning that $\omega \circ \psi = \mu \odot \xi'$ despite having $\omega \neq \mu \odot \nu'$ and $\psi \neq \nu \odot \xi'$. To see this, let $\mathbf{C}$ be a system constructed in the same way as $\mathbf{A}$ and $\mathbf{B}$ above, so it has the same von Neumann algebra and state. However the generator giving its dynamics can use different choices in place of k, g and l, h. As above, construct two couplings ω and ψ (giving balance of $\mathbf{A}$ and $\mathbf{B}$ with respect to ω, and of $\mathbf{B}$ and $\mathbf{C}$ with respect to ψ), but with entangled and mixed types not in overlapping parts of the two couplings respectively (i.e. the respective Y_n sets corresponding to these two types in the respective couplings should be disjoint), while the rest of each coupling is a κ_n of the product type. Then it can be verified using Proposition 3.4.5 that we indeed obtain $\omega \circ \psi = \mu \odot \xi'$, despite having $\omega \neq \mu \odot \nu'$ and $\psi \neq \nu \odot \xi'$. This illustrates that to have $\omega \circ \psi \neq \mu \odot \xi'$, we need sufficient "overlap" between ω and ψ, where this overlap condition has been made precise in Hilbert space terms (in the cyclic representations) by Proposition 3.4.5.

4.5. A reversing operation

Here we consider Θ-sqdb in Definition 3.5.3 and Corollary 3.5.6, as well as the two generalized detailed balance conditions suggested at the end of Section 3.5.

We will need the modular conjugation operator associated with μ (and ζ), which can here be obtained as the conjugate linear operator $J : H \to H$ defined by

$$J(e_p \otimes e_q) = e_q \otimes e_p$$

for all $p, q = 1, 2, 3, \ldots$. Furthermore,

$$j(\pi(a)) := J\pi(a)^* J = \pi'(a^\intercal) \tag{71}$$

for all $a \in B(\mathfrak{H})$, where $a^\intercal$ denotes the transpose of a in the basis $e_1, e_2, e_3, \ldots$.

Take Θ to be transposition in the basis $e_1, e_2, e_3, \ldots$, i.e.

$$\Theta(a) := a^\intercal$$

for all $a \in \mathscr{L}(\mathfrak{H})$. This is the standard choice of a reversing operation for $(\mathscr{L}(\mathfrak{H}), \zeta)$, used for example in [**29**, Section 2]. In the cyclic representation, Θ would be given by $\pi(a) \mapsto \pi(a^\intercal)$. It is readily confirmed from Eq. (71) that in this case the Θ-KMS dual of $\mathbf{B}$ is $\mathbf{B}^\Theta = (\mathscr{L}(\mathfrak{H}), \mathcal{T}', \zeta)$, i.e. in the cyclic representation we would have $\alpha_t^\Theta = \alpha_t'$ for all t.

For the diagonal coupling δ, obtained when κ_1 is of the entangled type with $Y_1 = \mathbb{N}_+$, then from Eqs. (69) and (64) we see that $\mathbf{B}$ and $\mathbf{B}^\Theta$ are in balance with respect to δ, i.e. $\mathbf{B}$ satisfies Θ-sqdb (Corollary 3.5.6), if and only if $l_p = 1 - l_p$, i.e. $l_p = 1/2$, for all p.

More generally, consider the situation where $\mathbf{B}$ satisfies Θ-sqdb, and $\mathbf{A}$ and $\mathbf{B}$ are in balance with respect to ω. It then follows from Eq. (69) that $k_p = 1/2$ for all p in every I_n such that κ_n is of the entangled or mixed type, but we need not have $k_p = 1/2$ for other values of p. This is therefore a strictly weaker condition on $\mathbf{A}$ than Θ-sqdb, as long as not all the $\kappa_n' s$ are of the entangled or mixed type.

Next consider the situation where $\mathbf{A}$ and $\mathbf{A}^\Theta$ are in balance with respect to ω, where again not all the $\kappa_n' s$ are of the entangled or mixed type. Then in a similar way we again see that $k_p = 1/2$ for all p in every I_n such that κ_n is of the entangled or mixed type, but we need not have $k_p = 1/2$ for other values of p. So again this is a strictly weaker condition than Θ-sqdb.

This illustrates the two conditions suggested at the end of Section 3.5, albeit in a very simple situation. Here the two conditions are essentially equivalent when applied to $\mathbf{A}$, but we expect this not to be the case in general.

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