Quadratic Hamilton–Poisson Systems on $\mathfrak{se}(1,1)^*_-$: the Inhomogeneous Case

D.I. Barrett, **R. Biggs** and **C.C. Remsing**

Abstract We consider equivalence, stability and integration of quadratic Hamilton–Poisson systems on the semi-Euclidean Lie–Poisson space $\mathfrak{se}(1,1)^*_-$. The inhomogeneous positive semidefinite systems are classified (up to affine isomorphism); there are 16 normal forms. For each normal form, we compute the symmetry group and determine the Lyapunov stability nature of the equilibria. Explicit expressions for the integral curves of a subclass of the systems are found. Finally, we identify several basic invariants of quadratic Hamilton–Poisson systems.

Keywords Hamilton–Poisson system · Lie–Poisson space · Lyapunov stability

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1 Introduction

The dual space of a Lie algebra admits a natural Poisson structure, called the Lie–Poisson structure. Such structures are in a one-to-one correspondence with

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linear Poisson structures [24] (i.e., those structures for which the Poisson bracket of two linear functions is linear). Many dynamical systems admit a Hamiltonian formulation in terms of Lie–Poisson brackets: for instance, the motion of a rigid body (and its generalizations) [20, 25] and (on infinite-dimensional Lie algebras) fluid dynamics in the form of Euler's equation for an ideal fluid [25]. Moreover, such systems arise naturally in the study of invariant optimal control problems [21, 7, 22].

The study of quadratic Hamilton–Poisson systems (especially on low-dimensional spaces) has seen a flurry of recent activity. We provide a brief overview here. Systems on the orthogonal space $\mathfrak{so}(3)^*_-$, as well as the Euclidean space $\mathfrak{se}(2)^*_-$, have been treated extensively. In particular, on $\mathfrak{so}(3)^*_-$ orthogonal equivalence and explicit integration of homogeneous systems is considered in [19] (see also [29]) whereas in [2, 5] affine equivalence, integration and stability of inhomogeneous systems have been studied (in a similar vein to this paper). Similar questions on $\mathfrak{se}(2)^*_-$ were considered in [4, 3, 6]. On the other hand, spectral and Lyapunov stability, as well as numerical integration, of homogeneous Hamilton–Poisson systems on $\mathfrak{se}(1,1)^*_-$ are considered in [9], and spectral stability and numerical integration on $\mathfrak{sl}(2,\mathbb{R})^*_-$ in [13]. A number of Hamilton–Poisson systems have also been studied from the viewpoint of invariant optimal control problems (see, e.g., [3, 6, 15, 28, 12, 11] and references therein). A thorough treatment of homogeneous systems on three-dimensional Lie–Poisson spaces has also recently been published [18] (see also [16]).

This paper serves as a sequel to the earlier work [14], in which the homogeneous systems on $\mathfrak{se}(1,1)^*_-$ were treated; here we are chiefly concerned with the inhomogeneous systems. Together, these papers form an extensive and systematic study of the classification, integration and stability of positive semidefinite quadratic Hamilton–Poisson systems on $\mathfrak{se}(1,1)^*_-$. (Our restriction to positive semidefinite quadratic forms is motivated by control theoretic considerations; see, e.g., [14, 17].) We expect that this study will complement existing work on other Lie–Poisson spaces and would be integral to any systematic treatment of (inhomogeneous) Hamilton–Poisson systems in three dimensions.

We start by classifying the inhomogeneous and positive semidefinite quadratic Hamilton–Poisson systems on $\mathfrak{se}(1,1)^*_-$. Sixteen normal forms are obtained, including five one-parameter families of systems, as well as a single two-parameter family of systems. We distinguish between those systems whose integral curves evolve on lines, on planes, or on neither lines nor planes. The latter group is further subdivided by separating out those systems for which the equilibria are the union of lines or planes. Some systems are shown to be equivalent to systems previously considered on the orthogonal space $\mathfrak{so}(3)^*_-$ and Euclidean space $\mathfrak{se}(2)^*_-$; these systems shall be excluded from our treatment of stability and integration.

For each normal form, we compute the symmetry group and determine the (Lyapunov) stability nature of its equilibria. To prove stability, the extended energy-Casimir method [26] is applied; instability either follows from spectral instability or by a direct approach.

With the exception of a subclass of systems, we find explicit expressions for all (maximal) integral curves. (Due to the complexity of the computations required, we exclude those nonplanar systems whose equilibria are not the union of lines or planes.) We provide proofs for typical cases. Most integral curves are expressed in terms of elementary functions. However, for one system the Jacobi elliptic functions are used. Also, it turns out that the Hamiltonian vector fields for two of the normal forms are not complete.

We conclude the paper by identifying some invariants of quadratic Hamilton–Poisson systems. These invariants may be used to form a "taxonomy" of systems.

2 Preliminaries

2.1 Lie–Poisson spaces

Let $\mathfrak{g}$ be an n-dimensional (real) Lie algebra. The dual space $\mathfrak{g}^*$ admits a natural linear Poisson structure, called the (minus) *Lie–Poisson structure* [24,25]. If $F, G \in C^\infty(\mathfrak{g}^*)$, then the Lie–Poisson bracket is given by

$$\{F,G\}(p) = -\left\langle \mathrm{ad}^*_{\mathbf{d}F(p)}\, p, \mathbf{d}G(p)\right\rangle = -\langle p, [\mathbf{d}F(p), \mathbf{d}G(p)]\rangle.$$

Here $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$ and $\mathrm{ad}^*_{\mathbf{d}F(p)}$ is the dual of the adjoint map $\mathrm{ad}_{\mathbf{d}F(p)} = [\mathbf{d}F(p), \cdot]$. (As $\mathbf{d}F(p)$ and $\mathbf{d}G(p)$ are linear functions on $\mathfrak{g}^*$, they are identified with elements of $\mathfrak{g}$.) The Lie–Poisson space $(\mathfrak{g}^*, \{\cdot,\cdot\})$ is denoted $\mathfrak{g}^*_-$. A *linear Poisson automorphism* is a linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ such that $\{F,G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$ for every $F, G \in C^\infty(\mathfrak{g}^*)$. Linear Poisson automorphisms are exactly the dual maps of Lie algebra automorphisms.

The *Hamiltonian vector field* $\vec{H}$ corresponding to a function $H \in C^\infty(\mathfrak{g}^*)$ is defined as $\vec{H}[F] = \{F, H\}$ for $F \in C^\infty(\mathfrak{g}^*)$. Explicitly, we have $\vec{H}(p) = \mathrm{ad}^*_{\mathbf{d}H(p)}\, p$. A *Casimir function* is a function $C \in C^\infty(\mathfrak{g}^*)$ such that $\vec{C} = 0$. (Casimir functions are constants of motion for any Hamiltonian vector field on $\mathfrak{g}^*$.) An *integral curve* of a Hamiltonian vector field $\vec{H}$ is an absolutely continuous curve $p(\cdot) : (a,b) \to \mathfrak{g}^*$ such that $\dot{p}(t) = \vec{H}(p(t))$ for almost every t. We say that $\vec{H}$ is *complete* if the domain of every integral curve of $\vec{H}$ can be extended to $\mathbb{R}$ (cf. [1]). An integral curve is *maximal* if it has maximal domain. The following lemma is easy to prove.

Lemma 1 *Let $p(\cdot) : (a,b) \to \mathfrak{g}^*$ be an integral curve of $\vec{H}$. If (i) $a = -\infty$ or $\lim_{t\to a} \|p(t)\| = \infty$ and (ii) $b = \infty$ or $\lim_{t\to b} \|p(t)\| = \infty$, then $p(\cdot)$ is maximal.*

A *quadratic Hamilton–Poisson system* is a pair $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$, where $\mathfrak{g}^*_-$ is a Lie–Poisson space and $H_{A,\mathcal{Q}}$ is a Hamiltonian function of the form

$$H_{A,\mathcal{Q}}(p) = L_A(p) + H_{\mathcal{Q}}(p) = \langle p, A\rangle + \mathcal{Q}(p).$$

Here $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a quadratic form on $\mathfrak{g}^*$. (In coordinates, we write $\mathcal{Q}(p) = \frac{1}{2} p Q p^\top$, where $Q \in \mathbb{R}^{n\times n}$.) When the space $\mathfrak{g}^*_-$ is fixed, $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ will be identified with its Hamiltonian $H_{A,\mathcal{Q}}$. We shall consider only those systems for which $\mathcal{Q}$ is positive semidefinite. If $A = 0$, then the system is said to be *homogeneous*; otherwise, it is called *inhomogeneous*.

Let $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ and $(\mathfrak{h}^*_-, H_{B,\mathcal{R}})$ be two quadratic Hamilton–Poisson systems. We say that $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are *affinely equivalent* (or A-equivalent) if there exists an affine isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ such that $\psi_* \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}}$. If ψ is a linear isomorphism, then $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are called *linearly equivalent* (or L-equivalent). It is easy to show that the following systems are all L-equivalent to $H_{A,\mathcal{Q}}$:

($\mathfrak{E}$1) $H_{A,\mathcal{Q}} \circ \psi$, where $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ is a linear Poisson automorphism.

($\mathfrak{C}2$) $H_{A,\mathcal{Q}} + C$, where C is a Casimir function.
($\mathfrak{C}3$) $H_{A,r\mathcal{Q}}$, where $r \neq 0$.

Affine equivalence of two inhomogeneous systems implies linear equivalence of the corresponding homogeneous systems.

Proposition 1 *If $\psi : p \mapsto \psi_0(p) + q$ is an affine isomorphism such that $\psi_* \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}}$, then $(\psi_0)_* \vec{H}_{\mathcal{Q}} = \vec{H}_{\mathcal{R}}$.*

Proof We have

$$\vec{H}_{A,\mathcal{Q}}(p) = \mathrm{ad}^*_{\mathbf{d}(L_A + H_{\mathcal{Q}})(p)}\, p = \mathrm{ad}^*_A\, p + \mathrm{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(p)}\, p = \vec{L}_A(p) + \vec{H}_{\mathcal{Q}}(p).$$

Accordingly, $\psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \psi_0 \cdot \vec{L}_A + \psi_0 \cdot \vec{H}_{\mathcal{Q}}$ and

$$\begin{aligned}\vec{H}_{B,\mathcal{R}}(\psi(p)) &= \vec{L}_B(\psi(p)) + \vec{H}_{\mathcal{R}}(\psi(p)) \\ &= (\vec{L}_B \circ \psi_0)(p) + \vec{L}_B(q) + (\vec{H}_{\mathcal{R}} \circ \psi_0)(p) + \vec{H}_{\mathcal{R}}(q) + F(p) + G(p)\end{aligned}$$

where $F(p) = \mathrm{ad}^*_{\mathbf{d}H_{\mathcal{R}}(\psi_0(p))}\, q$ and $G(p) = \mathrm{ad}^*_{\mathbf{d}H_{\mathcal{R}}(q)}\, \psi_0(p)$. Expanding terms in $(\psi_0 \cdot \vec{H}_{A,\mathcal{Q}})(p) - (\vec{H}_{B,\mathcal{R}} \circ \psi)(p) = 0$, we get

$$\begin{aligned}&(\psi_0 \cdot \vec{L}_A)(p) + (\psi_0 \cdot \vec{H}_{\mathcal{Q}})(p) - (\vec{L}_B \circ \psi_0)(p) - (\vec{H}_{\mathcal{R}} \circ \psi_0)(p) - F(p) - G(p) \\ &\quad = \vec{L}_B(q) + \vec{H}_{\mathcal{R}}(q).\end{aligned} \tag{1}$$

Taking $p = 0$ yields $\vec{L}_B(q) + \vec{H}_{\mathcal{R}}(q) = 0$. Interpreting both sides of (1) as maps $\mathfrak{g}^* \to \mathfrak{h}^*$, we have

$$T_0(\psi_0 \cdot \vec{L}_A) + T_0(\psi_0 \cdot \vec{H}_{\mathcal{Q}}) - T_0(\vec{L}_B \circ \psi_0) - T_0(\vec{H}_{\mathcal{R}} \circ \psi_0) - T_0F - T_0G = 0.$$

(Here T_0F is the tangent map of F at zero.) Elementary calculations show that $T_0(\psi_0 \cdot \vec{H}_{\mathcal{Q}}) = T_0(\vec{H}_{\mathcal{Q}} \circ \psi_0) = 0$. Furthermore, F and G can be shown to be linear; hence we make the identifications $T_0F \leftrightarrow F$ and $T_0G \leftrightarrow G$. (Likewise, $\psi_0 \cdot \vec{L}_A$ and $\vec{L}_B \circ \psi_0$ are linear.) Thus $\psi_0 \cdot \vec{L}_A - \vec{L}_B \circ \psi_0 - F - G = 0$, and so (1) becomes $\psi_0 \cdot \vec{H}_{\mathcal{Q}} = \vec{H}_{\mathcal{R}} \circ \psi_0$. That is, $(\psi_0)_* \vec{H}_{\mathcal{Q}} = \vec{H}_{\mathcal{R}}$. □

2.2 Stability

A point $p_e \in \mathfrak{g}^*$ is called an *equilibrium point* of a Hamiltonian vector field $\vec{H}$ if $\vec{H}(p_e) = 0$. An equilibrium point p_e is said to be (Lyapunov) *stable* if for every neighbourhood N of p_e there exists a neighbourhood $N' \subseteq N$ of p_e such that, for every integral curve $p(\cdot)$ of $\vec{H}$ with $p(0) \in N'$, we have $p(t) \in N$ for all $t > 0$. The point p_e is *spectrally stable* if all eigenvalues of the linearized dynamical system $\mathbf{D}\vec{H}(p_e)$ have nonpositive real parts. Every stable equilibrium point is spectrally stable. The point p_e is *unstable* (resp. *spectrally unstable*) if it is not stable (resp. spectrally stable).

The (extended) energy-Casimir method and continuous energy-Casimir method [26] provide sufficient conditions for stability of equilibria. We state simplified versions here.

Proposition 2 *Let p_e be an equilibrium point and let C be a Casimir function. If there exist $\lambda_0, \lambda_1 \in \mathbb{R}$ such that $\mathbf{d}(\lambda_0 H + \lambda_1 C)(p_e) = 0$ and the quadratic form $\mathbf{d}^2(\lambda_0 H + \lambda_1 C)(p_e)\big|_{W\times W}$ is positive definite, where $W = \ker \mathbf{d}H(p_e) \cap \ker \mathbf{d}C(p_e)$, then p_e is stable.*

Proposition 3 *If C is a Casimir function and $H^{-1}(H(p_e)) \cap C^{-1}(C(p_e)) = \{p_e\}$ in a neighbourhood of p_e, then p_e is stable.*

2.3 The Lie–Poisson space $\mathfrak{se}(1,1)^*_-$

The three-dimensional semi-Euclidean Lie algebra

$$\mathfrak{se}(1,1) = \left\{ x_1E_1 + x_2E_2 + x_3E_3 = \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & x_3 \\ x_2 & x_3 & 0 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

is the Lie algebra of the Lie group $\mathsf{SE}(1,1)$ of (orientation-preserving) isometries of the Minkowski plane. The nonzero commutator relations are $[E_2, E_3] = -E_1$ and $[E_3, E_1] = E_2$. Let (E_1^*, E_2^*, E_3^*) be the dual of the standard basis (E_1, E_2, E_3). We write elements $p = p_1E_1^* + p_2E_2^* + p_3E_3^* \in \mathfrak{se}(1,1)^*$ as row vectors. For convenience, we take $\|p\| = \sqrt{p_1^2 + p_2^2 + p_3^2}$. The group of linear Poisson automorphisms of $\mathfrak{se}(1,1)^*_-$ are

$$\left\{ p \mapsto p \begin{bmatrix} x & y & v \\ \sigma y & \sigma x & w \\ 0 & 0 & \sigma \end{bmatrix} : v, w, x, y \in \mathbb{R},\ \sigma \in \{-1, 1\},\ x^2 \neq y^2 \right\}.$$

Let $H : \mathfrak{se}(1,1)^* \to \mathbb{R}$ be a Hamiltonian function. The equations of motion take the following explicit form:

$$\begin{cases} \dot{p}_1 = p_2 \dfrac{\partial H}{\partial p_3} \\ \dot{p}_2 = p_1 \dfrac{\partial H}{\partial p_3} \\ \dot{p}_3 = -p_1 \dfrac{\partial H}{\partial p_2} - p_2 \dfrac{\partial H}{\partial p_1}. \end{cases}$$

The function $C : p \mapsto p_1^2 - p_2^2$ is a Casimir function on $\mathfrak{se}(1,1)^*_-$.

Remark 1 The vector field $\vec{H}$ may be written in the form $\vec{H} = \frac{1}{2}\nabla H \times \nabla C$.

Lemma 2 *The points $(0,0,\mu)$, $\mu \in \mathbb{R}$ are equilibrium points for any Hamilton–Poisson system H on $\mathfrak{se}(1,1)^*_-$. If $\frac{\partial H}{\partial p_3}(0,0,\mu) \neq 0$, then the state $(0,0,\mu)$ is (spectrally) unstable.*

Proof The linearization at $(0,0,\mu)$ of the vector field $\vec{H}$ has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\frac{\partial H}{\partial p_3}(0,0,\mu)$. Hence, if $\frac{\partial H}{\partial p_3}(0,0,\mu) \neq 0$, then the state $(0,0,\mu)$ is spectrally unstable. □

Lemma 3 *Let H be a Hamilton–Poisson system on $\mathfrak{se}(1,1)^*_-$ and let $p(\cdot)$ be an absolutely continuous curve such that $\dot{p}_1 = p_2 \frac{\partial H}{\partial p_3}$, $C(p(t)) = constant$ and $H(p(t)) = constant$. Then $p(\cdot)$ is an integral curve of $\vec{H}$.*

Proof By assumption, the first equation of motion is satisfied. Differentiating both sides of $C(p(t)) = p_1(t)^2 - p_2(t)^2$ yields $0 = 2p_1\dot{p}_1 - 2p_2\dot{p}_2$, i.e., $\dot{p}_2 = \frac{p_1\dot{p}_1}{p_2} = p_1\frac{\partial H}{\partial p_3}$. Hence the second equation of motion holds. Lastly, differentiate both sides of $H(p(t)) =$ constant, to get $\dot{p}_1\frac{\partial H}{\partial p_1} + \dot{p}_2\frac{\partial H}{\partial p_2} + \dot{p}_3\frac{\partial H}{\partial p_3} = 0$. Solving for $\dot{p}_3$ gives $\dot{p}_3 = -p_1\frac{\partial H}{\partial p_2} - p_2\frac{\partial H}{\partial p_1}$. Thus $\dot{p}(t) = \vec{H}(p(t))$. □

3 Classification

We classify all inhomogeneous quadratic Hamilton–Poisson systems (with positive semidefinite quadratic form) on $\mathfrak{se}(1,1)^*_-$. This classification is based on a classification of the homogeneous systems [14]. The following two results essentially comprise that classification; however, we state a slightly stronger version here. (Nevertheless, the proof is identical, so we shall not repeat it.)

Proposition 4 (cf. [14]) *Let $H_\mathcal{Q}$ be a positive semidefinite homogeneous quadratic Hamilton–Poisson system on $\mathfrak{se}(1,1)^*_-$. There exists a linear Poisson automorphism ψ and real numbers $r > 0$, $k \in \mathbb{R}$ such that $rH_\mathcal{Q} \circ \psi + kC = H_i$ for exactly one $i \in \{0,\ldots,5\}$, where*

$$H_0(p) = 0 \qquad H_1(p) = \tfrac{1}{2}p_1^2 \qquad H_2(p) = \tfrac{1}{2}(p_1 + p_2)^2$$
$$H_3(p) = \tfrac{1}{2}p_3^2 \qquad H_4(p) = \tfrac{1}{2}(p_1^2 + p_3^2) \qquad H_5(p) = \tfrac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right].$$

Corollary 1 *Every homogeneous quadratic Hamilton–Poisson system on $\mathfrak{se}(1,1)^*_-$ is L-equivalent to exactly one of the systems $H_0,\ldots,H_5$.*

For each of the systems $H_0,\ldots,H_5$, let $\mathsf{S}(H_i)$ denote the subgroup of linear Poisson automorphisms $\psi : \mathfrak{se}(1,1)^* \to \mathfrak{se}(1,1)^*$ satisfying $H_i \circ \psi = rH_i + kC$ for some $r > 0$ and $k \in \mathbb{R}$.

Lemma 4 *The subgroups $\mathsf{S}(H_i)$ are given by*

$$\mathsf{S}(H_0) : \begin{bmatrix} x & y & v \\ \sigma y & \sigma x & w \\ 0 & 0 & \sigma \end{bmatrix} \qquad \mathsf{S}(H_1) : \begin{bmatrix} x & 0 & v \\ 0 & \sigma x & w \\ 0 & 0 & \sigma \end{bmatrix}, \begin{bmatrix} 0 & y & v \\ \sigma y & 0 & w \\ 0 & 0 & \sigma \end{bmatrix}$$
$$\mathsf{S}(H_2) : \begin{bmatrix} x & y & v \\ y & x & w \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{S}(H_3) : \begin{bmatrix} x & y & 0 \\ \sigma y & \sigma x & 0 \\ 0 & 0 & \sigma \end{bmatrix}$$
$$\mathsf{S}(H_4) : \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1\sigma_2 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_1 & 0 \\ \sigma_2\sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} \qquad \mathsf{S}(H_5) : \begin{bmatrix} x & \sigma - x & 0 \\ \sigma - x & x & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here $\sigma, \sigma_1, \sigma_2 \in \{-1, 1\}$, $v, w, x, y \in \mathbb{R}$ and the determinant of each matrix is nonzero.

Proof We illustrate by finding $\mathsf{S}(H_1)$. We have

$$(H_1 \circ \psi)(p) = \tfrac{1}{2}p \begin{bmatrix} x^2 & \sigma xy & 0 \\ \sigma xy & y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} p^\top, \qquad \text{where } \psi : p \mapsto p \begin{bmatrix} x & y & v \\ \sigma y & \sigma x & w \\ 0 & 0 & \sigma \end{bmatrix}.$$

If $\psi \in \mathsf{S}(H_1)$, then either $y = 0$ or $x = 0$ and so ψ is of the given form. If $y = 0$, then $(H_1 \circ \psi)(p) = x^2 H_1(p)$ and so $\psi \in \mathsf{S}(H_1)$. Likewise, if $x = 0$, then $(H_1 \circ \psi)(p) = y^2 H_1(p) - \frac{y^2}{2}C(p)$ and so $\psi \in \mathsf{S}(H_1)$. □

Theorem 1 *Let $H_{A,\mathcal{Q}} = L_A + H_{\mathcal{Q}}$ be an inhomogeneous quadratic Hamilton–Poisson system on $\mathfrak{se}(1,1)^*_-$.*

(i) If $H_{\mathcal{Q}}$ is L-equivalent to $H_0(p) = 0$, then $H_{A,\mathcal{Q}}$ is A-equivalent to exactly one of the following systems:

$$H_1^{(0)}(p) = p_1$$
$$H_{2,\alpha}^{(0)}(p) = \alpha p_3.$$

(ii) If $H_{\mathcal{Q}}$ is L-equivalent to $H_1(p) = \frac{1}{2}p_1^2$, then $H_{A,\mathcal{Q}}$ is A-equivalent to exactly one of the following systems:

$$H_1^{(1)}(p) = p_1 + \tfrac{1}{2}p_1^2$$
$$H_2^{(1)}(p) = p_1 + p_2 + \tfrac{1}{2}p_1^2$$
$$H_{3,\alpha}^{(1)}(p) = \alpha p_3 + \tfrac{1}{2}p_1^2.$$

(iii) If $H_{\mathcal{Q}}$ is L-equivalent to $H_2(p) = \frac{1}{2}(p_1+p_2)^2$, then $H_{A,\mathcal{Q}}$ is A-equivalent to exactly one of the following systems:

$$H_1^{(2)}(p) = p_1 + \tfrac{1}{2}(p_1+p_2)^2$$
$$H_2^{(2)}(p) = p_1 + p_2 + \tfrac{1}{2}(p_1+p_2)^2$$
$$H_{3,\delta}^{(2)}(p) = \delta p_3 + \tfrac{1}{2}(p_1+p_2)^2.$$

(iv) If $H_{\mathcal{Q}}$ is L-equivalent to $H_3(p) = \frac{1}{2}p_3^2$, then $H_{A,\mathcal{Q}}$ is A-equivalent to exactly one of the following systems:

$$H_1^{(3)}(p) = p_1 + \tfrac{1}{2}p_3^2$$
$$H_2^{(3)}(p) = p_1 + p_2 + \tfrac{1}{2}p_3^2$$
$$H_3^{(3)}(p) = \tfrac{1}{2}p_3^2.$$

(v) If $H_{\mathcal{Q}}$ is L-equivalent to $H_4(p) = \frac{1}{2}(p_1^2+p_3^2)$, then $H_{A,\mathcal{Q}}$ is A-equivalent to exactly one of the following systems:

$$H_{1,\alpha}^{(4)}(p) = \alpha p_1 + \tfrac{1}{2}(p_1^2+p_3^2)$$
$$H_{2,\alpha_1,\alpha_2}^{(4)}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \tfrac{1}{2}(p_1^2+p_3^2).$$

(vi) If $H_{\mathcal{Q}}$ is L-equivalent to $H_5(p) = \frac{1}{2}[(p_1+p_2)^2 + p_3^2]$, then $H_{A,\mathcal{Q}}$ is A-equivalent to exactly one of the following systems:

$$H_{1,\alpha}^{(5)}(p) = \alpha p_1 + \tfrac{1}{2}\left[(p_1+p_2)^2 + p_3^2\right]$$
$$H_2^{(5)}(p) = p_1 - p_2 + \tfrac{1}{2}\left[(p_1+p_2)^2 + p_3^2\right]$$
$$H_{3,\alpha}^{(5)}(p) = \alpha(p_1+p_2) + \tfrac{1}{2}\left[(p_1+p_2)^2 + p_3^2\right].$$

Here $\alpha > 0$, $\alpha_1 \geq \alpha_2 > 0$ and $\delta \neq 0$ parametrize families of normal forms, each different value corresponding to a distinct (non-equivalent) normal form.

Proof Let $H_{A,\mathcal{Q}}$ be an inhomogeneous quadratic Hamilton–Poisson system. By Corollary 1 and ($\mathfrak{E}1$), ($\mathfrak{E}2$), ($\mathfrak{E}3$), we have that $H_{A,\mathcal{Q}}$ is A-equivalent to a system $H = L_B + H_i$ for some $B \in \mathfrak{se}(1,1)$ and $i \in \{0,\ldots,5\}$. By Proposition 1, $L_B + H_i$ is not A-equivalent to $L_{B'} + H_j$ for any $B' \in \mathfrak{se}(1,1)$ when $i \neq j$. Hence there are six cases to consider (corresponding to each H_i).

(*i*) Suppose $H = L_B + H_0$. There exists $\psi \in \mathsf{S}(H_0)$ such that $L_B \circ \psi \in \{L_{E_1}, L_{E_1+E_2}, L_{\alpha E_3} \,:\, \alpha > 0\}$. Indeed, let $B = \sum_{i=1}^{3} b_i E_i$. Suppose $b_3 = 0$. If $b_1^2 \neq b_2^2$, then

$$\psi : p \mapsto p \begin{bmatrix} \frac{b_1}{b_1^2-b_2^2} & -\frac{b_2}{b_1^2-b_2^2} & 0 \\ -\frac{b_2}{b_1^2-b_2^2} & \frac{b_1}{b_1^2-b_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an element of $\mathsf{S}(H_0)$ such that $L_B \circ \psi = L_{E_1}$. If $b_1^2 = b_2^2$, i.e., $b_1 = b \neq 0$ and $b_2 = \pm b$, then $\psi : p \mapsto p\,\mathrm{diag}(\frac{1}{b}, \pm\frac{1}{b}, \pm 1) \in \mathsf{S}(H_0)$ and $L_B \circ \psi = L_{E_1+E_2}$. On the other hand, suppose $b_3 \neq 0$. Then

$$\psi : p \mapsto p \begin{bmatrix} 1 & 0 & -\frac{b_1}{b_3} \\ 0 & \mathrm{sgn}(b_3) & -\mathrm{sgn}(b_3)\frac{b_2}{b_3} \\ 0 & 0 & \mathrm{sgn}(b_3) \end{bmatrix}$$

is an element of $\mathsf{S}(H_0)$ such that $L_B \circ \psi = L_{\alpha E_3}$, where $\alpha = |b_3| > 0$.

Consequently, H is A-equivalent to one of the systems $G_1(p) = p_1$, $G_2(p) = p_1 + p_2$ or $G_{3,\alpha}(p) = \alpha p_3$. The systems G_1 and G_2 are A-equivalent. Indeed,

$$\psi : p \mapsto p \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear isomorphism such that $\psi_* \vec{G}_1 = \vec{G}_2$. However, G_1 and $G_{3,\alpha}$ are not A-equivalent. Suppose there exists an affine isomorphism $\psi : p \mapsto p\Psi + q$, $\Psi = [\Psi_{ij}]$ such that $\psi_* \vec{G}_1 = \vec{G}_{3,\alpha}$. This yields the system of equations

$$\begin{cases} \alpha\Psi_{12}p_1 + (\alpha\Psi_{22} + \Psi_{31})p_2 + \alpha\Psi_{32}p_3 + \alpha q_2 = 0 \\ \alpha\Psi_{11}p_1 + (\alpha\Psi_{21} + \Psi_{32})p_2 + \alpha\Psi_{31}p_3 + \alpha q_1 = 0 \\ \Psi_{33}p_2 = 0. \end{cases}$$

By inspection, we have $\Psi_{31} = \Psi_{32} = \Psi_{33} = 0$, whence $\det \Psi = 0$, a contradiction. Likewise, $G_{3,\alpha}$ is A-equivalent to $G_{3,\alpha'}$ only if $\alpha = \alpha'$. Therefore H is A-equivalent to either $H_1^{(0)}(p) = p_1$ or $H_{2,\alpha}^{(0)}(p) = \alpha p_3$.

(*ii*) Suppose $H = L_B + H_1$. Like in case (*i*), there exists $\psi \in \mathsf{S}(H_1)$ such that $L_B \circ \psi \in \{L_{E_1+\beta E_2}, L_{\alpha E_3} \,:\, \alpha > 0,\ \beta \geq 0\}$. Hence H is A-equivalent to one of the systems $G_{1,\beta}(p) = p_1 + \beta p_2 + \frac{1}{2}p_1^2$ or $G_{2,\alpha}(p) = \alpha p_3 + \frac{1}{2}p_1^2$. The systems $G_{1,\beta}$, $\beta > 0$ and $G_{1,1}$ are A-equivalent. Indeed, $\psi : p \mapsto p\,\mathrm{diag}(1, \frac{1}{\beta}, \frac{1}{\beta})$ is a linear isomorphism such that $\psi_* \vec{G}_{1,\beta} = \vec{G}_{1,1}$. One can now verify that $G_{1,1}$ and $G_{2,\alpha}$ are not A-equivalent, and $G_{2,\alpha}$ is A-equivalent to $G_{2,\alpha'}$ only if $\alpha = \alpha'$. Hence H is A-equivalent to exactly one of $H_1^{(1)}(p) = p_1 + \frac{1}{2}p_1^2$, $H_2^{(1)}(p) = p_1 + p_2 + \frac{1}{2}p_1^2$ or $H_{3,\alpha}^{(1)}(p) = \alpha p_3 + \frac{1}{2}p_1^2$.

(*iii*) Suppose $H = L_B + H_2$. There exists $\psi \in \mathsf{S}(H_2)$ such that $L_B \circ \psi \in \{L_{E_1}, L_{E_1+\sigma E_2}, L_{\delta E_3} : \delta \neq 0,\ \sigma \in \{-1, 1\}\}$, and so H is A-equivalent to one of the systems $G_1(p) = p_1 + \frac{1}{2}(p_1 + p_2)^2$, $G_{2,\sigma}(p) = p_1 + \sigma p_2 + \frac{1}{2}(p_1 + p_2)^2$ or $G_{3,\delta}(p) = \delta p_3 + \frac{1}{2}(p_1 + p_2)^2$. We have that G_1 is A-equivalent to $G_{2,-1}$. Indeed,

$$: p \mapsto p \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear isomorphism such that $\psi_* \vec{G}_1 = \vec{G}_{2,-1}$. No two of the systems G_1, $G_{2,1}$ and $G_{3,\delta}$, $\delta \neq 0$ are A-equivalent. Thus H is A-equivalent to exactly one of $H_1^{(2)}(p) = p_1 + \frac{1}{2}(p_1 + p_2)^2$, $H_2^{(2)}(p) = p_1 + p_2 + \frac{1}{2}(p_1 + p_2)^2$ or $H_{3,\delta}^{(2)}(p) = \delta p_3 + \frac{1}{2}(p_1 + p_2)^2$.

(*iv*) Suppose $H = L_B + H_3$. There exists $\in \mathsf{S}(H_3)$ such that $L_B \circ \; = \{L_{E_1+\beta E_3}, L_{E_1+E_2+\gamma E_3}, L_{\alpha E_3} : \alpha > 0,\ \beta \geq 0,\ \gamma \in \mathbb{R}\}$. Thus H is A-equivalent to one of the systems $G_{1,\beta}(p) = p_1 + \beta p_3 + \frac{1}{2}p_3^2$, $G_{2,\gamma}(p) = p_1 + p_2 + \gamma p_3 + \frac{1}{2}p_3^2$ or $G_{3,\alpha}(p) = \alpha p_3 + \frac{1}{2}p_3^2$. Let $\psi : p \mapsto p + \beta E_3^*$, $\;' : p \mapsto p + \alpha E_3^*$ and $\;'' : p \mapsto p + \gamma E_3^*$. Then $\psi_* \vec{G}_{1,\beta} = \vec{G}_{1,0}$, $\;'_* \vec{G}_{2,\alpha} = \vec{G}_{2,0}$ and $\psi''_* \vec{G}_{3,\gamma} = \vec{G}_{3,0}$. No two of the systems $G_{1,0}$, $G_{2,0}$ and $G_{3,0}$ are A-equivalent. Therefore H is A-equivalent to exactly one of the systems $H_1^{(3)}(p) = p_1 + \frac{1}{2}p_3^2$, $H_2^{(3)}(p) = p_1 + p_2 + \frac{1}{2}p_3^2$ or $H_3^{(3)}(p) = \frac{1}{2}p_3^2$.

(*v*) Suppose $H = L_B + H_4$. There exists $\in \mathsf{S}(H_4)$ such that $L_B \circ \; \in \{L_{\beta E_1+\alpha E_2}, L_{\gamma E_1+\beta E_2+\alpha E_3} : \alpha > 0,\ \beta \geq 0,\ \gamma \in \mathbb{R}\}$. Thus H is A-equivalent to one of the systems $G_{1,\alpha,\beta}(p) = \beta p_1 + \alpha p_2 + \frac{1}{2}(p_1^2 + p_3^2)$ or $G_{2,\alpha,\beta,\gamma}(p) = \gamma p_1 + \beta p_2 + \alpha p_3 + \frac{1}{2}(p_1^2 + p_3^2)$. If $\psi : p \mapsto p + \alpha E_3^*$, then $\psi_* \vec{G}_{2,\alpha,\beta,\gamma} = \vec{G}_{2,0,\beta,\gamma}$. Likewise, if $\psi' : p \mapsto p\,\mathrm{diag}(-1,1,1)$, then $\psi'_* \vec{G}_{2,0,\beta,\gamma} = \vec{G}_{2,0,\beta,-\gamma}$. Accordingly, we have a family of potential normal forms $G_{2,0,\beta_1,\beta_2}(p) = \beta_1 p_1 + \beta_2 p_2 + \frac{1}{2}(p_1^2 + p_3^2)$, with $\beta_1, \beta_2 \geq 0$ and β_1, β_2 not both zero. If $\beta_2 > 0$, then $G_{2,0,\beta_1,\beta_2} = G_{1,\alpha,\beta}$, where $\alpha = \beta_2 > 0$ and $\beta = \beta_1 \geq 0$. If $\beta_1 > 0$, then

$$'' : p \mapsto p \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear isomorphism such that $\psi''_* \vec{G}_{2,0,\beta_1,\beta_2} = \vec{G}_{1,\alpha,\beta}$, where $\alpha = \beta_1 > 0$ and $\beta = \beta_2 \geq 0$. Let $G_{3,\alpha}(p) = \alpha p_1 + \frac{1}{2}(p_1^2 + p_3^2)$. Then $\psi''_* \vec{G}_{1,\alpha,0} = \vec{G}_{3,\alpha}$. Hence we have the potential normal forms $G_{1,\alpha_1,\alpha_2}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \frac{1}{2}(p_1^2 + p_3^2)$ and $G_{3,\alpha}$, where $\alpha, \alpha_1, \alpha_2 > 0$. If $\alpha_2 > \alpha_1$, then $\psi''_* \vec{G}_{1,\alpha_1,\alpha_2} = \vec{G}_{1,\alpha_2,\alpha_1}$, and so we may assume $\alpha_1 \geq \alpha_2 > 0$. No two of the systems G_{1,α_1,α_2}, $\alpha_1 \geq \alpha_2 > 0$ and $G_{3,\alpha}$, $\alpha > 0$ are A-equivalent. Thus H is A-equivalent to exactly one of $H_{1,\alpha}^{(4)}(p) = \alpha p_1 + \frac{1}{2}(p_1^2 + p_3^2)$ or $H_{1,\alpha_1,\alpha_2}^{(4)}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \frac{1}{2}(p_1^2 + p_3^2)$.

(*vi*) Suppose $H = L_B + H_5$. There exists $\in \mathsf{S}(H_5)$ such that $L_B \circ \; \in \{L_{\beta E_1+\gamma E_3}, L_{\delta E_1+\alpha E_2+\gamma E_3} : \alpha > 0,\ \beta \geq 0,\ \gamma \in \mathbb{R},\ \delta \neq 0\}$. Hence H is A-equivalent to one of the systems $G_{1,\beta,\gamma}(p) = \beta p_1 + \gamma p_3 + \frac{1}{2}[(p_1+p_2)^2 + p_3^2]$ or $G_{2,\alpha,\gamma,\delta}(p) = \delta p_1 + \alpha p_2 + \gamma p_3 + \frac{1}{2}[(p_1+p_2)^2 + p_3^2]$. If $\psi : p \mapsto p + \gamma E_3^*$, then $\psi_* \vec{G}_{1,\beta,\gamma} = \vec{G}_{1,\beta,0}$. (As $G_{1,0,0}$ is a homogeneous system, we assume $\beta = \alpha > 0$.) Likewise, $\psi_* \vec{G}_{2,\alpha,\gamma,\delta} = \vec{G}_{2,\alpha,0,\delta}$. Suppose $\delta^2 \neq \alpha^2$. Then

$$' : p \mapsto p \begin{bmatrix} \frac{\delta}{|\delta+\alpha|} & \frac{\delta}{|\delta+\alpha|} & 0 \\ \frac{\delta}{|\delta+\alpha|} & \frac{\delta}{|\delta+\alpha|} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear isomorphism such that $\psi'_* \vec{G}_{2,\alpha,\delta,0} = \vec{G}_{1,|\delta+\alpha|,0}$. On the other hand, suppose $\delta^2 = \alpha^2$. Let $G_3(p) = p_1 - p_2 + \frac{1}{2}[(p_1 + p_2)^2 + p_3^2]$ and $G_{4,\alpha}(p) = \alpha(p_1 + p_2) + \frac{1}{2}[(p_1 + p_2)^2 + p_3^2]$. If $\alpha = -\delta > 0$, then

$$\psi'' : p \mapsto p \begin{bmatrix} \frac{1+\delta}{2} & \frac{1-\delta}{2} & 0 \\ \frac{1-\delta}{2} & \frac{1+\delta}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear isomorphism such that $\psi''_* \vec{G}_{2,\alpha,\delta,0} = \vec{G}_3$. If $\alpha = \delta > 0$, then $G_{2,\alpha,\delta,0} = G_{4,\alpha}$. No two of the systems $G_{1,\alpha}$, $\alpha > 0$, G_3 and $G_{4,\alpha}$, $\alpha > 0$ are A-equivalent. Thus H is A-equivalent to exactly one of $H^{(5)}_{1,\alpha}(p) = \alpha p_1 + \frac{1}{2}[(p_1 + p_2)^2 + p_3^2]$, $H^{(5)}_2(p) = p_1 - p_2 + \frac{1}{2}[(p_1 + p_2)^2 + p_3^2]$ or $H^{(5)}_{3,\alpha}(p) = \alpha(p_1 + p_2) + \frac{1}{2}[(p_1 + p_2)^2 + p_3^2]$. □

We say that a system $H_{A,\mathcal{Q}}$ is *ruled* if the trace of every integral curve of $\vec{H}_{A,\mathcal{Q}}$ is contained in a line; *planar* if it is not ruled and if the trace of every integral curve lies in a plane; and *nonplanar*, otherwise. A nonplanar system $H_{A,\mathcal{Q}}$ is said to be of *type I* if the set of all equilibrium points of $\vec{H}_{A,\mathcal{Q}}$ is the union of lines or planes; otherwise, it is said to be of *type II*. The partition of the normal forms into these four classes is given in Table 1. (As a concluding remark to the paper, we discuss how these normal forms may be better organized according to some invariant properties.)

Remark 2 The classes of ruled, planar and nonplanar systems can be characterized in terms of the curvature and torsion of a system's integral curves. Indeed, a system is

– ruled, if and only if every integral curve has zero curvature;
– planar, if and only if every integral curve has zero torsion, and there exists an integral curve with nonzero curvature; and,
– nonplanar, if and only if there exists an integral curve with nonzero curvature and nonzero torsion.

(Although the curvature and torsion of a curve are not invariant under affine isomorphisms, whether they vanish or not is invariant.)

Table 1 Normal forms for inhomogeneous quadratic Hamilton–Poisson systems

Class	Systems
Ruled	$H^{(0)}_1$, $H^{(1)}_1$, $H^{(1)}_2$, $H^{(2)}_1$, $H^{(2)}_2$
Planar	$H^{(0)}_{2,\alpha}$
Nonplanar, type I	$H^{(1)}_{3,\alpha}$, $H^{(2)}_{3,\delta}$, $H^{(3)}_1$, $H^{(3)}_2$, $H^{(4)}_{1,\alpha}$, $H^{(5)}_{3,\alpha}$
Nonplanar, type II	$H^{(4)}_{2,\alpha_1,\alpha_2}$, $H^{(5)}_{1,\alpha}$, $H^{(5)}_2$

3.1 Symmetry groups

In this section we compute the symmetry group for each normal form. The *symmetry group* of a system $H_{A,\mathcal{Q}}$, denoted $\mathsf{Sym}(H_{A,\mathcal{Q}})$, is the group of all affine isomorphisms $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ such that $\psi_* \vec{H}_{A,\mathcal{Q}} = \vec{H}_{A,\mathcal{Q}}$. Throughout this section, we shall identify $p = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$ with $\tilde{p} = \begin{bmatrix} 1 & p_1 & p_2 & p_3 \end{bmatrix}$. An affine isomorphism $\psi : p \mapsto p\Psi + q$ is then written as $\psi : \tilde{p} \mapsto \tilde{p} \begin{bmatrix} 1 & q \\ 0 & \Psi \end{bmatrix}$.

Proposition 5 *The symmetry groups of the (nontrivial) homogeneous normal forms are given by*

$$\mathsf{Sym}(H_1) : \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 0 & x & v \\ 0 & y & 0 & w \\ 0 & 0 & 0 & xy \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & x & 0 & v \\ 0 & 0 & y & w \\ 0 & 0 & 0 & xy \end{bmatrix} \qquad \mathsf{Sym}(H_2) : \begin{bmatrix} 1 & a & -a & b \\ 0 & x & z-x & v \\ 0 & y & z-y & w \\ 0 & 0 & 0 & z^2 \end{bmatrix}$$

$$\mathsf{Sym}(H_3) : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & y & 0 \\ 0 & \sigma y & \sigma x & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix} \qquad \mathsf{Sym}(H_4) : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1\sigma_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_1\sigma_2 \end{bmatrix}$$

$$\mathsf{Sym}(H_5) : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & \sigma - x & 0 \\ 0 & \sigma - x & x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here $\sigma, \sigma_1, \sigma_2 \in \{-1, 1\}$, $a, b, v, w, x, y, z \in \mathbb{R}$ and the determinant of each matrix is nonzero.

Proof As a typical case, we find $\mathsf{Sym}(H_3)$. Let $\psi : p \mapsto \psi_0(p) + q$ (where $\psi_0(p) = p[\Psi_{ij}]$) be an affine isomorphism such that $\psi_* \vec{H}_3 = \vec{H}_3$. By Proposition 1, ψ_0 is a symmetry of H_3. In particular, we have

$$\psi_0 \cdot \vec{H}_3(E_3^*) = \vec{H}_3(\psi_0 \cdot E_3^*) \quad \Longleftrightarrow \quad \begin{bmatrix} \Psi_{33}\Psi_{32} & \Psi_{33}\Psi_{31} & 0 \end{bmatrix} = 0.$$

Suppose $\Psi_{33} = 0$; then $\psi_0 \cdot \vec{H}_3(E_1^* + E_3^*) = \vec{H}_3(\psi_0 \cdot (E_1^* + E_3^*))$ and $\psi_0 \cdot \vec{H}_3(E_2^* + E_3^*)) = \vec{H}_3(\psi_0 \cdot (E_2^* + E_3^*))$ imply that $\Psi_{13} = \Psi_{23} = 0$, a contradiction. Hence $\Psi_{33} \neq 0$ and $\Psi_{31} = \Psi_{32} = 0$. Again, as $\psi_0 \cdot \vec{H}_3(E_1^* + E_3^*) = \vec{H}_3(\psi_0 \cdot (E_1^* + E_3^*))$ and $\psi_0 \cdot \vec{H}_3(E_2^* + E_3^*) = \vec{H}_3(\psi_0 \cdot (E_2^* + E_3^*))$, we get $\Psi_{13} = \Psi_{23} = 0$, $\Psi_{33} = \sigma \in \{-1, 1\}$ and $\Psi_{21} = \sigma\Psi_{12}$, $\Psi_{22} = \sigma\Psi_{11}$. Relabelling Ψ_{11} and Ψ_{12} as x and y, respectively, yields

$$\psi_0 : p \mapsto p \begin{bmatrix} x & y & 0 \\ \sigma y & \sigma x & 0 \\ 0 & 0 & \sigma \end{bmatrix}. \tag{2}$$

It is now easy to show that $\psi_* \vec{H}_3 = \vec{H}_3$ implies $q = 0$, and so ψ is a linear isomorphism of the form (2). Conversely, every map of this form is a symmetry of H_3. □

Proposition 6 *The symmetry groups of the inhomogeneous normal forms are given below. (Throughout, we have $\sigma \in \{-1, 1\}$, $a, b, c, v, w, x, y, z \in \mathbb{R}$ and the determinant of each matrix is nonzero.)*

(*i*) *For the systems corresponding to* H_0:

$$\mathsf{Sym}(H_1^{(0)}): \begin{bmatrix} 1 & a & 0 & b \\ 0 & x & 0 & v \\ 0 & y & z & w \\ 0 & 0 & 0 & z \end{bmatrix} \qquad \mathsf{Sym}(H_{2,\alpha}^{(0)}): \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & x & y & 0 \\ 0 & y & x & 0 \\ 0 & 0 & 0 & z \end{bmatrix}.$$

(*ii*) *For the systems corresponding to* H_1:

$$\mathsf{Sym}(H_1^{(1)}): \begin{bmatrix} 1 & -1 & a & b \\ 0 & 0 & a & v \\ 0 & x & 0 & w \\ 0 & 0 & 0 & ax \end{bmatrix}, \begin{bmatrix} 1 & a & 0 & b \\ 0 & 1+a & 0 & v \\ 0 & 0 & x & w \\ 0 & 0 & 0 & (1+a)x \end{bmatrix}$$

$$\mathsf{Sym}(H_2^{(1)}): \begin{bmatrix} 1 & a & b & c \\ 0 & 1+a & 0 & v \\ 0 & 0 & 1+a & w \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a & b & c \\ 0 & 0 & 1+b & v \\ 0 & 1+b & 0 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathsf{Sym}(H_{3,\alpha}^{(1)}): \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x^2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 0 & x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x^2 \end{bmatrix}.$$

(*iii*) *For the systems corresponding to* H_2:

$$\mathsf{Sym}(H_1^{(2)}): \begin{bmatrix} 1 & 1(\sigma+a) & -a^2 & b \\ 0 & z(1+2\sigma a) & -2\sigma a z & v \\ 0 & z(1-z+2\sigma a) & z(z-2\sigma a) & w \\ 0 & 0 & 0 & z^2 \end{bmatrix}$$

$$\mathsf{Sym}(H_2^{(2)}): \begin{bmatrix} 1 & a & -a & b \\ 0 & x & 1-x & v \\ 0 & y & 1-y & w \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a & -(1+a) & b \\ 0 & x & -(x+1) & v \\ 0 & y & -(y+1) & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathsf{Sym}(H_{3,\delta}^{(2)}): \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & x & y & 0 \\ 0 & y & x & 0 \\ 0 & 0 & 0 & (x+y)^2 \end{bmatrix}.$$

(*iv*) *For the systems corresponding to* H_3:

$$\mathsf{Sym}(H_1^{(3)}): \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix} \qquad \mathsf{Sym}(H_2^{(3)}): \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 1-x & 0 \\ 0 & 1-x & x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(*v*) *For the systems corresponding to* H_4:

$$\mathsf{Sym}(H_{1,\alpha}^{(4)}): \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix} \qquad \mathsf{Sym}(H_{2,\alpha_1,\alpha_2}^{(4)}): \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(*vi*) *For the systems corresponding to* H_5:

$$\mathsf{Sym}(H^{(5)}_{1,\alpha}) : \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \qquad \mathsf{Sym}(H^{(5)}_{2}) : \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&0&-1&0\\0&-1&0&0\\0&0&0&1\end{bmatrix}$$

$$\mathsf{Sym}(H^{(5)}_{3,\alpha}) : \begin{bmatrix}1&0&0&0\\0&x&1-x&0\\0&1-x&x&0\\0&0&0&1\end{bmatrix}.$$

Proof The proof is analogous to that for Proposition 5. However, note that by Proposition 1, if $\psi : p \mapsto \psi_0(p) + q$ is an affine isomorphism such that $\psi_* \vec{H}_{A,\mathcal{Q}} = \vec{H}_{A,\mathcal{Q}}$, then ψ_0 must be a symmetry of $H_{\mathcal{Q}}$. (This substantially simplifies the calculations.) □

3.2 Equivalent systems on $\mathfrak{se}(2)^*_-$ and $\mathfrak{so}(3)^*_-$

The systems $H^{(4)}_{1,\alpha}$ and $H^{(4)}_{2,\alpha_1,\alpha_2}$ turn out to be affinely equivalent to systems already considered on the Euclidean space $\mathfrak{se}(2)^*_-$ and the orthogonal space $\mathfrak{so}(3)^*_-$. (Accordingly, we shall not treat the stability or integration of $H^{(4)}_{1,\alpha}$ and $H^{(4)}_{2,\alpha_1,\alpha_2}$.) We give explicit isomorphisms between the equivalent systems below. The Lie algebras $\mathfrak{se}(2)$ and $\mathfrak{so}(3)$ are given by

$$\mathfrak{se}(2) = \left\{ x_1\widetilde{E}_1 + x_2\widetilde{E}_2 + x_3\widetilde{E}_3 = \begin{bmatrix}0&0&0\\x_1&0&-x_3\\x_2&x_3&0\end{bmatrix} : x_1,x_2,x_3 \in \mathbb{R}\right\}$$

and

$$\mathfrak{so}(3) = \left\{ x_1\widehat{E}_1 + x_2\widehat{E}_2 + x_3\widehat{E}_3 = \begin{bmatrix}0&-x_3&x_2\\x_3&0&-x_1\\-x_2&x_1&0\end{bmatrix} : x_1,x_2,x_3 \in \mathbb{R}\right\}$$

respectively. The non-zero commutator relations are $[\widetilde{E}_2,\widetilde{E}_3] = \widetilde{E}_1$, $[\widetilde{E}_3,\widetilde{E}_1] = \widetilde{E}_2$ and $[\widehat{E}_2,\widehat{E}_3] = \widehat{E}_1$, $[\widehat{E}_3,\widehat{E}_1] = \widehat{E}_2$, $[\widehat{E}_1,\widehat{E}_2] = \widehat{E}_3$.

The system $(\mathfrak{se}(1,1)^*_-, H^{(4)}_{1,\alpha})$ is A-equivalent to both the system

$$\left(\mathfrak{se}(2)^*_-, H(\widetilde{p}) = \widetilde{p}_1 + \tfrac{1}{2}(\tfrac{1}{c_1}\widetilde{p}_2^2 + \tfrac{1}{c_2}\widetilde{p}_3^2)\right), \qquad c_1,c_2>0,\ \alpha = \sqrt{\tfrac{c_1}{c_2}}$$

and the system

$$\left(\mathfrak{so}(3)^*_-, H'(\widehat{p}) = \alpha'\widehat{p}_1 + \widehat{p}_1^2 + \tfrac{1}{2}\widehat{p}_2^2\right), \qquad \alpha' = \sqrt{2}\,\alpha$$

that were treated in [6] and [2], respectively. Indeed,

$$\psi : \mathfrak{se}(2)^* \to \mathfrak{se}(1,1)^*, \qquad \widetilde{p} \mapsto \widetilde{p}\begin{bmatrix}\frac{1}{\sqrt{c_1c_2}}&0&0\\0&0&-\frac{1}{\sqrt{c_1c_2}}\\0&-\frac{1}{c_2}&0\end{bmatrix} + \left[-\sqrt{\tfrac{c_1}{c_2}}\ 0\ 0\right]$$

and

$$\psi' : \mathfrak{so}(3)^* \to \mathfrak{se}(1,1)^*, \qquad \widehat{p} \mapsto \widehat{p} \begin{bmatrix} -\sqrt{2} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -\sqrt{2} & 0 \end{bmatrix} + \begin{bmatrix} -\sqrt{2}\,\alpha' & 0 & 0 \end{bmatrix}$$

are affine isomorphisms such that $\psi_* \vec{H} = \vec{H}^{(4)}_{1,\alpha}$ and $\psi'_* \vec{H}' = \vec{H}^{(4)}_{1,\alpha}$. On the other hand, $(\mathfrak{se}(1,1)^*_-, H^{(4)}_{2,\alpha_1,\alpha_2})$ is A-equivalent to the system

$$\left(\mathfrak{so}(3)^*_-, H(\widehat{p}) = \alpha'_1 \widehat{p}_1 + \alpha'_2 \widehat{p}_3 + \widehat{p}_1^2 + \tfrac{1}{2}\widehat{p}_2^2\right), \qquad \alpha'_i = \sqrt{2}\,\alpha_i$$

considered in [2]. Indeed,

$$\psi : \mathfrak{so}(3)^* \to \mathfrak{se}(1,1)^*, \qquad \widehat{p} \mapsto \widehat{p} \begin{bmatrix} -\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{bmatrix} + \begin{bmatrix} -\sqrt{2}\,\alpha'_1 & -\sqrt{2}\,\alpha'_2 & 0 \end{bmatrix}$$

is an affine isomorphism such that $\psi_* \vec{H} = \vec{H}^{(4)}_{2,\alpha_1,\alpha_2}$.

4 Ruled and planar systems

For the sake of completeness, we briefly treat the ruled and planar systems. The ruled systems have the following integral curves and equilibria:

$H_1^{(0)} : p(t) = (c_1, c_2, c_3 - c_2 t)$ $\qquad \mathsf{e}^{\eta,\mu} = (\eta, 0, \mu)$

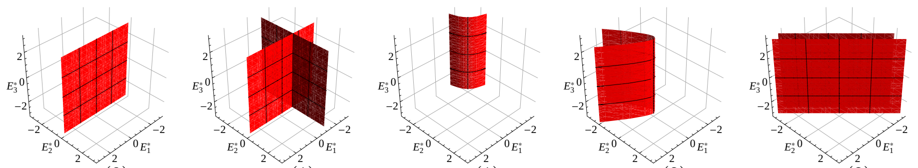
$H_1^{(1)} : p(t) = (c_1, c_2, c_3 - c_2(1 + c_1)t)$ $\qquad \mathsf{e}_1^{\eta,\mu} = (\eta, 0, \mu),\ \mathsf{e}_2^{\nu,\mu} = (-1, \nu, \mu)$

$H_2^{(1)} : p(t) = (c_1, c_2, c_3 - (c_1 + c_2 + c_1 c_2)t)$ $\qquad \mathsf{e}^{\eta,\mu} = (-(1 + e^{\eta}), -(1 + e^{-\eta}), \mu)$

$H_1^{(2)} : p(t) = (c_1 - c_2, c_2, c_3 - (c_1^2 + c_2)t)$ $\qquad \mathsf{e}^{\eta,\mu} = (\eta + \eta^2, -\eta^2, \mu)$

$H_2^{(2)} : p(t) = (c_1, c_2 - c_1, c_3 - c_2(c_2 + 1)t)$ $\qquad \mathsf{e}_1^{\eta,\mu} = (\eta, -(1 + \eta), \mu),\ \mathsf{e}_2^{\eta,\mu} = (\eta, -\eta, \mu).$

(Here $c_1, c_2, c_3, \eta, \mu, \nu \in \mathbb{R}$ and $\nu \neq 0$.) All equilibria for these systems are unstable; we graph these equilibria in Fig. 1.



(a) $H_1^{(0)}$ (b) $H_1^{(1)}$ (c) $H_2^{(1)}$ (d) $H_1^{(2)}$ (e) $H_2^{(2)}$

Fig. 1 Equilibria of ruled systems

The only planar system is $H^{(0)}_{2,\alpha}(p) = \alpha p_3$, $\alpha > 0$. The equations of motion are

$$\begin{cases} \dot{p}_1 = \alpha p_2 \\ \dot{p}_2 = \alpha p_1 \\ \dot{p}_3 = 0. \end{cases}$$

The equilibrium states of $\vec{H}^{(0)}_{2,\alpha}$ are $\mathsf{e}^\mu = (0,0,\mu)$; all equilibria are (spectrally) unstable. The integral curves are given by

$$\begin{cases} p_1(t) = p_1(0)\cosh(\alpha t) + p_2(0)\sinh(\alpha t) \\ p_2(t) = p_1(0)\sinh(\alpha t) + p_2(0)\cosh(\alpha t) \\ p_3(t) = p_3(0). \end{cases}$$

5 Nonplanar systems, type I

In this section we consider the stability and integration of the nonplanar, type I systems (see Table 1). The integration for each system is typically subdivided into several cases, where each case corresponds to a qualitatively different integral curve. Qualitative changes occur when the level sets corresponding to the Hamiltonian and Casimir functions are tangent. Since every Hamiltonian vector field $\vec{H}$ on $\mathfrak{se}(1,1)^*_-$ can be written in the form $\vec{H} = \frac{1}{2}\nabla H \times \nabla C$, it follows that these level sets are tangent exactly when $\vec{H}(p) = 0$, i.e., at equilibria. Thus we have a set $\{(H(p_e), C(p_e)) : p_e \in \mathfrak{se}(1,1)^*_-,\ \vec{H}(p_e) = 0\}$ of *critical energy states* corresponding to equilibria. The set of all energy states $\{(H(p), C(p)) : p \in \mathfrak{se}(1,1)^*_-\}$ is subdivided into a number of regions by the critical energy states. As a general rule, each qualitative case corresponds to a different region of energy states. (The integral curves corresponding to two different points in the same region can usually be continuously deformed into each other.)

Remark 3 For some cases, we find it useful to consider critical energy states corresponding to equilibria "at infinity." We say that (h_0, c_0) is a *generalized critical energy state* if there exist two curves f and g in $\mathfrak{se}(1,1)^*$ such that $\lim_{s\to\infty}[f(s) - g(s)] = 0$, $\vec{H}(g(s)) = 0$, $H(f(s)) = h_0$ and $C(f(s)) = c_0$. (Every critical energy state is a generalized critical energy state.)

For each system (except two) we graph the critical energy states. (The analysis of $H^{(1)}_{3,\alpha}$ and $H^{(2)}_{3,\delta}$ is straightforward, and hence we omit the graphs for these systems.) Critical states corresponding to stable equilibria are coloured in blue, whereas those corresponding to unstable equilibria are coloured in red. The generalized critical energy states are depicted in purple. For typical configurations (or rather typical energy states (h_0, c_0)) of the system, we graph the level sets $H^{-1}(h_0)$ and $C^{-1}(c_0)$ and their intersection. The equilibrium points for each system are also graphed. (As before, stable equilibria are blue and unstable equilibria red.)

Throughout this section, we parametrize equilibria by $\mu, \nu \in \mathbb{R}$ with $\nu \neq 0$.

5.1 The systems $H^{(1)}_{3,\alpha}$ and $H^{(2)}_{3,\delta}$

The equations of motion of the system $H^{(1)}_{3,\alpha}(p) = \alpha p_3 + \frac{1}{2}p_1^2$, $\alpha > 0$ are

$$\begin{cases} \dot{p}_1 = \alpha p_2 \\ \dot{p}_2 = \alpha p_1 \\ \dot{p}_3 = -p_1 p_2. \end{cases}$$

The equilibrium states of $\vec{H}^{(1)}_{3,\alpha}$ are $\mathsf{e}^{\mu} = (0, 0, \mu)$; all states are (spectrally) unstable. The integral curves are given by

$$\begin{cases} p_1(t) = p_1(0)\cosh(\alpha t) + p_2(0)\sinh(\alpha t) \\ p_2(t) = p_1(0)\sinh(\alpha t) + p_2(0)\cosh(\alpha t) \\ p_3(t) = \frac{1}{2\alpha}\left(p_1(0)^2 - p_1(t)^2\right) + p_3(0). \end{cases}$$

The equations of motion of the system $H^{(2)}_{3,\delta}(p) = \delta p_3 + \frac{1}{2}(p_1 + p_2)^2$, $\delta \neq 0$ are

$$\begin{cases} \dot{p}_1 = \delta p_2 \\ \dot{p}_2 = \delta p_1 \\ \dot{p}_3 = -(p_1 + p_2)^2. \end{cases}$$

The equilibrium states of $\vec{H}^{(2)}_{3,\delta}$ are $\mathsf{e}^{\mu} = (0, 0, \mu)$; all states are (spectrally) unstable. The integral curves are given by

$$\begin{cases} p_1(t) = p_1(0)\cosh(\delta t) + p_2(0)\sinh(\delta t) \\ p_2(t) = p_1(0)\sinh(\delta t) + p_2(0)\cosh(\delta t) \\ p_3(t) = \frac{1}{2\delta}\left[(p_1(0) + p_2(0))^2 - (p_1(t) + p_2(t))^2\right] + p_3(0). \end{cases}$$

5.2 The system $H^{(3)}_1$

The equations of motion of the system $H^{(3)}_1(p) = p_1 + \frac{1}{2}p_3^2$ are

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_2. \end{cases}$$

The equilibrium states of $\vec{H}^{(3)}_1$ are $\mathsf{e}^{\mu}_1 = (\mu, 0, 0)$ and $\mathsf{e}^{\nu}_2 = (0, 0, \nu)$.

Proposition 7 *The equilibrium states have the following behaviour:*

(*i*) *The states* e^{μ}_1, $\mu \in (-\infty, 0]$ *are unstable.*
(*ii*) *The states* e^{μ}_1, $\mu \in (0, \infty)$ *are stable.*
(*iii*) *The equilibrium states* e^{ν}_2 *are (spectrally) unstable.*

Proof (*i*) Consider the states e^{μ}_1, $\mu \in (-\infty, 0)$. The integral curve

$$\begin{cases} p_1(t) = \mu[1 + 2\operatorname{csch}^2(\sqrt{-\mu}\, t)] \\ p_2(t) = -2\mu\coth(\sqrt{-\mu}\, t)\operatorname{csch}(\sqrt{-\mu}\, t) \\ p_3(t) = 2\sqrt{-\mu}\, \operatorname{csch}(\sqrt{-\mu}\, t) \end{cases}$$

satisfies $\lim_{t\to-\infty} \|p(t) - \mathsf{e}^{\mu}_1\| = 0$. Accordingly, for every neighbourhood N of e^{μ}_1, there exists $t_1 < 0$ such that $p(t_1) \in N$. Furthermore, $\lim_{t\to 0} \|p(t) - \mathsf{e}^{\mu}_1\| = \infty$. Hence the states e^{μ}_1, $\mu \in (-\infty, 0)$ are unstable. Likewise, the integral curve $p(t) = (-\frac{2}{t^2}, \frac{2}{t^2}, \frac{2}{t})$ suffices to show that the state e^{0}_1 is unstable.

(ii) Let $H_\lambda = \lambda_0 H_1^{(3)} + \lambda_1 C$, where $\lambda_0 = \mu$ and $\lambda_1 = -\frac{1}{2}$. Then $\mathbf{d}H_\lambda(\mathsf{e}_1^\mu) = 0$ and the restriction of $\mathbf{d}^2 H_\lambda(\mathsf{e}_1^\mu) = \mathrm{diag}(-1,1,\mu)$ to $W \times W$ is positive definite, where $W = \ker \mathbf{d}H_1^{(3)}(\mathsf{e}_1^\mu) \cap \ker \mathbf{d}C(\mathsf{e}_1^\mu) = \mathrm{span}\{E_2^*, E_3^*\}$. Hence the states e_1^μ, $\mu \in (0,\infty)$ are stable.

(iii) As $\frac{\partial H_1^{(3)}}{\partial p_3}(\mathsf{e}_2^\nu) = \nu$, it follows from Lemma 2 that the states e_2^ν are spectrally unstable. □

Table 2 Index of cases for the integral curves of $H_1^{(3)}$

Conditions		Designation
$c_0 > 0$	$h_0 > \sqrt{c_0}$	Case I-a
	$h_0 = \sqrt{c_0}$	Case I-b
	$-\sqrt{c_0} < h_0 < \sqrt{c_0}$	Case I-c
	$h_0 = -\sqrt{c_0}$	Case I-d
	$h_0 < -\sqrt{c_0}$	Case I-e
$c_0 = 0$	$h_0 > 0$	Case II-a
	$h_0 = 0$	Case II-b
	$h_0 < 0$	Case II-c
$c_0 < 0$		Case III

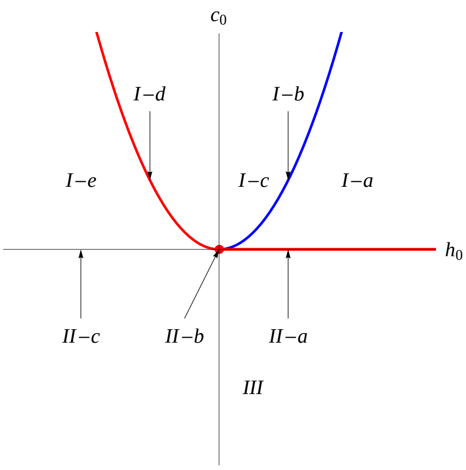


Fig. 2 Critical energy states for $H_1^{(3)}$

Table 2 lists the partition of cases used for integration. The critical energy states of the system $H_1^{(3)}$ are graphed in Fig. 2 and the typical configurations graphed in Fig. 3 and Fig. 4. The integral curves of $\vec{H}_1^{(3)}$ are expressed in terms of the Jacobi elliptic functions (see, e.g., [8]). Given a modulus $k \in [0,1]$ (and complementary modulus $k' = \sqrt{1-k^2}$), the basic Jacobi elliptic functions are defined as

$$\begin{aligned}\operatorname{sn}(x,k) &= \sin \operatorname{am}(x,k)\\ \operatorname{cn}(x,k) &= \cos \operatorname{am}(x,k)\\ \operatorname{dn}(x,k) &= \sqrt{1-k^2\operatorname{sn}^2(x,k)}.\end{aligned}$$

Here $\operatorname{am}(\cdot,k) = F(\cdot,k)^{-1}$ and $F(x,k) = \int_0^x \frac{dt}{\sqrt{1-k^2\sin^2 t}}$. (For the degenerate cases $k=0$ and $k=1$, we recover the circular and hyperbolic functions, respectively.) The functions $\operatorname{sn}(\cdot,k)$ and $\operatorname{cn}(\cdot,k)$ have period $4K$, whereas $\operatorname{dn}(\cdot,k)$ has period $2K$, where $K = F(\frac{\pi}{2},k)$. Furthermore, $\operatorname{sn}(\cdot,k)$ is odd, whereas $\operatorname{cn}(\cdot,k)$ and $\operatorname{dn}(\cdot,k)$ are even.

Theorem 2.1 (Case *I-a*) *Let $p(\cdot) : (-\varepsilon,\varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and $h_0 > \sqrt{c_0}$.*

(i) If $p_1(0) \le -\sqrt{c_0}$, then there exist $t_0 \in (0, \frac{2K}{\Omega})$ and $\sigma \in \{-1,1\}$ such that $p(t) = \bar{p}(t+t_0)$ for every $t \in (-\varepsilon,\varepsilon)$, where $\bar{p}(\cdot) : (0,\frac{2K}{\Omega}) \to \mathfrak{se}(1,1)^*$ is given by

$$\begin{cases} \bar{p}_1(t) = \dfrac{(\delta+h_0)\operatorname{dn}(\Omega t,k) + (\delta - h_0)}{\operatorname{dn}(\Omega t,k)-1}\\ \bar{p}_2(t) = 2\sigma\delta\,\dfrac{\operatorname{cn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)-1}\\ \bar{p}_3(t) = \sigma k^2 \Omega\,\dfrac{\operatorname{sn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)-1}.\end{cases}$$

Here $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{h_0+\delta}$ and $k = \sqrt{\frac{2\delta}{h_0+\delta}}$.

(ii) If $p_1(0) \ge \sqrt{c_0}$, then there exists $t_0 \in [-\frac{2K}{\Omega}, \frac{2K}{\Omega}]$ such that $p(t) = \bar{p}(t+t_0)$ for every $t \in (-\varepsilon,\varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is given by

$$\begin{cases} \bar{p}_1(t) = \sqrt{c_0}\,\dfrac{k'\operatorname{dn}(\Omega t,k)+1}{\operatorname{dn}(\Omega t,k)+k'}\\ \bar{p}_2(t) = k^2\sqrt{c_0}\,\dfrac{\operatorname{sn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)+k'}\\ \bar{p}_3(t) = k\sqrt{2\delta}\,\dfrac{\operatorname{cn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)+k'}.\end{cases}$$

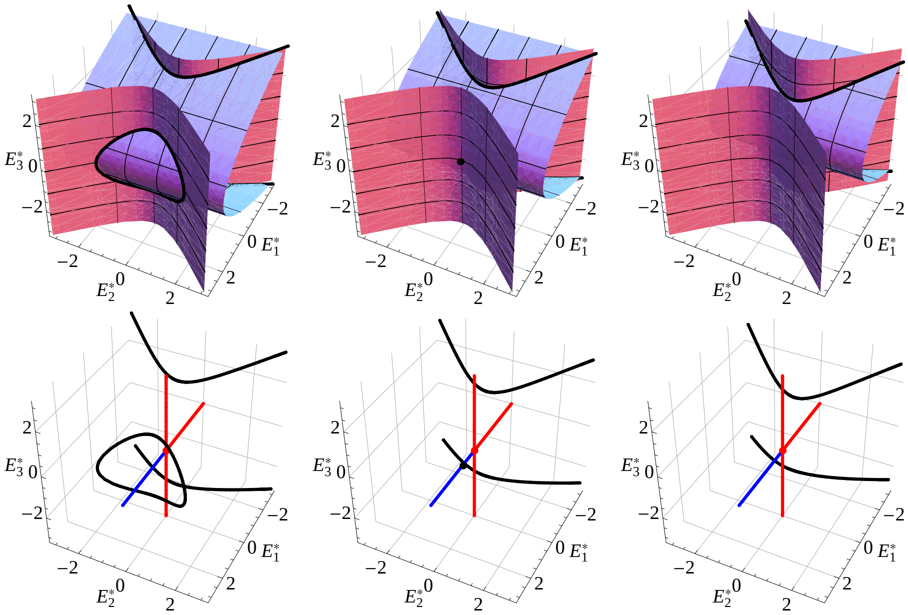
Here $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{h_0+\delta}$, $k = \sqrt{\frac{2\delta}{h_0+\delta}}$ and $k' = \sqrt{\frac{h_0-\delta}{h_0+\delta}}$.

Proof As $p_1(0)^2 \ge p_1(0)^2 - p_2(0)^2 = c_0$, we have that either $p_1(0) \le -\sqrt{c_0}$ or $p_1(0) \ge \sqrt{c_0}$. We describe how the expressions for $\bar{p}(\cdot)$ were found in the case $p_1(0) \le -\sqrt{c_0}$. (The expressions for $\bar{p}(\cdot)$ in (*ii*) were found in a similar fashion.) Let $\bar{p}(\cdot)$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(\bar{p}(0)) = h_0$, $C(\bar{p}(0)) = c_0$, $h_0 > \sqrt{c_0}$ and $\bar{p}_1(0) \le -\sqrt{c_0}$. As $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$, $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)$ and $c_0 = \bar{p}_1(t)^2 - \bar{p}_2(t)^2$, we get

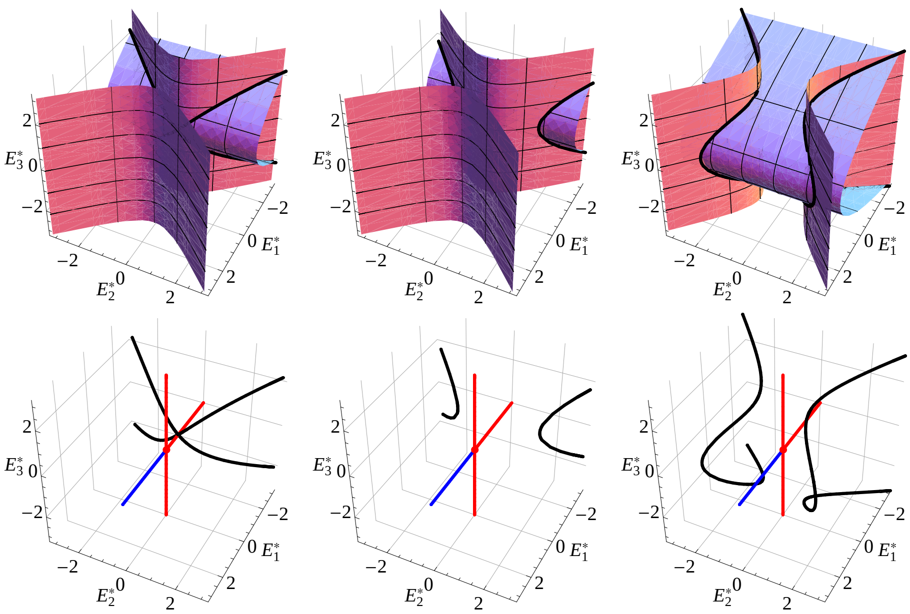
$$\dot{\bar{p}}_1 = \sigma_1\sqrt{(\bar{p}_1^2 - c_0)(2h_0 - 2\bar{p}_1)}$$

for some $\sigma_1 \in \{-1,1\}$. Equivalently,

$$\int \frac{\mathrm{d}\bar{p}_1}{\sqrt{(\bar{p}_1^2-c_0)(2h_0-2\bar{p}_1)}} = \int \sigma_1\,\mathrm{d}t. \tag{3}$$

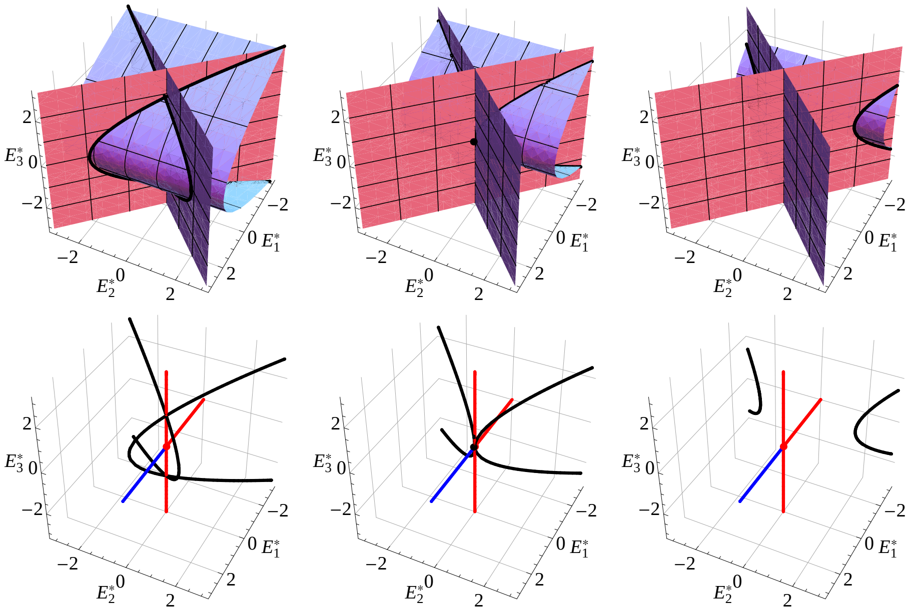


(a) Case I-a (b) Case I-b (c) Case I-c



(d) Case I-d (e) Case I-e (f) Case III

Fig. 3 Typical configurations of $H_1^{(3)}$, $c_0 \neq 0$



(a) Case *II-a* (b) Case *II-b* (c) Case *II-c*

Fig. 4 Typical configurations of $H_1^{(3)}$, $c_0 = 0$

We transform (3) to standard form and apply an integral formula (see, e.g., [8]). First, (3) may be rewritten as

$$\int \frac{\mathrm{d}\bar{p}_1}{\sqrt{[A_1(\bar{p}_1+\lambda_1)^2 + B_1(\bar{p}_1+\lambda_2)^2]\,[A_2(\bar{p}_1+\lambda_1)^2 + B_2(\bar{p}_1+\lambda_2)^2]}} = \int \sigma_1\,\mathrm{d}t$$

where $\lambda_1 = -(\delta + h_0)$, $\lambda_2 = \delta - h_0$, $A_1 = \frac{1}{2}(1 - \frac{h_0}{\delta}) < 0$, $B_1 = \frac{1}{2}(1 + \frac{h_0}{\delta}) > 0$, $A_2 = \frac{1}{2\delta} > 0$, $B_2 = -\frac{1}{2\delta} < 0$ and $\delta = \sqrt{h_0^2 - c_0}$. The change of variables $u = \frac{\bar{p}_1+\lambda_1}{\bar{p}_1+\lambda_2}$ yields

$$\int \frac{\mathrm{d}u}{\sqrt{-(u^2 + \frac{B_1}{A_1})(u^2 + \frac{B_2}{A_2})}} = \int \sigma_1(\lambda_2 - \lambda_1)\sqrt{-A_1A_2}\,\mathrm{d}t.$$

Thus we have

$$\int \frac{\mathrm{d}u}{\sqrt{\left(\frac{h_0+\delta}{h_0-\delta} - u^2\right)(u^2-1)}} = \int \sigma_1\sqrt{h_0-\delta}\,\mathrm{d}t. \tag{4}$$

We now apply the integral formula ([8])

$$\int_b^x \frac{\mathrm{d}u}{\sqrt{(a^2-u^2)(u^2-b^2)}} = \frac{1}{a}\,\mathrm{nd}^{-1}\left(\frac{x}{b}, \frac{\sqrt{a^2-b^2}}{a}\right), \qquad b \le x \le a$$

to the left-hand side of (4), for $a = \sqrt{\frac{h_0+\delta}{h_0-\delta}}$, $b = 1$ and $x = \frac{\bar{p}_1(t)+\lambda_1}{\bar{p}_1(t)+\lambda_2}$. (Here $\mathrm{nd}(x,k) = \frac{1}{\mathrm{dn}(x,k)}$.) We get

$$\frac{\bar{p}_1(t)+\lambda_1}{\bar{p}_1(t)+\lambda_2} = \mathrm{nd}\left(a\sqrt{h_0-\delta}\,t, \frac{\sqrt{a^2-1}}{a}\right).$$

(For convenience, we omit any translation in t.) Let $\Omega = a\sqrt{h_0 - \delta} = \sqrt{\delta + h_0}$ and $k = \frac{\sqrt{a^2-1}}{a} = \sqrt{\frac{2\delta}{\delta+h_0}}$, $k' = \sqrt{\frac{h_0-\delta}{h_0+\delta}}$. Then $\bar{p}_1(t)$ takes the form

$$\bar{p}_1(t) = \frac{(\delta + h_0)\operatorname{dn}(\Omega t, k) + (\delta - h_0)}{\operatorname{dn}(\Omega t, k) - 1}.$$

Using the equation $c_0 = \bar{p}_1(t)^2 - \bar{p}_2(t)^2$ and the identity $\operatorname{dn}^2(\Omega t, k) = k^2 \operatorname{cn}^2(\Omega t, k) + (k')^2$, we obtain

$$\bar{p}_2(t) = 2\sigma_2\delta \frac{\operatorname{cn}(\Omega t, k)}{\operatorname{dn}(\Omega t, k) - 1}$$

for some $\sigma_2 \in \{-1, 1\}$. Likewise, as $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)^2$, we have

$$\bar{p}_3(t) = \sigma_3\sqrt{2\delta}k \frac{\operatorname{sn}(\Omega t, k)}{\operatorname{dn}(\Omega t, k) - 1}$$

for some $\sigma_3 \in \{-1, 1\}$. We claim that $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_1^{(3)}$ if and only if $\sigma_2 = \sigma_3$. By Lemma 3, it suffices to show that $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ if and only if $\sigma_2 = \sigma_3$. Indeed,

$$\dot{\bar{p}}_1(t) - \bar{p}_2(t)\bar{p}_3(t) = \frac{2\delta\sqrt{2\delta}\, k(1 - \sigma_2\sigma_3)\operatorname{cn}(\Omega t, k)\operatorname{sn}(\Omega t, k)}{(\operatorname{dn}(\Omega t, k) - 1)^2}$$

and so $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ if and only if $\sigma_2 = \sigma_3$. In this case we have $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$. Since $\operatorname{dn}(\Omega t, k) = 1$ for $t \in \{\frac{2nK}{\Omega} : n \in \mathbb{Z}\}$ and $\bar{p}(\cdot)$ has period $\frac{2K}{\Omega}$, we may take the domain of $\bar{p}(\cdot)$ to be $(0, \frac{2K}{\Omega})$. Moreover, by Lemma 1, $\bar{p}(\cdot)$ is maximal.

It remains to be shown that any integral curve takes the form $t \mapsto \bar{p}(t + t_0)$ in each case.

(*i*) Let $\sigma = -\operatorname{sgn}(p_3(0)) \in \{-1, 1\}$. (If $p_3(0) = 0$, then $h_0 = p_1(0) \leq -\sqrt{c_0} < 0$, a contradiction.) We have $\operatorname{sgn}(\bar{p}_2|_{(0,K/\Omega)}(t)) = \sigma$ and $\operatorname{sgn}(\bar{p}_2|_{(K/\Omega,2K/\Omega)}(t)) = -\sigma$. Moreover, $\lim_{t\to 0}\bar{p}_2(t) = -\sigma\infty$ and $\lim_{t\to 2K/\Omega}\bar{p}_2(t) = \sigma\infty$. Therefore, since $\bar{p}_2(\cdot)$ is continuous, there exists $t_0 \in (0, \frac{2K}{\Omega})$ such that $\bar{p}_2(t_0) = p_2(0)$. Then

$$\bar{p}_1(t_0)^2 = c_0 + \bar{p}_2(t_0)^2 = c_0 + p_2(0)^2 = p_1(0)^2.$$

We have $\bar{p}_1(t_0), p_1(0) \leq -\sqrt{c_0}$, and so $\bar{p}_1(t_0) = p_1(0)$. Furthermore,

$$\bar{p}_3(t_0)^2 = 2(h_0 - \bar{p}_1(t_0)) = 2(h_0 - p_1(0)) = p_3(0)^2.$$

Since $\operatorname{sgn}(\bar{p}_3(t_0)) = -\sigma = \operatorname{sgn}(p_3(0))$, it follows that $\bar{p}_3(t_0) = p_3(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(*ii*) Let $\omega = \sqrt{2h_0 - 2\sqrt{c_0}}$. From the identity $h_0 = p_1(t) + \frac{1}{2}p_3(t)^2$ we have $p_3(t)^2 = 2h_0 - 2p_1(t) \leq 2h_0 - 2\sqrt{c_0} = \omega^2$, i.e., $-\omega \leq p_3(t) \leq \omega$. Likewise, $-\omega \leq \bar{p}_3(t) \leq \omega$. Moreover, $\bar{p}_3(0) = \omega$ and $\bar{p}_3(\frac{2K}{\Omega}) = -\omega$. Therefore, since $\bar{p}_3(\cdot)$ is continuous, there exists $t_1 \in [0, \frac{2K}{\Omega}]$ such that $\bar{p}_3(t_1) = p_3(0)$. Then $\bar{p}_1(t_1) = h_0 - \frac{1}{2}\bar{p}_3(t_1)^2 = h_0 - \frac{1}{2}p_3(0)^2 = p_1(0)$. Similarly,

$$\bar{p}_2^2(t_1)^2 = \bar{p}_1(t_1)^2 - c_0 = p_1(0)^2 - c_0 = p_2(0)^2$$

and so $\bar{p}_2(t_1) = \pm p_2(0)$. Since $\bar{p}_1(\cdot)$ and $\bar{p}_3(\cdot)$ are even and $\bar{p}_2(\cdot)$ is odd, we have $\bar{p}_1(-t_1) = \bar{p}_1(t_1)$, $\bar{p}_2(-t_1) = -\bar{p}_2(t_1)$ and $\bar{p}_3(t_1) = \bar{p}_3(t_1)$. Hence, there exists

$t_0 \in [-\frac{2K}{\Omega}, \frac{2K}{\Omega}]$ (i.e., $t_0 = t_1$ or $t_0 = -t_1$) such that $\bar{p}(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. □

Theorem 2.2 **(Case** ***I-b*****)** *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_1^{(3)}$ *such that* $H_1^{(3)}(p(0)) = h_0$, $C(p(0)) = c_0 > 0$, $h_0 = \sqrt{c_0}$ *and* $p_1(0) \leq -\sqrt{c_0}$. *There exist* $t_0 \in \left(-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}\right)$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$, *where* $\bar{p}(\cdot) : \left(-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}\right) \to \mathfrak{se}(1,1)^*$ *is given by*

$$\begin{cases} \bar{p}_1(t) = -h_0 \left[1 + 2\tan^2\left(\sqrt{h_0}\, t\right)\right] \\ \bar{p}_2(t) = -\sigma 2h_0 \sec\left(\sqrt{h_0}\, t\right) \tan\left(\sqrt{h_0}\, t\right) \\ \bar{p}_3(t) = 2\sigma\sqrt{h_0}\, \sec\left(\sqrt{h_0}\, t\right). \end{cases}$$

Theorem 2.3 **(Case** ***I-c*****)** *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_1^{(3)}$ *such that* $H_1^{(3)}(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ *and* $-\sqrt{c_0} < h_0 < \sqrt{c_0}$. *There exist* $t_0 \in \left(0, \frac{2K}{\Omega}\right)$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$, *where* $\bar{p}(\cdot) : \left(0, \frac{2K}{\Omega}\right) \to \mathfrak{se}(1,1)^*$ *is given by*

$$\begin{cases} \bar{p}_1(t) = \dfrac{(\delta + \sqrt{c_0})\,\mathrm{dn}(\Omega t, k) + (\delta - \sqrt{c_0})}{\mathrm{dn}(\Omega t, k) - 1} \\ \bar{p}_2(t) = \sigma k \sqrt{\delta(\delta + 2\sqrt{c_0})}\, \dfrac{\mathrm{cn}(\Omega t, k)\sqrt{\mathrm{dn}(\Omega t, k) + 1}}{\sqrt{\mathrm{dn}(\Omega t, k) + k'}\,[\mathrm{dn}(\Omega t, k) - 1]} \\ \bar{p}_3(t) = \sigma\sqrt{2(\delta + \sqrt{c_0} - h_0)}\, \dfrac{\sqrt{\mathrm{dn}(\Omega t, k) + k'}\sqrt{1 - \mathrm{dn}(\Omega t, k)}}{\mathrm{dn}(\Omega t, k) - 1}. \end{cases}$$

Here $\delta = \sqrt{2(c_0 - \sqrt{c_0} h_0)}$, $\Omega = \frac{1}{2}\sqrt{6\sqrt{c_0} - 2h_0 + 4\delta}$, $k = 2\sqrt{\frac{\delta}{3\sqrt{c_0} - h_0 + 2\delta}}$ *and* $k' = \sqrt{\frac{3\sqrt{c_0} - h_0 - 2\delta}{3\sqrt{c_0} - h_0 + 2\delta}}$.

Theorem 2.4 **(Case** ***I-d*****)** *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_1^{(3)}$ *such that* $H_1^{(3)}(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ *and* $h_0 = -\sqrt{c_0}$. *There exist* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$, *where* $\bar{p}(\cdot)$ *is given by*

$$\begin{cases} \bar{p}_1(t) = h_0 \left[1 + 2\,\mathrm{csch}^2\left(\sqrt{-h_0}\, t\right)\right] \\ \bar{p}_2(t) = -2\sigma h_0 \coth\left(\sqrt{-h_0}\, t\right) \mathrm{csch}\left(\sqrt{-h_0}\, t\right) \\ \bar{p}_3(t) = 2\sigma\, \mathrm{csch}\left(\sqrt{-h_0}\, t\right). \end{cases}$$

Furthermore, $\bar{p}|_{(-\infty,0)}(\cdot)$ *and* $\bar{p}|_{(0,\infty)}(\cdot)$ *are maximal.*

Proof The expression for $\bar{p}(\cdot)$ was obtained by taking the limit $h_0 \to -\sqrt{c_0}$ of the expressions in Theorem 2.3. Let $\bar{p}_-(\cdot) = \bar{p}|_{(-\infty,0)}(\cdot)$ and $\bar{p}_+(\cdot) = \bar{p}|_{(0,\infty)}(\cdot)$. By Lemma 1, we have that $\bar{p}_-(\cdot)$ and $\bar{p}_+(\cdot)$ are maximal. We prove that any integral curve is of the form $t \mapsto \bar{p}_-(t + t_0)$ or $t \mapsto \bar{p}_+(t + t_0)$. Let $\sigma = \mathrm{sgn}(p_2(0)) \in$

$\{-1,1\}$ and $\varsigma = \sigma \operatorname{sgn}(p_3(0)) \in \{-1,1\}$. (If $p_2(0) = 0$ or $p_3(0) = 0$, then $p(0)$ is an equilibrium point of $\vec{H}_1^{(3)}$.) We have $\operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t)) = \sigma$. Moreover,

$$\lim_{t\to 0} \bar{p}_{+,2}(t) = \sigma\infty, \; \lim_{t\to\infty} \bar{p}_{+,2}(t) = 0, \qquad \lim_{t\to-\infty} \bar{p}_{-,2}(t) = 0, \; \lim_{t\to 0} \bar{p}_{-,2}(t) = \sigma\infty.$$

Suppose $\operatorname{sgn}(p_3(0)) = 1$; then $\varsigma = \sigma$. Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous and $\operatorname{sgn}(p_2(0)) = \operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t))$, there exists

$$t_0 \in \begin{cases} (-\infty, 0) & \text{if } \sigma = -1 \\ (0, \infty) & \text{if } \sigma = +1 \end{cases}$$

such that $\bar{p}_{\sigma,2}(t_0) = p_2(0)$. Then

$$\bar{p}_{\sigma,1}(t_0)^2 = \bar{p}_{\sigma,2}(t_0)^2 + c_0 = p_2(0)^2 + c_0 = p_1(0)^2.$$

We have $p_1(0), \bar{p}_{\sigma,1}(t_0) \leq -\sqrt{c_0} < 0$, and so $\bar{p}_{\sigma,1}(t_0) = p_1(0)$. Moreover,

$$\bar{p}_{\sigma,3}(t_0)^2 = 2h_0 - 2\bar{p}_{\sigma,1}(t_0) = 2h_0 - 2p_1(0) = p_3(0)^2.$$

As $\operatorname{sgn}(\bar{p}_{\sigma,3}(t_0)) = 1 = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{\sigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}_\varsigma(t+t_0)$ and $t \mapsto p(t)$ are integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

On the other hand, suppose $\operatorname{sgn}(p_3(0)) = -1$; then $\varsigma = -\sigma$. Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous and $\operatorname{sgn}(p_2(0)) = \operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t))$, there exists

$$t_0 \in \begin{cases} (-\infty, 0) & \text{if } \sigma = +1 \\ (0, \infty) & \text{if } \sigma = -1 \end{cases}$$

such that $\bar{p}_{-\sigma,2}(t_0) = p_2(0)$. From $\bar{p}_{\sigma,1}(t_0)^2 = \bar{p}_{\sigma,2}(t_0)^2 = p_2(0)^2 = p_1(0)^2$ we again get $\bar{p}_{-\sigma,1}(t_0) = p_1(0)$. Similarly, as $\bar{p}_{\sigma,3}(t_0)^2 = 2h_0 - 2\bar{p}_{\sigma,1}(t_0) = 2h_0 - 2p_1(0) = p_3(0)^2$ and $\operatorname{sgn}(\bar{p}_{-\sigma,3}(t_0)) = -1 = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{-\sigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore $t \mapsto \bar{p}_\varsigma(t + t_0)$ and $t \mapsto p(t)$ both solve the same Cauchy problem, and so are identical. □

Theorem 2.5 (Case *I-e***)** *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and $h_0 < -\sqrt{c_0}$. There exist $t_0 \in \left(-\frac{2K}{\Omega}, \frac{2K}{\Omega}\right)$ and $\sigma \in \{-1,1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \left(-\frac{2K}{\Omega}, \frac{2K}{\Omega}\right) \to \mathfrak{se}(1,1)^*$ is given by*

$$\begin{cases} \bar{p}_1(t) = \dfrac{(h_0 + \delta)\operatorname{cn}(\Omega t, k) + (h_0 - \delta)\operatorname{dn}(\Omega t, k)}{\operatorname{dn}(\Omega t, k) + \operatorname{cn}(\Omega t, k)} \\ \bar{p}_2(t) = \sigma \dfrac{2\delta}{\operatorname{dn}(\Omega t, k) + \operatorname{cn}(\Omega t, k)} \\ \bar{p}_3(t) = -\sigma \dfrac{\sqrt{2\delta} k' \operatorname{sn}(\Omega t, k)}{\operatorname{dn}(\Omega t, k) + \operatorname{cn}(\Omega t, k)}. \end{cases}$$

Here $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{\delta - h_0}$, $k = \sqrt{\frac{h_0+\delta}{h_0-\delta}}$ and $k' = \sqrt{\frac{2\delta}{\delta - h_0}}$.

Theorem 2.6 **(Case** *II-a***)** *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = h_0 > 0$ and $C(p(0)) = 0$.*

(i) If $p_1(0) < 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot)$ is given by

$$\begin{cases} \bar{p}_1(t) = -h_0 \operatorname{csch}^2\left(\sqrt{\frac{h_0}{2}}\, t\right) \\ \bar{p}_2(t) = \sigma h_0 \operatorname{csch}^2\left(\sqrt{\frac{h_0}{2}}\, t\right) \\ \bar{p}_3(t) = \sigma \sqrt{2h_0}\, \coth\left(\sqrt{\frac{h_0}{2}}\, t\right). \end{cases}$$

Furthermore, $\bar{p}|_{(-\infty,0)}(\cdot)$ and $\bar{p}|_{(0,\infty)}(\cdot)$ are maximal.

(ii) If $p_1(0) > 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is given by

$$\begin{cases} \bar{p}_1(t) = h_0 \operatorname{sech}^2\left(\sqrt{\frac{h_0}{2}}\, t\right) \\ \bar{p}_2(t) = -\sigma h_0 \operatorname{sech}^2\left(\sqrt{\frac{h_0}{2}}\, t\right) \\ \bar{p}_3(t) = \sigma \sqrt{2h_0}\, \tanh\left(\sqrt{\frac{h_0}{2}}\, t\right). \end{cases}$$

Theorem 2.7 **(Case** *II-b***)** *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = C(p(0)) = 0$ and $p_1(0) < 0$. There exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot)$ is given by*

$$\begin{cases} \bar{p}_1(t) = -\dfrac{2}{t^2} \\ \bar{p}_2(t) = \dfrac{2\sigma}{t^2} \\ \bar{p}_3(t) = \dfrac{2\sigma}{t}. \end{cases}$$

Furthermore, $\bar{p}|_{(-\infty,0)}(\cdot)$ and $\bar{p}|_{(0,\infty)}(\cdot)$ are maximal.

Theorem 2.8 **(Case** *II-c***)** *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = h_0 < 0$ and $C(p(0)) = 0$. There exist $t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}) \to \mathfrak{se}(1,1)^*$ is given by*

$$\begin{cases} \bar{p}_1(t) = h_0 \sec^2\left(\sqrt{-\frac{h_0}{2}}\, t\right) \\ \bar{p}_2(t) = \sigma h_0 \sec^2\left(\sqrt{-\frac{h_0}{2}}\, t\right) \\ \bar{p}_3(t) = \sigma \sqrt{-2h_0}\, \tan\left(\sqrt{-\frac{h_0}{2}}\, t\right). \end{cases}$$

Theorem 2.9 **(Case** *III***)** *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = h_0$ and $C(p(0)) = c_0 < 0$. There exist $t_0 \in \left(0, \frac{4K}{\Omega}\right)$ and*

$\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t+t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : (-\frac{2K}{\Omega}, \frac{2K}{\Omega}) \to \mathfrak{se}(1,1)^*$ is given by

$$\begin{cases} \bar{p}_1(t) = \dfrac{(h_0 + \delta)\,\mathrm{cn}(\Omega t, k) + (h_0 - \delta)}{\mathrm{cn}(\Omega t, k) + 1} \\ \bar{p}_2(t) = \sigma \Omega^2 \dfrac{\mathrm{dn}(\Omega t, k)}{\mathrm{cn}(\Omega t, k) + 1} \\ \bar{p}_3(t) = \sigma \dfrac{\Omega\,\mathrm{sn}(\Omega t, k)}{\mathrm{cn}(\Omega t, k) + 1}. \end{cases}$$

Here $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{2\delta}$ and $k = \sqrt{\frac{\delta + h_0}{2\delta}}$.

5.3 The system $H_2^{(3)}$

The equations of motion of the system $H_2^{(3)}(p) = p_1 + p_2 + \frac{1}{2}p_3^2$ are

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -(p_1 + p_2). \end{cases}$$

The equilibrium states of $\vec{H}_2^{(3)}$ are $\mathsf{e}_1^\mu = (\mu, -\mu, 0)$ and $\mathsf{e}_2^\nu = (0, 0, \nu)$.

Proposition 8 *The equilibrium states have the following behaviour:*

(*i*) *The states e_1^μ are unstable.*
(*ii*) *The states e_2^ν are (spectrally) unstable.*

Proof (*i*) Consider the states e_1^μ, $\mu \neq 0$. The integral curve $p(t) = (\mu e^{\delta t}, -\mu e^{\delta t}, -\delta)$, $\delta > 0$ satisfies $\|p(0) - \mathsf{e}_1^\mu\| = \delta$. Accordingly, for any open neighbourhood N of e_1^μ, there exists $\delta > 0$ such that $p(0) \in V$. Since $\lim_{t\to\infty} \|p(t)\| = \infty$, it follows that the states e_1^μ, $\mu \neq 0$ are unstable. Likewise, the integral curve $p(t) = (\delta e^{\delta t}, -\delta e^{\delta t}, -\delta)$ suffices to show that the state e_1^0 is unstable.

(*ii*) As $\frac{\partial H_2^{(3)}}{\partial p_3}(\mathsf{e}_2^\nu) = \nu$, it follows from Lemma 2 that the states e_2^ν are spectrally unstable. □

Table 3 Index of cases for the integral curves of $H_2^{(3)}$

Conditions		Designation
$c_0 > 0$	$h_0 > 0$	Case *I-a*
	$h_0 = 0$	Case *I-b*
	$h_0 < 0$	Case *I-c*
$c_0 = 0$	$h_0 > 0$	Case *II-a*
	$h_0 = 0$	Case *II-b*
	$h_0 < 0$	Case *II-c*

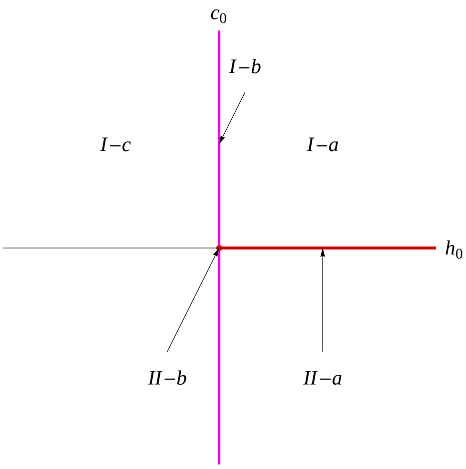


Fig. 5 Critical energy states for $H_2^{(3)}$

Remark 4 The states $(0, c_0)$, $c_0 \neq 0$ and $(h_0, 0)$, $h_0 \geq 0$ are generalized critical energy states of $H_2^{(3)}$ (see Remark 3). Indeed, the points $(h_0, 0)$, $h_0 \geq 0$ are critical energy states in the usual sense. On the other hand, the curves $f(s) = \big(-\frac{c_0 s^4+4}{4s^2}, \frac{c_0 s^4-4}{4s^2}, -\frac{2}{s}\big)$ and $g(s) = \big(-\frac{c_0 s^2}{4}, \frac{c_0 s^2}{4}, 0\big)$ suffice to show that $(0, c_0)$, $c_0 \neq 0$ are generalized critical energy states.

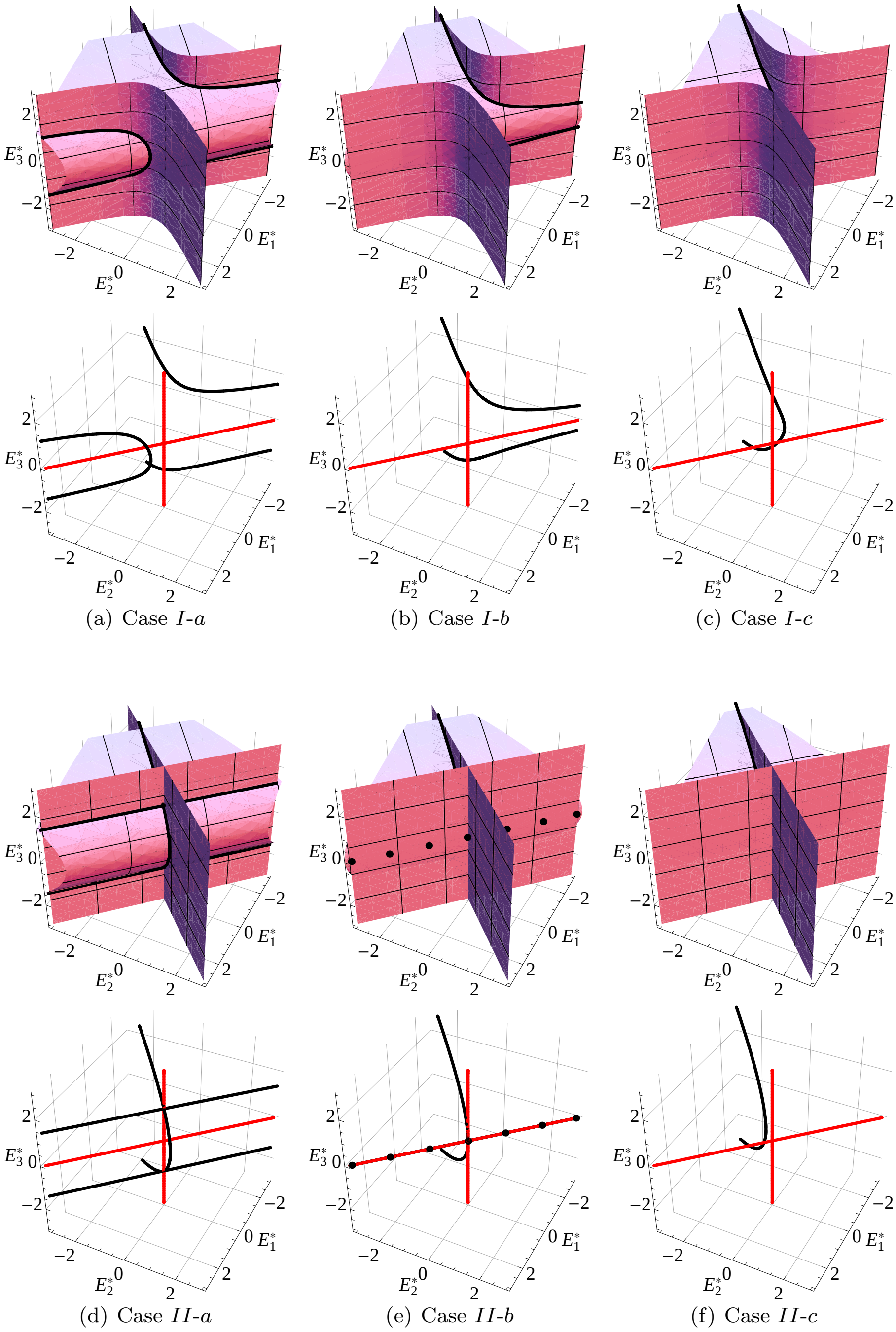
The map $\psi : (p_1, p_2, p_3) \mapsto (p_2, p_1, p_3)$ is a symmetry of $H_2^{(3)}$ such that $C \circ \psi = -C$. Accordingly, we may assume $c_0 \geq 0$. For the cases $c_0 > 0$ and $c_0 = 0$, there are several further subcases (see Table 3). The (generalized) critical energy states of $H_2^{(3)}$ are graphed in Fig. 5; the typical configurations are graphed in Fig. 6.

Theorem 3.1 (Case I**-**a**, case** II**-**a**)** *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_2^{(3)}$ *such that* $H_2^{(3)}(p(0)) = h_0 > 0$ *and* $C(p(0)) = c_0 \geq 0$*.*

(i) *If* $p_1(0) \leq -\sqrt{c_0}$*, then there exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$*, where* $\bar{p}(\cdot)$ *is given by*

$$\begin{cases} \bar{p}_1(t) = -\dfrac{1}{2h_0}\left[h_0^2 \operatorname{csch}^2\big(\sqrt{\tfrac{h_0}{2}}\, t\big) + c_0 \sinh^2\big(\sqrt{\tfrac{h_0}{2}}\, t\big)\right] \\ \bar{p}_2(t) = -\dfrac{1}{2h_0}\left[h_0^2 \operatorname{csch}^2\big(\sqrt{\tfrac{h_0}{2}}\, t\big) - c_0 \sinh^2\big(\sqrt{\tfrac{h_0}{2}}\, t\big)\right] \\ \bar{p}_3(t) = -\sqrt{2h_0}\, \coth\big(\sqrt{\tfrac{h_0}{2}}\, t\big). \end{cases}$$

Furthermore, $\bar{p}|_{(-\infty,0)}(\cdot)$ *and* $\bar{p}|_{(0,\infty)}(\cdot)$ *are maximal.*



(a) Case I-a (b) Case I-b (c) Case I-c

(d) Case II-a (e) Case II-b (f) Case II-c

Fig. 6 Typical configurations of $H_2^{(3)}$

(*ii*) *If* $p_1(0) \geq \sqrt{c_0}$*, then there exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$*, where* $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ *is given by*

$$\begin{cases} \bar{p}_1(t) = \dfrac{1}{2h_0}\left[h_0^2 \operatorname{sech}^2\left(\sqrt{\tfrac{h_0}{2}}\, t\right) + c_0 \cosh^2\left(\sqrt{\tfrac{h_0}{2}}\, t\right)\right] \\ \bar{p}_2(t) = \dfrac{1}{2h_0}\left[h_0^2 \operatorname{sech}^2\left(\sqrt{\tfrac{h_0}{2}}\, t\right) - c_0 \cosh^2\left(\sqrt{\tfrac{h_0}{2}}\, t\right)\right] \\ \bar{p}_3(t) = -\sqrt{2h_0}\, \tanh\left(\sqrt{\tfrac{h_0}{2}}\, t\right). \end{cases}$$

(*iii*) *If* $c_0 = 0$ *and* $p_1(0) + p_2(0) = 0$ *(with* $p_1(0)$ *and* $p_2(0)$ *not both zero), then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma, \varsigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$*, where* $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ *is given by*

$$\begin{cases} \bar{p}_1(t) = \varsigma e^{-\sigma\sqrt{2h_0}\, t} \\ \bar{p}_2(t) = -\varsigma e^{-\sigma\sqrt{2h_0}\, t} \\ \bar{p}_3(t) = \sigma\sqrt{2h_0}. \end{cases}$$

Proof Standard computations yield the expressions for $\bar{p}(\cdot)$ shown. We prove that for case (*i*) every integral curve takes the form $t \mapsto \bar{p}_-(t + t_0)$ or $t \mapsto \bar{p}_+(t + t_0)$, where $\bar{p}_-(\cdot) = \bar{p}|_{(-\infty,0)}(\cdot)$ and $\bar{p}_+(\cdot) = \bar{p}|_{(0,\infty)}(\cdot)$. (The arguments for (*ii*) and (*iii*) are analogous.) By Lemma 1, we have that $\bar{p}_-(\cdot)$ and $\bar{p}_+(\cdot)$ are maximal. Let $\varsigma = -\operatorname{sgn}(p_3(0)) \in \{-1, 1\}$. (If $p_3(0) = 0$, then $p_1(0) > 0$, a contradiction.) We have

$$\lim_{t\to 0} \bar{p}_{+,2}(t) = -\infty,\ \lim_{t\to\infty} \bar{p}_{+,2}(t) = \infty, \qquad \lim_{t\to 0} \bar{p}_{-,2}(t) = -\infty,\ \lim_{t\to-\infty} \bar{p}_{-,2}(t) = \infty.$$

Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous, there exists

$$t_0 \in \begin{cases} (-\infty, 0) & \text{if } \varsigma = -1 \\ (0, \infty) & \text{if } \varsigma = +1 \end{cases}$$

such that $\bar{p}_{\varsigma,2}(t_0) = p_2(0)$. Then

$$\bar{p}_{\varsigma,1}(t_0)^2 = \bar{p}_{\varsigma,2}(t_0)^2 + c_0 = p_2(0)^2 + c_0 = p_1(0)^2.$$

We have $\bar{p}_{\varsigma,1}(t_0), p_1(0) \leq -\sqrt{c_0} \leq 0$, and so $\bar{p}_{\varsigma,1}(t_0), p_1(0) < 0$. (If $c_0 = 0$ and $p_1(0) = 0$, then $p(0)$ is an equilibrium point.) Hence $\bar{p}_{\varsigma,1}(t_0) = p_1(0)$. Furthermore,

$$\bar{p}_{\varsigma,3}(t_0)^2 = 2(h_0 - \bar{p}_{\varsigma,1}(t_0) - \bar{p}_{\varsigma,2}(t_0)) = 2(h_0 - p_1(0) - p_2(0)) = p_3(0)^2.$$

As $\operatorname{sgn}(\bar{p}_{\varsigma,3}(t_0)) = -\varsigma = \operatorname{sgn}(p_3(0))$, it follows that $\bar{p}_{\varsigma,3}(t_0) = p_3(0)$. Therefore, as $t \mapsto \bar{p}_\varsigma(t + t_0)$ and $t \mapsto p(t)$ are integral curves of $\vec{H}_2^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. □

Theorem 3.2 (Case ***I-b*****, case** ***II-b*****)** *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_2^{(3)}$ *such that* $H_2^{(3)}(p(0)) = 0$ *and* $C(p(0)) = c_0 \geq 0$. *There exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$, *where* $\bar{p}(\cdot)$ *is given by*

$$\begin{cases} \bar{p}_1(t) = -\dfrac{4 + c_0 t^4}{4t^2} \\ \bar{p}_2(t) = -\dfrac{4 - c_0 t^4}{4t^2} \\ \bar{p}_3(t) = -\dfrac{2}{t}. \end{cases}$$

Furthermore, $\bar{p}|_{(-\infty,0)}(\cdot)$ *and* $\bar{p}|_{(0,\infty)}(\cdot)$ *are maximal.*

Theorem 3.3 (Case ***I-c*****)** *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_2^{(3)}$ *such that* $H_2^{(3)}(p(0)) = h_0 < 0$ *and* $C(p(0)) = c_0 \geq 0$. *There exists* $t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$, *where* $\bar{p}(\cdot) : (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}) \to \mathfrak{se}(1,1)^*$ *is given by*

$$\begin{cases} \bar{p}_1(t) = \dfrac{1}{2h_0}\left[h_0^2 \sec^2\left(\sqrt{-\tfrac{h_0}{2}}\, t\right) + c_0 \cos^2\left(\sqrt{-\tfrac{h_0}{2}}\, t\right)\right] \\ \bar{p}_2(t) = \dfrac{1}{2h_0}\left[h_0^2 \sec^2\left(\sqrt{-\tfrac{h_0}{2}}\, t\right) - c_0 \cos^2\left(\sqrt{-\tfrac{h_0}{2}}\, t\right)\right] \\ \bar{p}_3(t) = \sqrt{-2h_0}\tan\left(\sqrt{-\tfrac{h_0}{2}}\, t\right). \end{cases}$$

5.4 The system $H_{3,\alpha}^{(5)}$

The equations of motion of the system $H_{3,\alpha}^{(5)}(p) = \alpha(p_1 + p_2) + \frac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right]$, $\alpha > 0$ are

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -(p_1 + p_2)(p_1 + p_2 + \alpha). \end{cases}$$

The equilibrium states of $\vec{H}_{3,\alpha}^{(5)}$ are $\mathsf{e}_1^\mu = (\mu, -\mu, 0)$, $\mathsf{e}_2^\mu = (\mu, -(\alpha + \mu), 0)$ and $\mathsf{e}_3^\nu = (0, 0, \nu)$.

Proposition 9 *The equilibrium states have the following behaviour*:

(*i*) *The states* e_1^μ *are unstable.*
(*ii*) *The states* e_2^μ *are stable.*
(*iii*) *The states* e_3^ν *are* (*spectrally*) *unstable.*

Proof (*i*) Consider the states e_1^μ, $\mu \neq 0$. We have that $p(t) = (\mu e^{\delta t}, -\mu e^{\delta t}, -\delta)$ is an integral curve of $\vec{H}_{3,\alpha}^{(5)}$ (for any $\delta > 0$) such that $\|p(0) - \mathsf{e}_1^\mu\| = \delta$. Accordingly, for any neighbourhood N of e_1^μ there exists $\delta > 0$ such that $p(0) \in N$. Furthermore, $\lim_{t\to\infty} \|p(t)\| = \infty$. Therefore the states e_1^μ, $\mu \neq 0$ are unstable. Similarly, the integral curve $p(t) = (\delta e^{\delta t}, -\delta e^{\delta t}, -\delta)$ suffices to show that the state e_1^0 is unstable.

(ii) Consider the states e_2^{μ}. Let $H_\lambda = \lambda_0 H^{(5)}_{3,\alpha} + \lambda_1 C$, where $\lambda_0 = 1$ and $\lambda_1 = 0$. Then $\mathbf{d}H_\lambda(\mathsf{e}_2^{\mu}) = 0$ and the restriction of

$$\mathbf{d}^2 H_\lambda(\mathsf{e}_2^{\mu}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to $W \times W$ is positive definite. Here $W = \ker \mathbf{d}H^{(5)}_{3,\alpha}(\mathsf{e}_2^{\mu}) \cap \ker \mathbf{d}C(\mathsf{e}_2^{\mu}) = \operatorname{span}\{E_1^* - \frac{\mu}{\alpha+\mu}E_2^*, E_3^*\}$ when $\alpha + \mu \neq 0$, and $W = \operatorname{span}\{E_2^*, E_3^*\}$ when $\alpha + \mu = 0$. Thus the states e_2^{μ} are stable.

(iii) Consider the states e_3^{ν}. We have $\frac{\partial H^{(5)}_{3,\alpha}}{\partial p_3}(\mathsf{e}_3^{\nu}) = \nu$, hence by Lemma 2 the states e_3^{ν} are (spectrally) unstable. □

Table 4 Index of cases for the integral curves of $H^{(5)}_{3,\alpha}$

Conditions		Designation
$c_0 > 0$	$h_0 > 0$	Case *I-a*
	$h_0 = 0$	Case *I-b*
	$h_0 < 0$	Case *I-c*
$c_0 = 0$	$h_0 > 0$	Case *II-a*
	$h_0 = 0$	Case *II-b*
	$h_0 < 0$	Case *II-c*

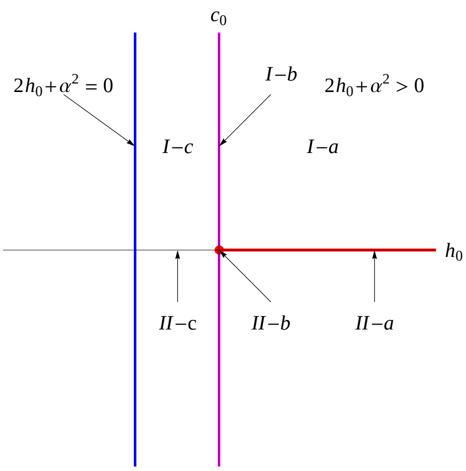


Fig. 7 Critical energy states for $H^{(5)}_{3,\alpha}$

Remark 5 The states $(0, c_0)$, $c_0 \neq 0$ and $(h_0, 0)$, $h_0 \geq 0$ are generalized critical energy states of $H^{(5)}_{3,\alpha}$ (see Remark 3). Indeed, $(h_0, 0)$, $h_0 \geq 0$ are critical energy states in the usual sense, whereas $f(s) = \left(-\frac{c_0(1+\alpha^2 s^2)^2+4\alpha^2}{4\alpha(\alpha^2 s^2+1)}, \frac{c_0(1+\alpha^2 s^2)^2-4\alpha^2}{4\alpha(\alpha^2 s^2+1)}, -\frac{2\alpha^2 s}{\alpha^2 s^2+1}\right)$

and $g(s) = \left(-\frac{c_0(\alpha^2 s^2+1)}{4\alpha}, \frac{c_0(\alpha^2 s^2+1)}{4\alpha}, 0\right)$ are sufficient to show that the states $(0, c_0)$, $c_0 \neq 0$ are generalized critical energy states.

The cylinders $(H^{(5)}_{3,\alpha})^{-1}(h_0)$ degenerate to a line exactly when $2h_0 + \alpha^2 = 0$; hence we assume $2h_0+\alpha^2 > 0$. The map $\psi : (p_1, p_2, p_3) \mapsto (p_2, p_1, p_3)$ is a symmetry of $H^{(5)}_{3,\alpha}$ such that $C \circ \psi = -C$, and so we may assume $c_0 \geq 0$. The cases $c_0 > 0$ and $c_0 = 0$ are further subdivided into several subcases (see Table 4). Fig. 7 illustrates the (generalized) critical energy states of $H^{(5)}_{3,\alpha}$; the typical configurations of the system are graphed in Fig. 8.

Theorem 4.1 (Case *I*-*a*, case *II*-*a*) *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}^{(5)}_{3,\alpha}$ such that $C(p(0)) = c_0 \geq 0$, $H^{(5)}_{3,\alpha}(p(0)) = h_0 > 0$ and $2h_0 + \alpha^2 > 0$.*

(i) If $p_1(0) \geq \sqrt{c_0}$ and $p_1(0) + p_2(0) \neq 0$, then there exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is given by

$$\begin{cases} \bar{p}_1(t) + \bar{p}_2(t) = \dfrac{2h_0}{\alpha + \sqrt{2h_0+\alpha^2}\cosh(\sqrt{2h_0}\,t)} \\ \bar{p}_1(t) - \bar{p}_2(t) = \dfrac{c_0}{2h_0}(\alpha + \sqrt{2h_0+\alpha^2}\cosh(\sqrt{2h_0}\,t)) \\ \bar{p}_3(t) = \dfrac{\sqrt{2h_0}\sqrt{2h_0+\alpha^2}(\alpha - \sqrt{2h_0+\alpha^2})\tanh\left(\sqrt{\frac{h_0}{2}}\,t\right)}{h_0 + \left(h_0 + \alpha^2 - \alpha\sqrt{2h_0+\alpha^2}\right)\tanh^2\left(\sqrt{\frac{h_0}{2}}\,t\right)}. \end{cases}$$

(ii) If $p_1(0) \leq -\sqrt{c_0}$ and $p_1(0) + p_2(0) \neq 0$, then there exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is given by

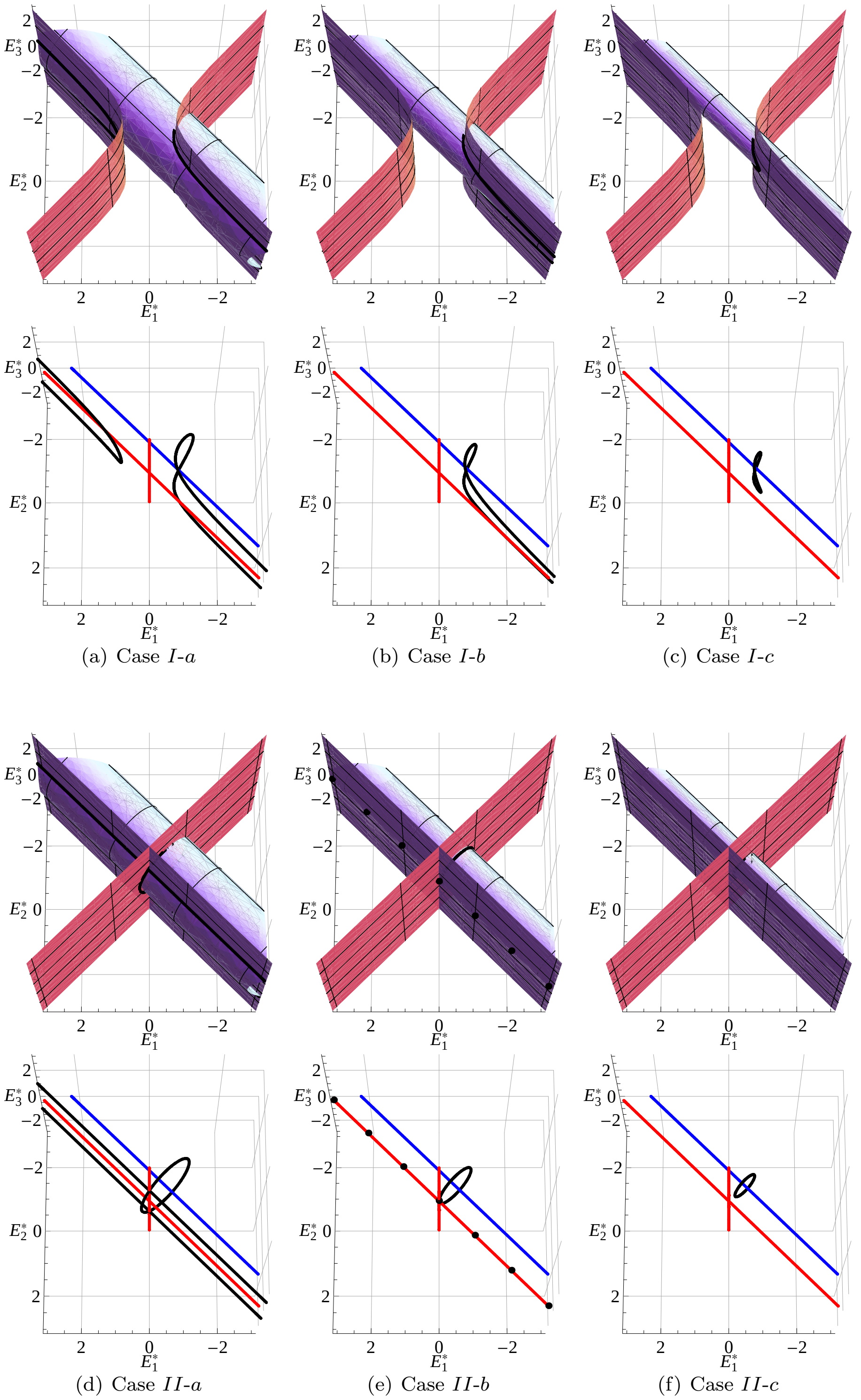
$$\begin{cases} \bar{p}_1(t) + \bar{p}_2(t) = \dfrac{2h_0}{\alpha - \sqrt{2h_0+\alpha^2}\cosh(\sqrt{2h_0}\,t)} \\ \bar{p}_1(t) - \bar{p}_2(t) = \dfrac{c_0}{2h_0}(\alpha - \sqrt{2h_0+\alpha^2}\cosh(\sqrt{2h_0}\,t)) \\ \bar{p}_3(t) = -\dfrac{\sqrt{2h_0}\sqrt{2h_0+\alpha^2}(\alpha + \sqrt{2h_0+\alpha^2})\tanh\left(\sqrt{\frac{h_0}{2}}\,t\right)}{h_0 + \left(h_0 + \alpha^2 + \alpha\sqrt{2h_0+\alpha^2}\right)\tanh^2\left(\sqrt{\frac{h_0}{2}}\,t\right)}. \end{cases}$$

(iii) If $c_0 = 0$ and $p_1(0) + p_2(0) = 0$ (with $p_1(0)$ and $p_2(0)$ not both zero), then there exists $t_0 \in \mathbb{R}$ and $\sigma, \varsigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is given by

$$\begin{cases} \bar{p}_1(t) + \bar{p}_2(t) = 0 \\ \bar{p}_1(t) - \bar{p}_2(t) = 2\varsigma e^{-\sigma\sqrt{2h_0}\,t} \\ \bar{p}_3(t) = \sigma\sqrt{2h_0}. \end{cases}$$

Proof We briefly describe how the expressions for $\bar{p}(\cdot)$ were found in (i). (A similar approach may be used for (ii), whereas the integration for case (iii) is straightforward.) Let $\omega = \sqrt{2h_0 + \alpha^2}$ and parametrize the cylinder $(H^{(5)}_{3,\alpha})^{-1}(h_0)$ by θ and z as follows:

$$\begin{cases} \bar{p}_1 + \bar{p}_2 = \omega\cos\theta - \alpha \\ \bar{p}_1 - \bar{p}_2 = z \\ \bar{p}_3 = \omega\sin\theta. \end{cases}$$



(a) Case I-a (b) Case I-b (c) Case I-c

(d) Case II-a (e) Case II-b (f) Case II-c

Fig. 8 Typical configurations of $H^{(5)}_{3,\alpha}$

From $\dot{\bar{p}}_3 = -(\bar{p}_1 + \bar{p}_2)(\bar{p}_1 + \bar{p}_2 + \alpha)$, we get $\dot{\theta} = \alpha - \omega \cos\theta$. Hence

$$\theta(t) = -2\tan^{-1}\left[\frac{(\omega-\alpha)}{\sqrt{2h_0}}\tanh\left(\sqrt{\tfrac{h_0}{2}}\,t\right)\right].$$

(For convenience, we have omitted any translation in the independent variable.) Using the identity $\bar{p}_1(t)^2 - \bar{p}_2(t)^2 = c_0$ and solving for $\bar{p}_1(t)$ and $\bar{p}_2(t)$ yields the given expressions.

It remains to be shown that every integral curve takes the form $t \mapsto \bar{p}(t+t_0)$.

(*i*) We have $p_1(0)+p_2(0) > 0$. (If $p_1(0)+p_2(0) < 0$, then $p_1(0) < 0$, a contradiction.) Thus $\omega^2 = (p_1(t)+p_2(t)+\alpha)^2+p_3(t)^2$ implies that $-\omega < p_3(t) < \omega$. Similarly, $-\omega < \bar{p}_3(t) < \omega$. Furthermore, $\lim_{t\to-\infty}\bar{p}_3(t) = \omega$ and $\lim_{t\to\infty}\bar{p}_3(t) = -\omega$. Since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$(\bar{p}_1(t_0) + \bar{p}_2(t_0) + \alpha)^2 = \omega^2 - \bar{p}_3(t_0)^2 = \omega^2 - p_3(0)^2 = (p_1(0) + p_2(0) + \alpha)^2$$

and so $\bar{p}_1(t_0) + \bar{p}_2(t_0) + \alpha = \pm(p_1(0) + p_2(0) + \alpha)$. But $p_1(0) + p_2(0) + \alpha > 0$ and $\bar{p}_1(t_0) + \bar{p}_2(t_0) + \alpha > 0$, and so $\bar{p}_1(t_0) + \bar{p}_2(t_0) = p_1(0) + p_2(0)$. Thus, from

$$(\bar{p}_1(t_0) - \bar{p}_2(t_0))(\bar{p}_1(t_0) + \bar{p}_2(t_0)) = c_0 = (p_1(0) - p_2(0))(p_1(0) + p_2(0))$$

we get $\bar{p}_1(t_0) = p_1(0)$ and $\bar{p}_2(t_0) = p_2(0)$. Therefore, as $t \mapsto \bar{p}(t+t_0)$ and $t \mapsto p(t)$ are integral curves of $\vec{H}^{(5)}_{3,\alpha}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(*ii*) We have $-\alpha-\omega \leq p_1(t)+p_2(t) < 0$. (From $\omega^2 = (p_1(t)+p_2(t)+\alpha)^2+p_3(t)^2$ we get $p_1(t) + p_2(t) \geq -\alpha - \omega^2$. Also, if $p_1(t) + p_2(t) > 0$, then $p_1(t) > 0$, a contradiction.) Furthermore, $\bar{p}_1(0) + \bar{p}_2(0) = -\alpha - \omega^2$. Since $t \mapsto \bar{p}_1(t) + \bar{p}_2(t)$ is continuous, there exists $t_1 \in \mathbb{R}$ such that $\bar{p}_1(t_1) + \bar{p}_2(t_1) = p_1(0) + p_2(0)$. Then

$$\bar{p}_1(t_1)^2 - \bar{p}_2(t_1)^2 = c_0 = p_1(0)^2 - p_2(0)^2$$

implies that $\bar{p}_1(t_1) = p_1(0)$ and $\bar{p}_2(t_1) = p_2(0)$. Similarly,

$$\bar{p}_3(t_1)^2 = \omega^2 - (\bar{p}_1(t_1) + \bar{p}_2(t_1) + \alpha)^2 = \omega^2 - (p_1(0) + p_2(0) + \alpha)^2 = p_3(0)^2$$

and so $\bar{p}_3(t_1) = \pm p_3(0)$. Since $\bar{p}_1(\cdot)$, $\bar{p}_2(\cdot)$ are even and $\bar{p}_3(\cdot)$ is odd, we have $\bar{p}_1(-t_1) = \bar{p}_1(t_1)$, $\bar{p}_2(-t_1) = \bar{p}_2(t_1)$ and $\bar{p}_3(-t_1) = -\bar{p}_3(t_1)$. Hence there exists $t_0 \in \mathbb{R}$ (either $t_0 = -t_1$ or $t_0 = t_1$) such that $\bar{p}(t_0) = p(0)$. Therefore $t \mapsto \bar{p}(t+t_0)$ and $t \mapsto p(t)$ both solve the same Cauchy problem, and hence are identical.

(*iii*) Let $\sigma = \operatorname{sgn}(p_3(0))$ and $\varsigma = \operatorname{sgn}(p_1(0))$. (If $p_1(0) = 0$ or $p_3(0) = 0$, then $p(0)$ is an equilibrium point.) We have

$$\lim_{t\to-\infty}\bar{p}_1(t) = \begin{cases}\varsigma\infty & \text{if } \sigma = +1\\ 0 & \text{if } \sigma = -1\end{cases} \quad \text{and} \quad \lim_{t\to\infty}\bar{p}_1(t) = \begin{cases}0 & \text{if } \sigma = +1\\ \varsigma\infty & \text{if } \sigma = -1.\end{cases}$$

Hence, as $\operatorname{sgn}(\bar{p}_1(t)) = \operatorname{sgn}(p_1(t))$ for every t and $\bar{p}_1(\cdot)$ is continuous, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_1(t_0) = p_1(0)$. Then $\bar{p}_2(t_0) = -\bar{p}_1(t_0) = -p_1(0) = p_2(0)$ and $\bar{p}_3(t_0) = \sigma\sqrt{2h_0} = p_3(0)$. Therefore $t \mapsto p(t)$ and $t \mapsto \bar{p}(t+t_0)$ both solve the same Cauchy problem, and hence are identical. □

Theorem 4.2 (Case I**-**b**, case** II**-**b**)** *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}^{(5)}_{3,\alpha}$ *such that* $C(p(0)) = c_0 \geq 0$ *and* $H^{(5)}_{3,\alpha}(p(0)) = 0$*. There exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$*, where* $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ *is given by*

$$\begin{cases} \bar{p}_1(t) + \bar{p}_2(t) = -\dfrac{2\alpha}{\alpha^2 t^2 + 1} \\ \bar{p}_1(t) - \bar{p}_2(t) = -\dfrac{c_0}{2\alpha}(\alpha^2 t^2 + 1) \\ \bar{p}_3(t) = -\dfrac{2\alpha^2 t^2}{\alpha^2 t^2 + 1}. \end{cases}$$

Theorem 4.3 (Case I**-**c**, case** II**-**c**)** *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}^{(5)}_{3,\alpha}$ *such that* $C(p(0)) = c_0 \geq 0$*,* $H^{(5)}_{3,\alpha}(p(0)) = h_0 < 0$ *and* $2h_0 + \alpha^2 > 0$*. There exists* $t_0 \in \mathbb{R}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$*, where* $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ *is given by*

$$\begin{cases} \bar{p}_1(t) + \bar{p}_2(t) = \dfrac{2h_0}{\alpha - \sqrt{2h_0 + \alpha^2}\cos(\sqrt{-2h_0}\, t)} \\ \bar{p}_1(t) - \bar{p}_2(t) = \dfrac{c_0}{2h_0}(\alpha - \sqrt{2h_0 + \alpha^2}\cos(\sqrt{-2h_0}\, t)) \\ \bar{p}_3(t) = \dfrac{\sqrt{-2h_0}\sqrt{2h_0 + \alpha^2}(\alpha + \sqrt{2h_0 + \alpha^2})\tan\left(\sqrt{-\frac{h_0}{2}}\, t\right)}{h_0 - \left(h_0 + \alpha^2 + \alpha\sqrt{2h_0 + \alpha^2}\right)\tan^2\left(\sqrt{-\frac{h_0}{2}}\, t\right)}. \end{cases}$$

6 Nonplanar systems, type II

For the nonplanar, type II systems we consider only stability. As before, we graph the critical energy states for each system, the level sets $H^{-1}(h_0)$ and $C^{-1}(c_0)$ and their intersection (for typical configurations), as well as the equilibrium points.

Throughout this section, we again parametrize equilibria by $\mu, \nu \in \mathbb{R}$, $\nu \neq 0$.

6.1 The system $H^{(5)}_{1,\alpha}$

The equations of motion for $H^{(5)}_{1,\alpha}(p) = \alpha p_1 + \frac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right]$, $\alpha > 0$ are

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -\alpha p_2 - (p_1 + p_2)^2. \end{cases}$$

The equilibrium states are $\mathsf{e}_1^\mu = (\frac{1}{\alpha}\mu(\mu + \alpha), -\frac{1}{\alpha}\mu^2, 0)$ and $\mathsf{e}_2^\nu = (0, 0, \nu)$. In Fig. 9 we graph the critical energy states of this system. The typical configurations are graphed in Fig. 10 and Fig. 11.

Proposition 10 *The equilibrium states have the following behaviour*:

(i) *The states* e_1^μ, $\mu \in (-\infty, -\frac{\alpha}{3})$ *are stable.*
(ii) *The state* e_1^μ, $\mu = -\frac{\alpha}{3}$ *is unstable.*
(iii) *The states* e_1^μ, $\mu \in (-\frac{\alpha}{3}, 0)$ *are (spectrally) unstable.*

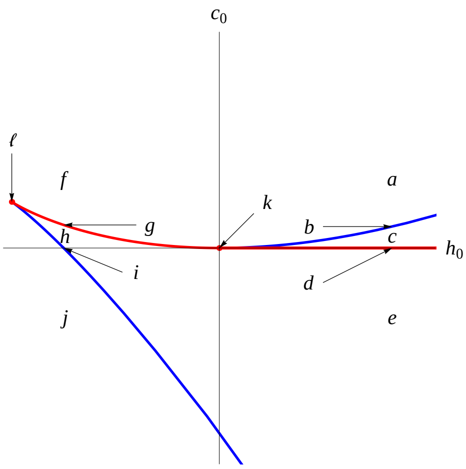


Fig. 9 Critical energy states for $H^{(5)}_{1,\alpha}$

(*iv*) *The state* e^{μ}_1, $\mu = 0$ *is unstable.*
(*v*) *The states* e^{μ}_1, $\mu \in (0, \infty)$ *are stable.*
(*vi*) *The states* e^{ν}_2 *are (spectrally) unstable.*

Proof (*i*) Let $H_\lambda = \lambda_0 H^{(5)}_{1,\alpha} + \lambda_1 C$, where $\lambda_0 = 1$ and $\lambda_1 = -\frac{\alpha}{2\mu}$. Then $\mathbf{d}H_\lambda(\mathsf{e}^{\mu}_1) = 0$ and $\mathbf{d}^2 H_\lambda(\mathsf{e}^{\mu}_1)\big|_{W\times W}$ is positive definite, where $W = \ker \mathbf{d}H^{(5)}_{1,\alpha}(\mathsf{e}^{\mu}_1) \cap \ker \mathbf{d}C(\mathsf{e}^{\mu}_1) = \operatorname{span}\left\{E^*_1 - \frac{\mu+\alpha}{\mu}E^*_2, E^*_3\right\}$. Hence the states e^{μ}_1, $\mu \in (-\infty, -\frac{\alpha}{3})$ are stable.

(*ii*) The integral curve

$$\begin{cases} p_1(t) = \alpha\left(\dfrac{6}{45 + \alpha t(\alpha t - 6)} - \dfrac{2}{(\alpha t - 3)^2} - \dfrac{2}{9}\right) \\ p_2(t) = \alpha\left(\dfrac{6}{45 + \alpha t(\alpha t - 6)} - \dfrac{2}{(\alpha t - 3)^2} - \dfrac{1}{9}\right) \\ p_3(t) = -\dfrac{72\alpha}{(\alpha t - 3)(\alpha t(\alpha t - 6) + 45)} \end{cases}$$

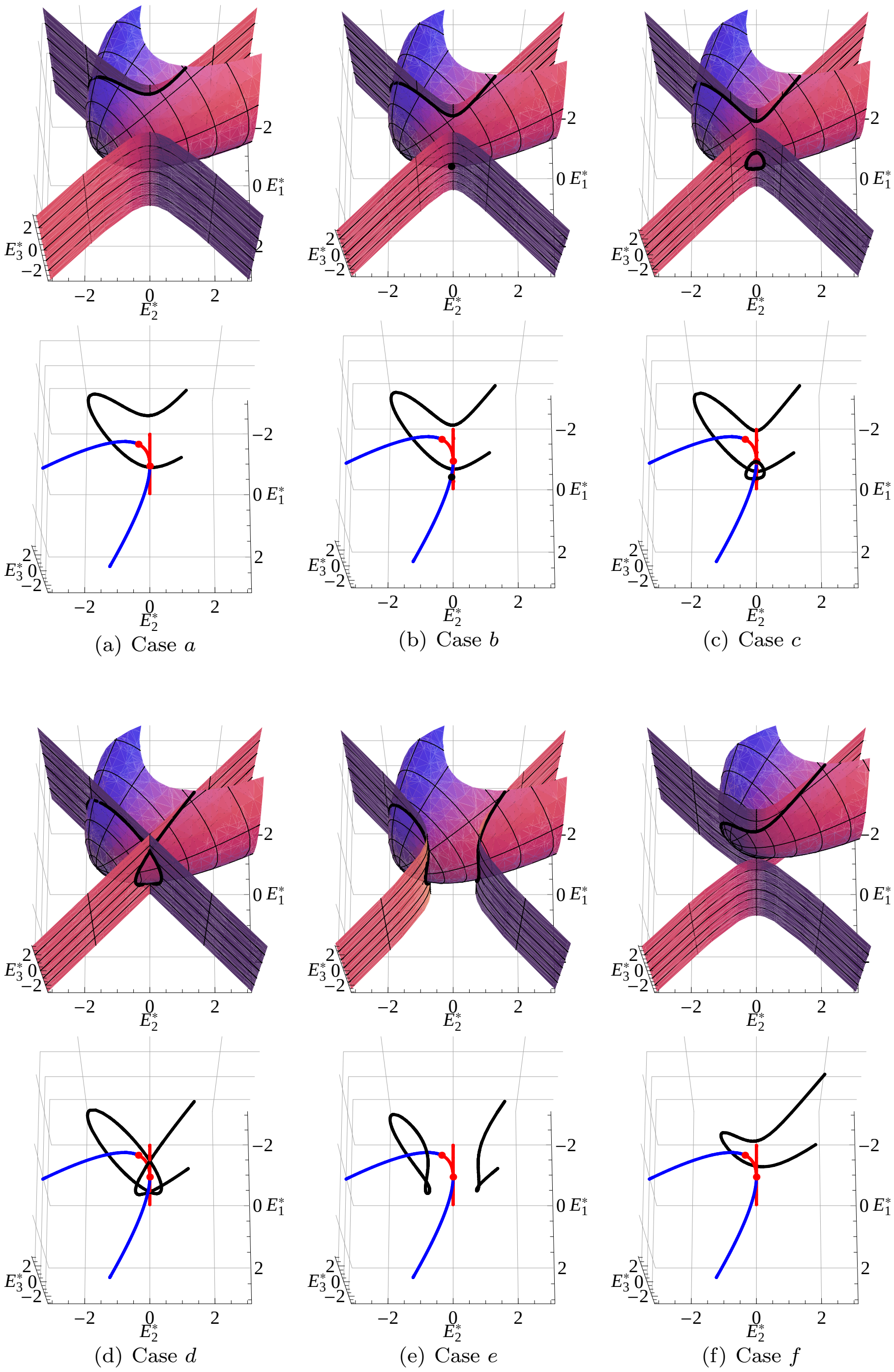
satisfies $\lim_{t\to-\infty} \|p(t) - \mathsf{e}^{\mu}_1\| = 0$. Let $\varepsilon = \frac{1}{2}\|p(0) - \mathsf{e}^{\mu}_1\| > 0$. Then for every neighbourhood N of e^{μ}_1 contained in the ε-ball B_ε about e^{μ}_1, there exists $t_1 < 0$ such that $p(t_1) \in N$. However, $p(0) \notin B_\varepsilon$, and so the state e^{μ}_1, $\mu = -\frac{\alpha}{3}$ is unstable.

(*iii*) The linearization of the system at has eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3} = \pm\sqrt{-\mu(\alpha + 3\mu)}$. There exists an eigenvalue with positive real part exactly when $\mu \in (-\frac{\alpha}{3}, 0)$. Hence the states e^{μ}_1, $\mu \in (-\frac{\alpha}{3}, 0)$ are spectrally unstable.

(*iv*) We have that $p(t) = \left(-\frac{2}{\alpha t^2}, \frac{2}{\alpha t^2}, \frac{2}{t}\right)$ is an integral curve of $\vec{H}^{(5)}_{1,\alpha}$ such that $\lim_{t\to-\infty} \|p(t) - \mathsf{e}^{\mu}_1\| = 0$. Accordingly, for every neighbourhood N of e^{μ}_1 there exists $t_1 < 0$ such that $p(t_1) \in N$. Furthermore, $\lim_{t\to 0} \|p(t)\| = \infty$. Thus the state e^{μ}_1, $\mu = 0$ is unstable.

(*v*) The function H_λ of (*i*) suffices to show that the states e^{μ}_1, $\mu \in (0, \infty)$ are stable.

(*vi*) As $\frac{\partial H^{(5)}_{1,\alpha}}{\partial p_3}(\mathsf{e}^{\nu}_2) = \nu$, it follows from Lemma 2 that the states e^{ν}_2 are spectrally unstable. □



(a) Case a (b) Case b (c) Case c

(d) Case d (e) Case e (f) Case f

Fig. 10 Typical configurations of $H_{1,\alpha}^{(5)}$ (cases a through f)

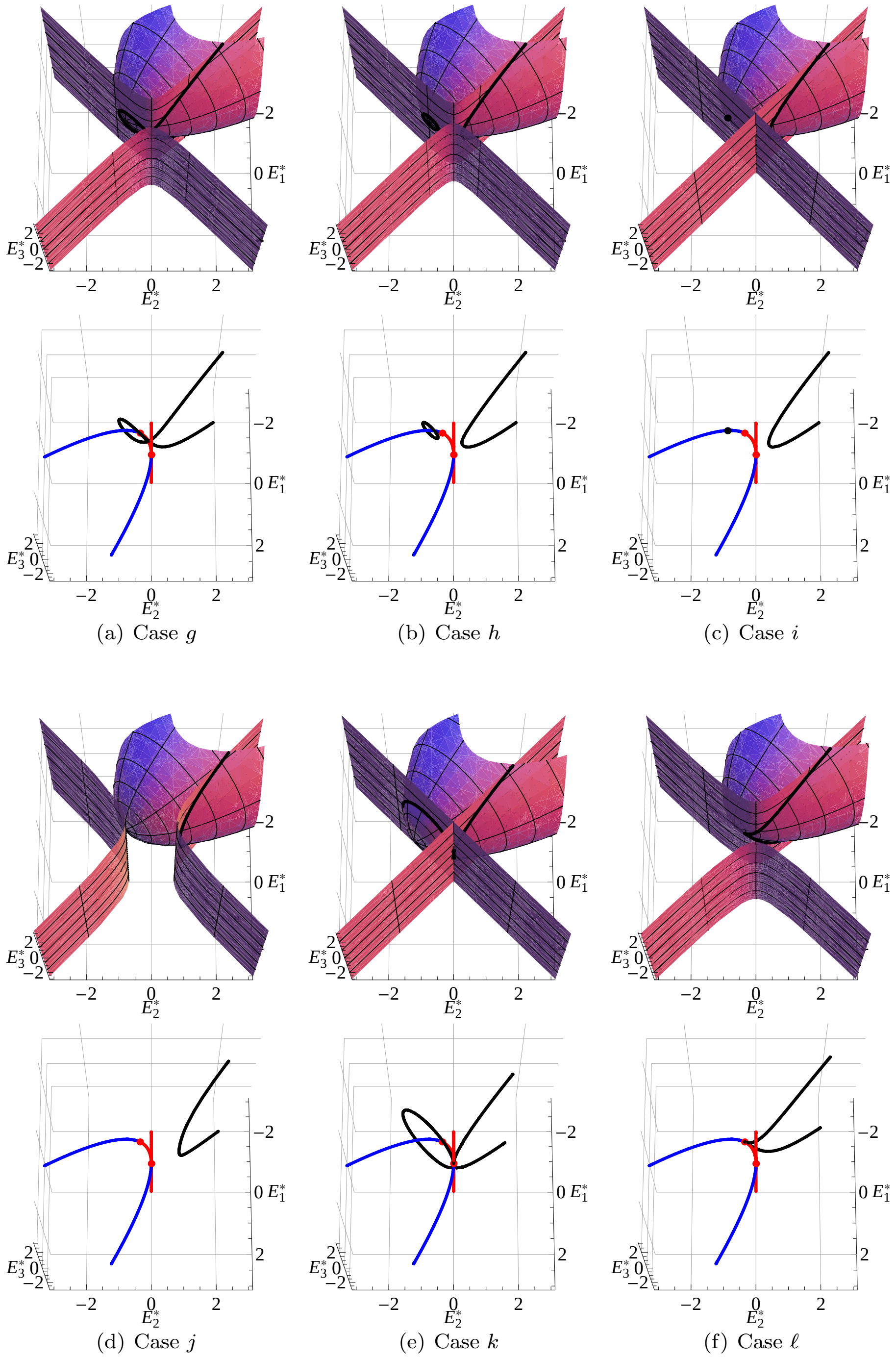


Fig. 11 Typical configurations of $H^{(5)}_{1,\alpha}$ (cases g through ℓ)

6.2 The system $H_2^{(5)}$

The equations of motion for $H_2^{(5)}(p) = p_1 - p_2 + \frac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right]$ are

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_1(p_1 + p_2 - 1) - p_2(p_1 + p_2 + 1). \end{cases}$$

The equilibrium states are $\mathsf{e}_1^\mu = (\frac{1}{2}(\mu+\mu^2), \frac{1}{2}(\mu-\mu^2), 0)$ and $\mathsf{e}_2^\nu = (0, 0, \nu)$. The critical energy states for this system are graphed in Fig. 12; the typical configurations are graphed in Fig. 13 (by symmetry, we may assume $c_0 \geq 0$).

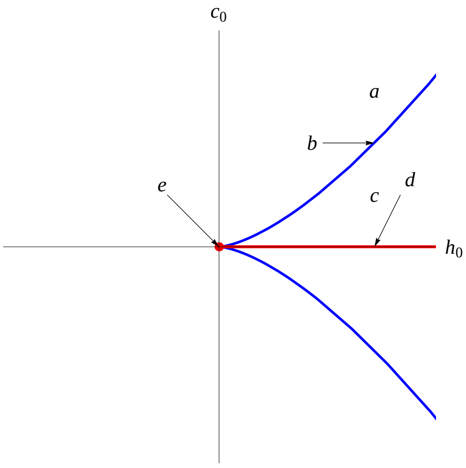


Fig. 12 Critical energy states for $H_2^{(5)}$

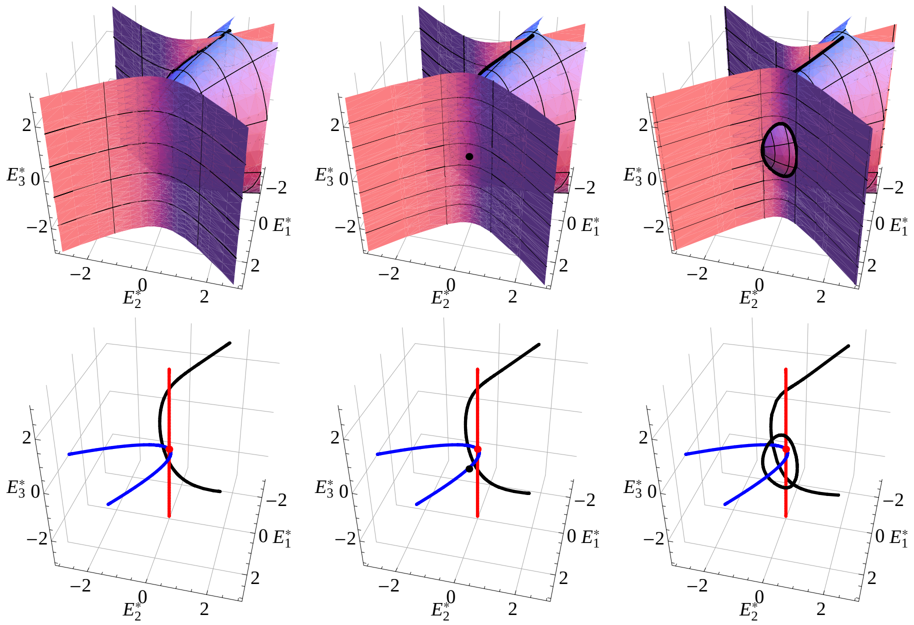
Proposition 11 *The equilibrium states have the following behaviour*:

(*i*) *The states* e_1^μ, $\mu \neq 0$ *are stable.*
(*ii*) *The state* e_1^μ, $\mu = 0$ *is unstable.*
(*iii*) *The states* e_2^ν *are* (*spectrally*) *unstable.*

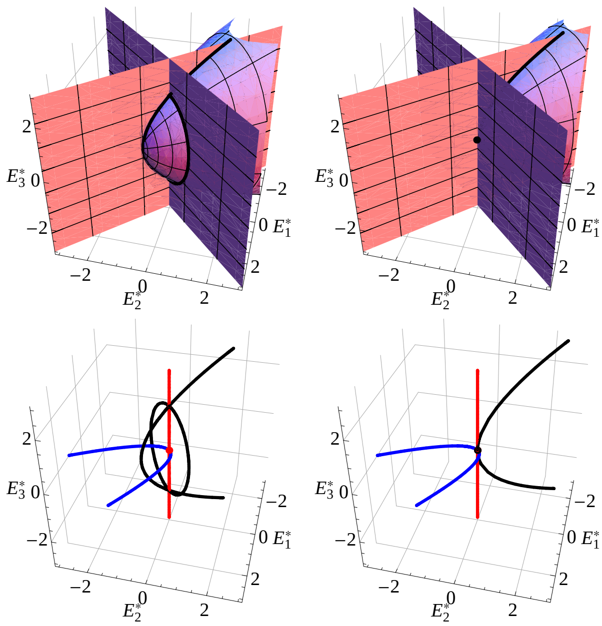
Proof (*i*) Suppose $\mu \neq 1$. Let $H_\lambda = \lambda_0 H_2^{(5)} + \lambda_1 C$, where $\lambda_0 = -\mu$ and $\lambda_1 = 1$. We have $\mathbf{d}H_\lambda(\mathsf{e}_1^\mu) = 0$ and $\mathbf{d}^2 H_\lambda(\mathsf{e}_1^\mu)\big|_{W\times W}$ positive definite, where $W = \ker \mathbf{d}H_2^{(5)}(\mathsf{e}_1^\mu) \cap \ker \mathbf{d}C(\mathsf{e}_1^\mu) = \operatorname{span}\left\{E_1^* + \frac{\mu+1}{\mu-1}E_2^*, E_3^*\right\}$. Therefore the states e_1^μ, $\mu \notin \{0, 1\}$ are stable. Suppose $\mu = 1$. We have $H_2^{(5)}(\mathsf{e}_1^1) = \frac{3}{2}$ and $C(\mathsf{e}_1^1) = 1$. It is straightforward to show that locally about e_1^1 we have $(H_2^{(5)})^{-1}(\frac{3}{2}) \cap C^{-1}(1) = \{\mathsf{e}_1^1\}$. Hence, by the continuous energy-Casimir method, the state e_1^1 is stable.

(*ii*) We have that $p(t) = (-\frac{1}{t^2}, \frac{1}{t^2}, \frac{2}{t})$ is an integral curve of $\vec{H}_2^{(5)}$ such that $\lim_{t\to-\infty} \|p(t)\| = 0$. Accordingly, for every neighbourhood V of e_1^μ there exists $t_1 < 0$ such that $p(t_1) \in V$. Furthermore, $\lim_{t\to 0} \|p(t)\| = \infty$. Thus the state e_1^μ, $\mu = 0$ is unstable.

(*iii*) As $\frac{\partial H_2^{(5)}}{p_3}(\mathsf{e}_2^\nu) = \nu$, it follows from Lemma 2 that the states e_2^ν are spectrally unstable. □



(a) Case a (b) Case b (c) Case c



(d) Case d (e) Case e

Fig. 13 Typical configurations of $H_2^{(5)}$

7 Concluding remark

In Section 3 we partitioned the quadratic (inhomogeneous) systems into four classes, *viz.*, the ruled, planar, nonplanar type I and nonplanar type II systems. Clearly, if two systems are affinely equivalent, then they must belong to the same class. We identify several additional invariants that facilitate the identification of the normal form of a system.

The dimension of the symmetry group is a simple invariant. (Likewise, for an inhomogeneous system $H_{A,\mathcal{Q}}$, the dimension of $\mathsf{Sym}(H_{\mathcal{Q}})$ is also an invariant.) Regarding the equilibria, we have that the set of equilibria for each system is the union of (a finite number of) lines, curves, planes or surfaces. We define the *equilibrium index* of a system to be a tuple (i, j, k, ℓ), where i is the number of lines; j the number of curves that are not lines; k the number of planes; and ℓ the number of surfaces that are not planes. Clearly, equivalent systems have the same equilibrium index. Moreover, for an inhomogeneous system $H_{A,\mathcal{Q}}$, the equilibrium index of the corresponding homogeneous system $H_{\mathcal{Q}}$ is another invariant.

More invariants may be found by identifying the type of quadratic constants of motion that a system admits. We say that a system has *spherical symmetry* if it admits a constant of motion of the form $K(p) = \mathcal{Q}(p - q)$, where $\mathcal{Q}$ is a positive definite quadratic form. Likewise, we say that a system has one of the following types of symmetry if it admits a constant of motion of the form $K(p) = \mathcal{Q}(p - q)$, where $\mathcal{Q}$ is a quadratic form with the corresponding signature:

- *hyperboloidal symmetry*: signature $(0, 2, 1)$.
- *hyp-cylindrical symmetry*: signature $(1, 1, 1)$.
- *cylindrical symmetry*: signature $(1, 2, 0)$.
- *planar symmetry*: signature $(2, 1, 0)$.

(The signature of $\mathcal{Q}$ is the triple (n_0, n_+, n_-), where n_0 is the number of zero eigenvalues; n_+ the number of positive eigenvalues; and n_- the number of negative eigenvalues.) Clearly, every system on $\mathfrak{se}(1,1)^*_-$ admits the Casimir function as a hyp-cylindrical symmetry.

Equivalent systems must have the same types of symmetry. Accordingly, these invariants may be useful in a more general classification of inhomogeneous systems. For instance, as $H^{(5)}_{3,\alpha}$ does not have a spherical or planar symmetry, it cannot be equivalent to any system on $\mathfrak{so}(3)^*_-$ or the Heisenberg Lie–Poisson space $(\mathfrak{h}_3)^*_-$. (However, as it has cylindrical and hyperboloidal symmetries, we cannot rule out the possibility of it being equivalent to a system on $\mathfrak{se}(2)^*_-$ or the pseudo-orthogonal Lie–Poisson space $\mathfrak{so}(2,1)^*_-$.)

In most cases, these above-mentioned invariants are sufficient to uniquely determine the equivalence class of a system. In Table 6 we list the partition of the inhomogeneous normal forms according to the invariants discussed above. (For the sake of completeness, the homogeneous normal forms are likewise partitioned in Table 5.)

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Table 5 Taxonomy of homogeneous systems

Type	dim(Sym(H_Q))	Eq. index of H_Q	Spherical	Hyperboloidal	Cylindrical	Planar	Systems
Ruled	5	$(0,2,0,0)$			•	•	H_1
	7	$(0,1,0,0)$			•	•	H_2
Planar	2	$(1,1,0,0)$		•		•	H_3
Nonplanar, type I	0	$(3,0,0,0)$	•	•	•		H_4
	1	$(2,0,0,0)$		•	•		H_5

Table 6 Taxonomy of inhomogeneous systems

Type	dim(Sym($H_{A,Q}$))	dim(Sym(H_Q))	Eq. index of $H_{A,Q}$	Eq. index of H_Q	Spherical	Hyperboloidal	Cylindrical	Planar	Systems
Ruled	5	5	$(0,0,0,1)$	$(0,2,0,0)$			•	•	$H_2^{(1)}$
	5	7	$(0,0,0,1)$	$(0,1,0,0)$			•	•	$H_1^{(2)}$
	5	5	$(0,2,0,0)$	$(0,2,0,0)$			•	•	$H_1^{(1)}$
	6	7	$(0,2,0,0)$	$(0,1,0,0)$			•	•	$H_2^{(2)}$
	7	12	$(0,1,0,0)$	-			•	•	$H_1^{(0)}$
Planar	4	12	$(1,0,0,0)$	-		•		•	$H_{2,\alpha}^{(0)}$
Nonplanar, type I	0	2	$(2,0,0,0)$	$(1,1,0,0)$		•			$H_1^{(3)}$
	0	0	$(3,0,0,0)$	$(3,0,0,0)$	•	•	•		$H_{1,\alpha}^{(4)}$
	1	2	$(2,0,0,0)$	$(1,1,0,0)$		•			$H_2^{(3)}$
	1	1	$(3,0,0,0)$	$(2,0,0,0)$		•	•		$H_{3,\alpha}^{(5)}$
	2	5	$(1,0,0,0)$	$(0,2,0,0)$					$H_{3,\alpha}^{(1)}$
	3	7	$(1,0,0,0)$	$(0,1,0,0)$					$H_{3,\delta}^{(2)}$
Nonplanar, type II	0	1	$(1,0,1,0)$	$(2,0,0,0)$		•	•		$H_{1,\alpha}^{(5)}$, $H_2^{(5)}$
	0	0	$(1,0,2,0)$	$(3,0,0,0)$	•	•	•		$H_{2,\alpha_1,\alpha_2}^{(4)}$