Galerkin finite element approximation of general linear second order hyperbolic equations

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Abstract

In this article we derive error estimates for the Galerkin approximation of a general linear second order hyperbolic partial differential equation. The results can be applied to a variety of cases e.g. vibrating systems of linked elastic bodies. The results generalize the work of Baker [1] and also allow for viscous type damping. Splitting the proofs for the semidiscrete and fully discrete cases not only simplifies the proofs but less restrictive regularity assumptions are required.

1 Introduction

In this paper we consider the Galerkin finite element approximation of a solution of a general second order hyperbolic equation. This includes the general second order wave equation but also vibrating beams, plates etc., since the variational forms are the same as for the wave equation. Even systems of linked elastic bodies can be accommodated, see Section 7. There is a restriction to the theory however and this relates to damping. Damping at the boundary or interfaces is for example not allowed.

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The general second order hyperbolic partial differential equation is formulated in Section 2 in variational form and is referred to as Problem G. Conditions for the existence of a unique solution are given.

Convergence of the Galerkin approximation for the wave equation was considered in the book [2] and article [3] both published in 1973. In [2] a sketch of a proof of convergence for the undamped case is presented. In [3] error estimates are derived for the undamped case and a problem with boundary damping. These error estimates are for the semi discrete approximation and the fully discrete approximation.

Dupont [3] made the assumption that the exact solution has a second order time derivative with respect to the energy norm, which is rather restrictive. In 1976 Baker derived error estimates with "minimal smoothness requirements on the solution", but no damping [1]. This article had a significant impact on the literature and other authors adapted his methods or were influenced by it, see e.g. [4], [5], [6], [7] and [8].

We generalize the results of Baker [1]. The spaces $\mathcal{L}^2(\Omega)$ and $H_0^1(\Omega)$ are replaced by general Hilbert spaces W and V, where V is a subset of W. In the general problem the $\mathcal{L}^2$-norm is replaced by the general inertia norm $\|\cdot\|_W$ and the well known energy norm for the wave equation (equivalent to the H^1 norm) is replaced by a general energy norm $\|\cdot\|_V$. Naturally certain properties of the spaces and their norms need to be assumed in a general setting (see Section 2). An attempt to obtain error estimates for a general hyperbolic problem was made in [9] and [10]. The assumptions made were also rather restrictive. The differentiability assumption is the same as in [3] and modal damping is assumed.

As mentioned, Problem G in Section 2 is the abstract variational problem in Hilbert space. (The problem in [1] is a special case.) So called viscous type damping is included. It is natural to ask whether this restriction is necessary. We believe that boundary or interface damping requires a different approach based on the fact that existence results are influenced by the type of damping.

The existence result in Section 2 is presented in variational form. This is more convenient for the finite element method as one can compare conditions for existence to conditions for convergence. We see that sufficient conditions for existence are also used to prove convergence but they seem not to be sufficient and more regularity of solutions is assumed.

This article is not merely about generalization but other contributions to

the theory are also made. Consider for example the significance of the semi-discrete approximation. In [3], [1] and other publications (e.g. [5], [6] and [7]) error estimates are derived for the semi discrete approximation and then for the fully discrete approximation without using the results already obtained. In some cases the semi-discrete approximation is ignored. We derive an error estimate for the semi-discrete approximation with respect to the inertia norm and then an estimate for the error in approximating the solution of the semi-discrete problem by the fully discrete approximation with respect to the same norm. The final estimate follows trivially by the triangle inequality. This approach has two advantages. It is not necessary to assume the existence of a third or fourth order derivative for the exact solution and the convergence analysis for the fully discrete approximation is simplified.

In [1] error estimates are derived "using L^2-projections of the initial data as starting values". The fact that the starting values for the approximation are a projection, is used to derive estimates (see Section 4). But it is not always possible to determine the projections exactly. It is not clear in such an event how the error estimates can be interpreted. In our approach projections are not used as initial approximations in the fundamental estimate. The errors for the initial approximations are dealt with at a later stage (see Section 5).

It is well known that the smoothness of a solution is important in convergence analysis. In this paper we attempt to pinpoint exactly where the regularity in space and time is used in the analysis. Regarding time, see Sections 4 and 6 and for spatial regularity, see Section 5. An issue that deserves attention here is the differentiability of the projection of the exact solution. It is possible that the exact solution does not have a second order derivative with respect to the energy norm but its projection may have a second order derivative (see Section 3).

To summarize, error estimates for the semi-discrete approximation are presented in Section 5 and those for the fully discrete appear in Section 6. Finally, applications are discussed in Section 7.

Remark To compare results in different articles is time consuming due to differences in notation. We have used the notation from [2].

2 Existence and uniqueness of solutions

In this section we consider the general second order hyperbolic equation or general linear vibration problem in variational form. Let X, W and V denote Hilbert spaces such that $V \subset W \subset X$.

Notation

X has inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$.

W has inner product c and norm $\|\cdot\|_W$.

V has inner product b and norm $\|\cdot\|_V$.

Let J be a bounded or unbounded interval, either an open interval containing zero or of the form $[0,\tau)$ or $[0,\infty)$. Let Y be any Hilbert space and consider a function u on J with values in Y.

We write $u'(t) \in Y$ if the derivative exists with respect to the norm of Y;

$u^{(k)} \in \mathcal{L}^2\big(J;Y\big)$ if $u^{(k)}(t) \in Y$ for each t and $\int_J \|u^{(k)}\|_Y^2 < \infty$;

$u \in C^k\big(J;Y\big)$ if $u^{(k)} \in C\big(J;Y\big)$.

Consider a bilinear form a defined on W and the following general problem.

Problem G
Given a function $f : J \to X$, determine a function $u \in C(J;V)$ such that u' is continuous at 0 and for each $t \in J$

$$u(t) \in V, \quad u'(t) \in V, \quad u''(t) \in W,$$

$$c\big(u''(t),v\big) + a\big(u'(t),v\big) + b\big(u(t),v\big) = \big(f(t),v\big)_X \quad \text{for each } v \in V, \tag{2.1}$$

$$\text{while} \quad u(0) = u_0\,, \quad u'(0) = u_1\,.$$

Assumptions The following assumptions are used for existence theory as well as convergence theory.

E1 V is dense in W,

E2 There exists a constant C_b such that $\|v\|_W \leq C_b\|v\|_V$ for each $v \in V$,

E3 There exists a constant C_c such that $\|v\|_X \leq C_c\|v\|_W$ for each $v \in W$,

E4 The bilinear form a is nonnegative, symmetric and bounded on W, i.e.

$$|a(u,v)| \leq K_a\|u\|_W\|v\|_W.$$

If the bilinear form a satisfies Assumption E4 we refer to the damping as viscous type since it is valid for viscous damping in the wave equation and other vibration problems. (In [11] it is referred to as weak damping.) The assumption is not valid for boundary damping; it does not hold for the model in [3].

Remark In the rest of this paper, except Section 7, b and c are inner products for the spaces V and W respectively and a is a bilinear form satisfying assumption A4.

For the wave equation $\|\cdot\|_X$ is just the $\mathcal{L}^2$-norm and in many applications the norms $\|\cdot\|_X$ and $\|\cdot\|_W$ are equivalent.

Definition

$E_b = \{\, x \in V \,\big|\, \text{there exists a } y \in W \text{ such that } c(y,v) = b(x,v) \text{ for all } v \in V \,\}.$

Theorem 2.1. *Suppose Assumptions E1, E2, E3 and E4 hold, then there exists a unique solution*

$$u \in C^1\big(J;V\big) \cap C^2\big(J;W\big),$$

for Problem G for each $u_0 \in E_b$, $u_1 \in V$ and each $f \in C^1\big(J;X\big)$. If $f = 0$ then $u \in C^1\big((-\infty,\infty);V\big) \cap C^2\big((-\infty,\infty);W\big)$.

Proof. See [11]. □

Problem G is equivalent to a first order differential equation in the product space $V \times W$: $y' = Ay$ with $y(0) = \langle u_0, u_1\rangle$. The operator A is determined by the three bilinear forms c, b and a and $\mathcal{D}(A) = E_b \times V$. The construction of A is given in [11]. Due to Assumptions E1, E2 and E4, A is the infinitesimal generator of a C_0 group of contractions.

Remarks

1. The definition of E_b implies that E_b is the set of equilibrium states.

2. Existence results may be found in other publications e.g. [12, 1972] and [13, 1977] but the result in [11, 2002] is convenient for this paper.

3 Galerkin approximation

In this section we consider the Galerkin approximation. Assumptions E1, E2, E3 and E4 are assumed to hold for the general convergence theory. Suppose S^h is a finite dimensional subspace of V. (The symbol h has no meaning at this stage, but it is customary to use the notation.)

Problem G^h
Given a function $f : J \to X$, find a function $u_h \in C^2(J)$ such that for each $t \in J$

$$c\big(u_h''(t), v\big) + a\big(u_h'(t), v\big) + b\big(u_h(t), v\big) = \big(f(t), v\big)_X \quad \text{for each } v \in S^h, \quad (3.1)$$

$$\text{while} \quad u_h(0) = u_0^h\,, \quad u_h'(0) = u_1^h\,.$$

The initial values u_0^h and u_1^h are elements of S^h as close as possible to u_0 and u_1, see Section 5.

Since c is an inner product and S^h is finite dimensional, existence is no problem if f is continuous (see Section 6).

Theorem 3.1. *If $f \in C(J, X)$, then there exists a unique solution $u_h \in C^2(J)$ for Problem G^h for each u_0^h and u_1^h in S^h. If $f = 0$ then $u_h \in C^2\big((-\infty, \infty)\big)$.*

To find an estimate for the discretization error $e_h(t) = u(t) - u_h(t)$, a projection is used. The projection operator P_h is defined by

$$b(u - P_h u, v) = 0 \ \text{ for each } v \in S^h.$$

If no confusion is possible we will write P for P_h. We also use the symbol P to denote the projection Pu of a function u, i.e. $(Pu)(t) = Pu(t)$ for each $t \in J$. Using the projection we split the error. Let $e(t) = Pu(t) - u_h(t)$ and $e_p(t) = u(t) - Pu(t)$. Then

$$e_h(t) = e_p(t) + e(t). \quad (3.2)$$

Estimates for the norm of $e_p(t)$ are obtained from interpolation theory in applications (see Sections 5 and 7). The problem is to determine an estimate for $e(t)$, the difference between the projection $Pu(t)$ and the Galerkin approximation $u_h(t)$. Our aim is to estimate this error in terms of the projection error and errors for the initial conditions.

The convergence results that can be obtained, depend on the properties of the solution. But these may vary from problem to problem. The weakest assumption is the result of the existence theorem: if u is a solution of Problem G, then $u \in C^1(J;V) \cap C^2(J;W)$.

Lemma 3.1. *If $u \in C^1(J,V)$, then $Pu \in C^1(J)$ and $(Pu)'(t) = Pu'(t)$.*

Proof. As the projection operator P is a bounded linear operator with norm less than one, we have that

$$\|(\delta t)^{-1}\left(Pu(t+\delta t) - Pu(t)\right) - Pu'(t)\|_V \leq \|(\delta t)^{-1}(u(t+\delta t) - u(t)) - u'(t)\|_V.$$

Clearly $Pu \in C^1\left(J,V\right)$ and $(Pu)'(t) = Pu'(t)$. □

We need the following property of the solution for the convergence analysis.

Assumption A1
The solution $u \in C(J,V)$ of Problem G has the property that $(Pu) \in C^2(J)$.

Remark In [3] the assumption $u \in C^2(J,V)$ is used and it is then possible to prove that $Pu \in C^2(J)$. Baker [1] derived error estimates for the undamped case using a weaker assumption, but assumed A1 without explicitly mentioning it. The assumption in [3] is sufficient to prove A1 but not necessary.

It is important to note that we use the fact that $(Pu)' = Pu'$ but not $(Pu)'' = Pu''$.

If the solution u of Problem G satisfies Assumption A1, then $e_p \in C^2(J,W)$. This is easy to see since $e_p = u - Pu$ and since norms in S^h are equivalent, $Pu \in C^2(J,W)$.

Proposition 3.1. *If the solution u of Problem G satisfies Assumption A1, then*

$$c(e_h''(t), v) + a(e_h'(t), v) + b(e(t), v) = 0 \text{ for all } v \in S^h.$$

Proof. By subtracting (3.1) from (2.1) we obtain the following identity. For any $v \in S^h$,

$$c(u''(t) - u_h''(t), v) + a(u'(t) - u_h'(t), v) + b(u(t) - u_h(t), v) = 0.$$

Add and subtract Pu in the bilinear form b above and we have

$$c(e_h''(t), v) + a(e_h'(t), v) + b(u(t) - Pu(t), v) + b(Pu(t) - u_h(t), v) = 0,$$

for all $v \in S^h$. Since P is a projection with respect to the inner product b, the result follows. □

4 Fundamental estimate

The following lemma is a generalization of an estimate for $e = Pu - u_h$ in [1] and we also include damping.

Lemma 4.1. *If the solution u of Problem G satisfies Assumption A1, then for $t \in [0, T]$,*

$$\begin{aligned} \|e(t)\|_W \quad \leq \quad & \sqrt{2}\left(\|e(0)\|_W + 3T\|e_h'(0)\|_W + 3\int_0^T \|e_p'(t)\|_W \right. \\ & \left. +3K_aT\|e_h(0)\|_W + 3K_a\int_0^T \|e_p(t)\|_W \right). \end{aligned}$$

Proof. In Proposition 3.1 we have

$$c(e_h''(t), v) + a(e_h'(t), v) + b(e(t), v) = 0 \text{ for all } v \in S^h.$$

Now, if v is a function on $[0, T]$ with values in S^h, then

$$c(e_h''(t), v(t)) + a(e_h'(t), v(t)) + b(e(t), v(t)) = 0. \tag{4.1}$$

Let $v'(t) = e(t)$, then

$$\begin{aligned} & \frac{d}{dt}\left[\frac{1}{2}\, c(e, e) - \frac{1}{2}\, b(v, v) - c(e_h', v) - a(e_h, v)\right] \\ & \quad = c(e', e) - b(e, v) - c(e_h'', v) - a(e_h', v) - c(e_h', e) - a(e_h, e) \\ & \quad = -c(e_p', e) - a(e_h, e) \end{aligned} \tag{4.2}$$

from (4.1).

Choose $v(\tau) = 0$ for some $\tau \in (0, T)$, and integrate (4.2) between 0 and τ. We obtain

$$-\left[\frac{1}{2}\, c(e(0), e(0)) - \frac{1}{2}\, b(v(0), v(0)) - c(e_h'(0), v(0)) - a(e_h(0), v(0))\right]$$
$$+\frac{1}{2}\, c(e(\tau), e(\tau)) = -\int_0^\tau c(e_p', e) - \int_0^\tau a(e_p, e) - \int_0^\tau a(e, e).$$

Since $a(e, e) \geq 0$ and $b(v(0), v(0)) \geq 0$, we have

$$\begin{aligned} \frac{1}{2}\, c(e(\tau), e(\tau)) \;\; &\leq \;\; \frac{1}{2}\, c(e(0), e(0)) - c(e_h'(0), v(0)) - a(e_h(0), v(0)) \\ &\quad - \int_0^\tau c(e_p', e) - \int_0^\tau a(e_p, e) \end{aligned} \tag{4.3}$$

and consequently

$$\begin{aligned} c(e(\tau), e(\tau)) \;\; &\leq \;\; c(e(0), e(0)) + 2\int_0^\tau c(e_h'(0), e) - 2\int_0^\tau c(e_p', e) \\ &\quad +2\int_0^\tau a(e_h(0), e) - 2\int_0^\tau a(e_p, e). \end{aligned} \tag{4.4}$$

Estimates for the terms on the right are now required. We use Young's inequality (see e.g. [14]) and denote $\max\limits_{t\in[0,T]}$ by max.

$$\begin{aligned} \left|2\int_0^\tau c(e_h'(0), e)\right| \;\; &\leq \;\; 2\int_0^\tau |c(e_h'(0), e)| \\ &\leq \;\; 2\int_0^\tau \|e_h'(0)\|_W \; \|e\|_W \\ &\leq \;\; 2T \max \|e(t)\|_W \|e_h'(0)\|_W \\ &\leq \;\; \frac{1}{9}\big[\max \|e(t)\|_W\big]^2 + 9T^2 \|e_h'(0)\|_W^2. \end{aligned}$$

Similarly

$$\begin{aligned} \left|2\int_0^\tau a(e_h(0), e)\right| \;\; &\leq \;\; 2K_a T \max \|e(t)\|_W \|e_h(0)\|_W \\ &\leq \;\; \frac{1}{9}\big[\max \|e(t)\|_W\big]^2 + 9{K_a}^2 T^2 \|e_h(0)\|_W^2, \end{aligned}$$

also using E4. Next

$$\begin{aligned} \left|2\int_0^\tau c(e_p', e)\right| \;\; &\leq \;\; 2 \max \|e(t)\|_W \int_0^\tau \|e_p'\|_W \\ &\leq \;\; \frac{1}{9}\big[\max \|e(t)\|_W\big]^2 + \left(3\int_0^\tau \|e_p'\|_W\right)^2 \end{aligned}$$

and

$$\begin{aligned}\left|2\int_0^\tau a(e_p,e)\right| &\le 2K_a \max\|e(t)\|_W \int_0^\tau \|e_p\|_W \\ &\le \frac{1}{9}\big[\max\|e(t)\|_W\big]^2 + \left(3K_a\int_0^\tau \|e_p\|_W\right)^2.\end{aligned}$$

Using these estimates in (4.4), we obtain

$$\begin{aligned}\|e(\tau)\|_W^2 &\le \frac{4}{9}\max\|e(t)\|_W^2 + \|e(0)\|_W^2 + 9T^2\|e_h'(0)\|_W^2 + 9\int_0^T \|e_p'(t)\|_W^2 \\ &\quad +9K_a^2T^2\|e_h(0)\|_W^2 + 9K_a^2\int_0^T \|e_p(t)\|_W^2.\end{aligned}$$

Since this is true for all $\tau \in (0,T)$, we have that

$$\begin{aligned}\max\|e(t)\|_W^2 &\le \frac{1}{2}\max\|e(t)\|_W^2 + \|e(0)\|_W^2 + 9T^2\|e_h'(0)\|_W^2 + 9\int_0^T \|e_p'(t)\|_W^2 \\ &\quad +9K_a^2T^2\|e_h(0)\|_W^2 + 9K_a^2\int_0^T \|e_p(t)\|_W^2,\end{aligned}$$

and hence

$$\begin{aligned}\max\|e(t)\|_W^2 &\le 2\Bigg(\|e(0)\|_W^2 + 9T^2\|e_h'(0)\|_W^2 + 9\int_0^T \|e_p'(t)\|_W^2 + 9K_a^2T^2\|e_h(0)\|_W^2 \\ &\qquad +9K_a^2\int_0^T \|e_p(t)\|_W^2\Bigg) \\ &\le 2\Bigg(\|e(0)\|_W + 3T\|e_h'(0)\|_W + 3\int_0^T \|e_p'(t)\|_W + 3K_aT\|e_h(0)\|_W \\ &\qquad +3K_a\int_0^T \|e_p(t)\|_W\Bigg)^2.\end{aligned}$$

□

Suppose that there is no damping, i.e. $a = 0$ in (4.4). Then, using the same arguments as in the proof above, we have the following corollary.

Corollary 4.1. *If the solution u of Problem G with $a = 0$ satisfies Assumption A1, then for $t \in [0,T]$,*

$$\|e(t)\|_W \le \sqrt{2}\|e(0)\|_W + 2T\|e_h'(0)\|_W + 4\sqrt{T}\max_{t\in[0,T]}\|e_p'(t)\|_W.$$

Now, consider the undamped case with the additional assumption that $c(u_1 - u_1^h, v(0)) = c(e_h'(0), v(0)) = 0$. The estimate in (4.3) reduces to

$$\frac{1}{2}\, c(e(\tau), e(\tau)) \leq \frac{1}{2}\, c(e(0), e(0)) - \int_0^\tau c(e_p', e)$$

and we have the following result.

Corollary 4.2. *Suppose the solution u of Problem G with $a = 0$ satisfies Assumption A1. If $c(u_1 - u_1^h, v) = 0$ for all $v \in S^h$, then for $t \in [0, T]$,*

$$\|e(t)\|_W \leq \sqrt{2}\|e(0)\|_W + 2\sqrt{T} \max_{t\in[0,T]} \|e_p'(t)\|_W.$$

This is a generalization of the estimate for $e = Pu - u_h$ in (3.13) in [1]. However, it is not necessary to make the assumption at this stage.

5 Convergence and error estimates for the semi-discrete approximation

The error estimates follow readily from Lemma 4.1.

Theorem 5.1. *If the solution u of Problem G satisfies Assumption A1, then*

$$\begin{aligned} \|u(t) - u_h(t)\|_W \;\leq\; & \|e_p(t)\|_W + \sqrt{2}\left(\|Pu_0 - u_0\|_W + 3T\|u_1 - u_1^h\|_W + 3\int_0^T \|e_p'(t)\|_W \right. \\ & \left. +(1 + 3K_aT)\|u_0 - u_0^h\|_W + 3K_a \int_0^T \|e_p(t)\|_W \right), \end{aligned} \tag{5.1}$$

for each $t \in [0, T]$.

Proof. From (3.2), $\|u(t) - u_h(t)\|_W \leq \|e_p(t)\|_W + \|e(t)\|_W$. The result follows from Lemma 4.1 since $\|Pu_0 - u_0^h\|_W \leq \|Pu_0 - u_0\| + \|u_0 - u_0^h\|$. □

In applications it is necessary to consider the errors on the right hand side of the estimate (5.1).

To formulate an assumption regarding the error when an element of V is approximated by an element of S^h, we suppose h is a parameter related to the dimension n of S^h and $h \to 0$ as $n \to \infty$.

Assumption A2
There exists a subspace H of V and a positive integer α such that if $w \in H$, then

$$\inf_{v \in S^h} \|w - v\|_V \leq \widehat{C} h^\alpha |||w|||_H,$$

where $|||w|||_H$ is a norm or semi-norm for H.

Using Assumption A2, estimates for the projection errors in (5.1) can be obtained. For example, $\|e_p'(t)\|_W = \|u'(t) - Pu'(t)\|_W \leq C_b \|u'(t) - Pu'(t)\|_V$ by Assumption E2 and Lemma 3.1. Therefore

$$\|e_p'(t)\|_W \leq C_b \widehat{C} h^\alpha |||u'(t)|||_H.$$

If we now choose $u_0^h = Pu_0$ and $u_1^h = Pu_1$ as in [1], then we have for example

$$\|u_0 - u_0^h\|_W \leq C_b \widehat{C} h^\alpha |||u_0|||_H, \tag{5.2}$$

and we are done. This is in general not possible. However, the estimate in Assumption A2 is derived from the interpolation error in applications. If we choose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$, where Π denotes the interpolation operator, then we have the same estimate as in (5.2).

Theorem 5.2. *Suppose Assumption A2 holds and $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. If the solution u of Problem G satisfies Assumption A1, $u(t) \in H$ and $u'(t) \in H$, then*

$$\begin{aligned} \|u(t) - u_h(t)\|_W \quad \leq \quad & C_b \widehat{C} h^\alpha |||u(t)|||_H + \sqrt{2} C_b \widehat{C} h^\alpha \big(3T \max |||u'(t)|||_H \\ & + 3K_a T \max |||u(t)|||_H + (2 + 3K_a T) |||u_0|||_H \\ & + 3T |||u_1|||_H \big), \end{aligned}$$

for each $t \in [0, T]$.

Remark In applications it is sometimes possible to use the Aubin-Nitche trick (see [15], [16]) to obtain higher order estimates for the projection error. But the improved projection errors serve no purpose if the interpolants are used to approximate the initial values u_0 and u_1. If the starting values u_0^h and u_1^h are projections of u_0 and u_1 with respect to the inner product c (as in [1]), then the result above can be improved. This is in general not possible.

6 Fully discrete approximation

6.1 A system of ordinary differential equations

In this section we consider the discretization of Problem G^h. Suppose S^h is the span of the set $\{\phi_1,\ \phi_2, \cdots,\ \phi_n\}$. Problem G^h is equivalent to a system of ordinary differential equations as we prove below.

Notation
Matrices K, C, M and the vector F are defined by

$$K_{ij} = b(\phi_j, \phi_i),\ C_{ij} = a(\phi_j, \phi_i),\ M_{ij} = c(\phi_j, \phi_i) \ \text{ and } \ F_i(t) = (f(t), \phi_i)_X.$$

If a function w has values in S^h, i.e. $w = \sum_{j=1}^n w_j\phi_j$, then we define a function

$$\bar{w} = (w_1, w_2, \ldots, w_n) = [w_1\ w_2\ \ldots\ w_n]^t,$$

with values in R_n. It is convenient to use d^h and v^h for initial conditions in this section instead of u_0^h and u_1^h. We define the vectors $\bar{d} = (d_1, d_2, \ldots, d_n)$ and $\bar{v} = (v_1, v_2, \ldots, v_n)$ where $d^h = \sum_{j=1}^n\ d_j\phi_j$ and $v^h = \sum_{j=1}^n\ v_j\phi_j$.

Problem ODE
Determine $\bar{u} \in C^2(J)$ such that

$$M\bar{u}'' + C\bar{u}' + K\bar{u} = F(t) \quad \text{with} \ \ \bar{u}(0) = \bar{d} \ \text{ and } \ \bar{u}'(0) = \bar{v}.$$

Proposition 6.1. *Suppose M, K, C, F, $\bar{d}$ and $\bar{v}$ are defined as above. Then, the function u_h is a solution of Problem G^h if and only if the function $\bar{u}$ is a solution of Problem ODE.*

Proposition 6.2. *If $F \in C(J)$, then Problem ODE has a unique solution for each pair of vectors $\bar{d}$ and $\bar{v}$.*

Proof. Since c is an inner product, M is invertible and the differential equation in Problem ODE may be written in the form $\bar{u}'' + M^{-1}C\bar{u}' + M^{-1}K\bar{u} = M^{-1}F(t)$. It follows from the theory of linear differential equations that the initial value problem has a unique solution. □

Proof of Theorem 3.1.
Clearly $F \in C(J)$ if $f \in C(J, X)$. The result follows from Propositions 6.1 and 6.2. □

Remark For existence the continuity of F is sufficient but for $\bar{u}$ to have derivatives of order $2+k$, we need to assume that F has derivatives of order k. (Note that $F \in C^k(J)$ if $f \in C^k(J, X)$.)

6.2 Fully discrete Galerkin scheme

A finite difference method is used to approximate the solution of the system in Problem G^h (or Problem ODE). We follow [1] with an obvious modification to include the damping term. Suppose the interval $[0, T]$ is divided into N steps of length $\tau = T/N$ and denote the approximation of $u_h(t_k)$ by u_k^h. Our aim is to estimate the difference between the solution of Problem G and the fully discrete approximation

$$u(t_k) - u_k^h = [u(t_k) - u_h(t_k)] + [u_h(t_k) - u_k^h].$$

An estimate for the error $u(t_k) - u_h(t_k)$ was obtained in Section 5 and an estimate for the error $u_h(t_k) - u_k^h$ is required. To be precise, an estimate for $\|u(t_k) - u_h(t_k)\|_W$ is derived in Theorem 5.2 and an estimate for the other error must be with respect to $\|\cdot\|_W$. Although norms are equivalent in a finite dimensional space, the dimension of S^h is not fixed and equivalence of norms may not be used. For this reason we consider the fully discrete problem in variational form as in [3], [1] and [14].

Notation For any sequence $\{x_k\} \subset R_n$:

$$\delta_t x_k = \tau^{-1}[x_{k+1} - x_k],$$

$$x_{k+\frac{1}{2}} = \frac{1}{2}[x_{k+1} + x_k].$$

Problem G^h-D
Find a sequence $\{u_k^h\} \subset S^h$ such that for $k = 0, 1, 2, \ldots, N-1$,

$$\delta_t u_k^h = v_{k+\frac{1}{2}}, \tag{6.1}$$

$$c(\delta_t v_k, \psi) + a(v_{k+\frac{1}{2}}, \psi) + b(u_{k+\frac{1}{2}}^h, \psi) = \frac{1}{2}([f(t_k) + f(t_{k+1})], \psi)_X \tag{6.2}$$

for each $\psi \in S^h$, while $u_0^h = u_h(0) = d^h$ and $v_0 = u_h'(0) = v^h$.

Following [1] we prove that the solution of Problem G^h-D is well defined. In the process an algorithm is derived. No confusion can arise if we write u_k for u_k^h.

Proposition 6.3. *Problem G^h-D has a unique solution for any pair of vectors d^h and v^h in S^h.*

Proof. For any $\psi \in S^h$,

$$\begin{aligned}
\frac{\tau}{2} b(u_{k+1} + u_k, \psi) &= \frac{\tau^2}{2} b(\tau^{-1}[u_{k+1} - u_k], \psi) + \frac{\tau}{2} b(u_k, \psi) + \frac{\tau}{2} b(u_k, \psi) \\
&= \frac{\tau^2}{2} b(\frac{1}{2}[v_{k+1} + v_k], \psi) + \tau b(u_k, \psi) \\
&= \frac{\tau^2}{4} b(v_{k+1}, \psi) + \frac{\tau^2}{4} b(v_k, \psi) + \tau b(u_k, \psi).
\end{aligned}$$

Substitution into (6.2) yields

$$\begin{aligned}
&c(v_{k+1}, \psi) - c(v_k, \psi) + \frac{\tau}{2} a(v_{k+1} + v_k, \psi) + \frac{\tau^2}{4} b(v_{k+1}, \psi) + \frac{\tau^2}{4} b(v_k, \psi) + \tau b(u_k, \psi) \\
&\quad = \frac{\tau}{2}([f(t_k) + f(t_{k+1})], \psi)_X
\end{aligned}$$

or

$$\begin{aligned}
&c(v_{k+1}, \psi) + \frac{\tau}{2} a(v_{k+1}, \psi) + \frac{\tau^2}{4} b(v_{k+1}, \psi) \\
&\quad = c(v_k, \psi) - \frac{\tau}{2} a(v_k, \psi) - \frac{\tau^2}{4} b(v_k, \psi) - \tau b(u_k, \psi) \\
&\qquad + \frac{\tau}{2}([f(t_k) + f(t_{k+1})], \psi)_X. \qquad (6.3)
\end{aligned}$$

for any $\psi \in S^h$. The bilinear forms b and c are positive definite and as a consequence $c + \frac{\tau}{2} a + \frac{\tau^2}{4} b$ is positive definite. Therefore v_{k+1} is uniquely determined by (6.3) and $u_{k+1} = u_k + \tau v_{k+\frac{1}{2}}$ from (6.1). □

The following algorithm may be used.

Problem FD
Find a sequence $\{\bar{u}_k\} \subset R_n$ such that for each k,

$$\begin{aligned}
\bar{u}_{k+1} &= \bar{u}_k + \tau \bar{v}_{k+\frac{1}{2}}, \\
(M + \frac{\tau}{2} C + \frac{\tau^2}{4} K)\bar{v}_{k+1} &= (M - \frac{\tau}{2} C - \frac{\tau^2}{4} K)\bar{v}_k - \tau K \bar{u}_k + \frac{\tau}{2}[F(t_k) + F(t_{k+1})]
\end{aligned}$$

with $\bar{u}_0 = \bar{d}$ and $\bar{v}_0 = \bar{v}$.

6.3 Error estimates

In this subsection we derive an estimate for the error $u_h(t_k) - u_k^h$. Recall that it must be with respect to $\|\cdot\|_W$. Usually the error $u(t_k) - u_k^h$ is estimated directly, see e.g. [3], [1], [14], [5], [6] and [7]. To do this, the assumption is made that the exact solution has derivatives $u^{(k)} \in \mathcal{L}^2([0,T],V)$ for $k \leq 4$ or $k \leq 3$. This assumption is very restrictive.

To obtain our general result, we adapt the proof in [1] (which is for the wave equation).

Truncation

Consider Problem G^h-D. Substituting $t = t_k$ and $t = t_{k+1}$ respectively into Problem G^h (Equation (3.1)), we obtain

$$\begin{aligned} &c(\tau^{-1}[v_h(t_{k+1}) - v_h(t_k)], \psi) + \frac{1}{2}\, a([v_h(t_{k+1}) + v_h(t_k)], \psi) + \frac{1}{2}\, b([u_h(t_{k+1}) + u_h(t_k)], \psi) \\ &\quad = \frac{1}{2}\, ([f(t_{k+1}) + f(t_k)], \psi)_X + c(\rho_k, \psi) \end{aligned} \tag{6.4}$$

where $v_h(t) = u_h'(t)$ and

$$\rho_k = \tau^{-1}[v_h(t_{k+1}) - v_h(t_k)] - \frac{1}{2}\, [v_h'(t_{k+1}) + v_h'(t_k)]. \tag{6.5}$$

We denote the errors by e_k and q_k:

$$e_k = u_h(t_k) - u_k \quad \text{and} \quad q_k = u_h'(t_k) - v_k.$$

Note that in our approach $e_0 = 0$ and $q_0 = 0$.

By subtracting (6.2) from (6.4) we obtain

$$c(\tau^{-1}[q_{k+1} - q_k], \psi) + a(\frac{1}{2}[q_{k+1} + q_k], \psi) + b(\frac{1}{2}[e_{k+1} + e_k], \psi) = c(\rho_k, \psi),$$

which can be rewritten as follows

$$c(\delta_t q_k, \psi) + a(q_{k+\frac{1}{2}}, \psi) + b(e_{k+\frac{1}{2}}, \psi) \;\; = \;\; c(\rho_k, \psi). \tag{6.6}$$

Substitution of u_h and v_h into (6.1) yields

$$\tau^{-1}[u_h(t_{k+1}) - u_h(t_k)] = \frac{1}{2}([v_h(t_{k+1}) + v_h(t_k)] + \sigma_k, \tag{6.7}$$

where (obviously)

$$\sigma_k = \tau^{-1}[u_h(t_{k+1}) - u_h(t_k)] - \frac{1}{2}([v_h(t_{k+1}) + v_h(t_k)]. \tag{6.8}$$

It follows from (6.1) and (6.7) that

$$\delta_t e_k \quad = \quad q_{k+\frac{1}{2}} + \sigma_k. \tag{6.9}$$

Stability

Consider any integer $n > 0$. It follows from (6.9) that

$$e_n = \tau \sum_{k=0}^{n-1} \delta_t e_k = \tau \sum_{k=0}^{n-1} q_{k+\frac{1}{2}} + \tau \sum_{k=0}^{n-1} \sigma_k.$$

Consequently

$$\begin{aligned} e_{n+\frac{1}{2}} &= \frac{1}{2}(e_{n+1} + e_n) \\ &= \frac{\tau}{2}\left(\sum_{k=0}^{n} q_{k+\frac{1}{2}} + \sum_{k=0}^{n-1} q_{k+\frac{1}{2}}\right) + \frac{\tau}{2}\left(\sum_{k=0}^{n} \sigma_k + \sum_{k=0}^{n-1} \sigma_k\right). \end{aligned} \tag{6.10}$$

But from (6.9) we also have

$$\begin{aligned} \delta_t e_n &= \frac{1}{2}[q_{n+1} + q_n] + \sigma_n \\ &= \frac{\tau}{2}\left[\sum_{k=0}^{n} \delta_t q_k + \sum_{k=0}^{n-1} \delta_t q_k\right] + \sigma_n. \end{aligned} \tag{6.11}$$

Let $s_0 = 0$ and $s_n = \tau \sum_{k=0}^{n-1} e_{k+\frac{1}{2}}$, then

$$s_{n+\frac{1}{2}} = \frac{\tau}{2}\left[\sum_{k=0}^{n} e_{k+\frac{1}{2}} + \sum_{k=0}^{n-1} e_{k+\frac{1}{2}}\right]. \tag{6.12}$$

In the analysis that follows, ψ denotes an arbitrary element of S^h. Using

(6.10), (6.11) and (6.12), we have for $n = 1,\ 2,\ \ldots,\ N-1$

$$\begin{aligned}
&c(\delta_t e_n, \psi) + a(e_{n+\frac{1}{2}}, \psi) + b(s_{n+\frac{1}{2}}, \psi) \\
&\quad = \frac{\tau}{2}\, c\left(\left[\sum_{k=0}^{n} \delta_t q_k + \sum_{k=0}^{n-1} \delta_t q_k\right], \psi\right) + c(\sigma_n, \psi) \\
&\qquad + \frac{\tau}{2}\, a\left(\left[\sum_{k=0}^{n} q_{k+\frac{1}{2}} + \sum_{k=0}^{n-1} q_{k+\frac{1}{2}}\right], \psi\right) \\
&\qquad + \frac{\tau}{2}\, b\left(\left[\sum_{k=0}^{n} e_{k+\frac{1}{2}} + \sum_{k=0}^{n-1} e_{k+\frac{1}{2}}\right], \psi\right).
\end{aligned}$$

Using (6.6), it follows that

$$\begin{aligned}
&c(\delta_t e_n, \psi) + a(e_{n+\frac{1}{2}}, \psi) + b(s_{n+\frac{1}{2}}, \psi) \\
&\quad = \frac{\tau}{2}\, c\left(\left[\sum_{k=0}^{n} \rho_k + \sum_{k=0}^{n-1} \rho_k\right], \psi\right) + c(\sigma_n, \psi). \qquad (6.13)
\end{aligned}$$

Let

$$\epsilon_n \;=\; \frac{\tau}{2}\rho_n + \tau \sum_{k=0}^{n-1} \rho_k + \sigma_n \qquad (6.14)$$

for $n = 1, 2, \ldots N-1$, then

$$c(\delta_t e_n, \psi) + a(e_{n+\frac{1}{2}}, \psi) + b(s_{n+\frac{1}{2}}, \psi) = c(\epsilon_n, \psi).$$

Choose $\psi = e_{n+\frac{1}{2}}$, then

$$\begin{aligned}
\|e_{n+1}\|^2 - \|e_n\|^2 + 2\tau a(e_{n+\frac{1}{2}}, e_{n+\frac{1}{2}}) + \|s_{n+1}\|_V^2 - \|s_n\|_V^2 \\
= 2\tau c(\epsilon_n, e_{n+\frac{1}{2}}). \qquad (6.15)
\end{aligned}$$

Consider any integer ν with $2 \le \nu \le N$. Using (6.15) and the fact that $a(e_{n+\frac{1}{2}}, e_{n+\frac{1}{2}}) \ge 0$, we have

$$\sum_{n=1}^{\nu-1} \left[\|e_{n+1}\|^2 - \|e_n\|^2 + \|s_{n+1}\|_V^2 - \|s_n\|_V^2\right] \le 2\tau \sum_{n=1}^{\nu-1} c(\epsilon_n, e_{n+\frac{1}{2}}).$$

It follows that

$$\begin{aligned}
&\|e_\nu\|_W^2 - \|e_1\|_W^2 + \|s_\nu\|_V^2 - \|s_1\|_V^2 \\
&\quad \le 4T\tau \sum_{n=1}^{\nu-1} \|\epsilon_n\|_W^2 + \frac{\tau}{4T} \sum_{n=1}^{\nu-1} \|e_{n+\frac{1}{2}}\|_W^2,
\end{aligned} \tag{6.16}$$

using the Cauchy-Schwartz inequality and Young's inequality.

Now, consider $n = 0$. Let $k = 0$ in (6.6):

$$c(\delta_t q_0, \psi) + a(q_{\frac{1}{2}}, \psi) + b(e_{\frac{1}{2}}, \psi) = c(\rho_0, \psi). \tag{6.17}$$

Also, let $k = 0$ in (6.9):

$$\delta_t e_0 = q_{\frac{1}{2}} + \sigma_0.$$

Since $q_0 = 0$ and $e_0 = 0$ we have $\delta_t e_0 = \frac{e_1}{\tau}$, $e_{\frac{1}{2}} = \frac{e_1}{2}$, $\delta_t q_0 = \frac{q_1}{\tau}$ and $q_{\frac{1}{2}} = \frac{q_1}{2}$ and hence

$$q_1 = \frac{2}{\tau}\, e_1 - 2\sigma_0.$$

Making the appropriate substitutions into (6.17) we obtain

$$\begin{aligned}
&\frac{2}{\tau^2}\, c(e_1, \psi) + \frac{1}{\tau}\, a(e_1, \psi) + \frac{1}{2}\, b(e_1, \psi) \\
&\qquad = c(\rho_0, \psi) + \frac{2}{\tau}\, c(\sigma_0, \psi) + a(\sigma_0, \psi).
\end{aligned} \tag{6.18}$$

From the definition of s_0 and s_1, we have $e_1 = \frac{2}{\tau}\, s_1$. Substituting this into the bilinear form b in (6.18) (and multiplying by $\frac{\tau^2}{2}$) yields

$$c(e_1, \psi) + \frac{\tau}{2}\, a(e_1, \psi) + \frac{\tau}{2} b(s_1, \psi) = \frac{\tau^2}{2}\, c(\rho_0, \psi) + \tau\, c(\sigma_0, \psi) + \frac{\tau^2}{2}\, a(\sigma_0, \psi).$$

Choose $\psi = e_1$, then

$$\begin{aligned}
&\|e_1\|_W^2 + \|s_1\|_V^2 + \frac{\tau}{2} a(e_1, e_1) \\
&\quad = \frac{\tau^2}{2}\, c(\rho_0, e_1) + \tau\, c(\sigma_0, e_1) + \frac{\tau^2}{2}\, a(\sigma_0, e_1).
\end{aligned}$$

Since $a(e_1, e_1) \geq 0$,

$$\begin{aligned}
\|e_1\|_W^2 + \|s_1\|_V^2 &\leq \frac{\tau^2}{2}\|\rho_0\|_W\|e_1\|_W + \tau\|\sigma_0\|_W\|e_1\|_W + \frac{\tau^2}{2}K_a\|\sigma_0\|_W\|e_1\|_W \\
&\leq \tau^4\|\rho_0\|_W^2 + \frac{1}{16}\|e_1\|_W^2 + 4\tau^2\|\sigma_0\|_W^2 + \frac{1}{16}\|e_1\|_W^2 \\
&\quad + \tau^4 K_a^2\|\sigma_0\|_W^2 + \frac{1}{16}\|e_1\|_W^2 \\
&= \frac{3}{16}\|e_1\|_W^2 + \tau^4\|\rho_0\|_W^2 + (4\tau^2 + \tau^4 K_a)\|\sigma_0\|_W^2, \qquad (6.19)
\end{aligned}$$

using the Cauchy-Schwartz and Young's inequalities.

To proceed, we combine (6.16) and (6.19):

$$\begin{aligned}
\|e_\nu\|_W^2 &\leq 4T\tau\sum_{n=1}^{\nu-1}\|\epsilon_n\|_W^2 + \frac{\tau}{4T}\sum_{n=1}^{\nu-1}\|e_{n+\frac{1}{2}}\|_W^2 \\
&\quad + \frac{3}{16}\|e_1\|_W^2 + \tau^4\|\rho_0\|_W^2 + (4\tau^2 + \tau^4 K_a)\|\sigma_0\|_W^2 \\
&\leq 4T\tau\sum_{n=1}^{\nu-1}\|\epsilon_n\|_W^2 + \frac{1}{2}\max\|e_n\|_W^2 + \tau^4\|\rho_0\|_W^2 + (4\tau^2 + \tau^4 K_a)\|\sigma_0\|_W^2.
\end{aligned}$$

The stability result follows.

Lemma 6.1. *Stability*

$$\max\|e_n\|_W^2 \leq 8T\tau\sum_{n=0}^{N-1}\|\epsilon_n\|_W^2 + 2\tau^4\|\rho_0\|_W^2 + (8\tau^2 + 2\tau^4 K_a)\|\sigma_0\|_W^2.$$

Convergence

It remains to determine estimates for ρ_k and σ_k. To find an estimate for ρ_k, we consider the following two parts separately

$$\begin{aligned}
\rho_A &= \tau^{-1}[v_h(t_{k+1}) - v_h(t_k)] - v_h'(t_k + \frac{\tau}{2}) \quad \text{and} \\
\rho_B &= -\frac{1}{2}[v_h'(t_{k+1}) + v_h'(t_k)] + v_h'(t_k + \frac{\tau}{2}).
\end{aligned}$$

Recall that $v_h(t_k) = \sum_{i=1}^n v_i(t_k)\phi_i$. Using Taylor expansions we obtain

$$\begin{aligned}
&\tau^{-1}[v_i(t_{k+1}) - v_i(t_k)] - v_i'(t_k + \frac{\tau}{2}) \\
&= \frac{1}{4\tau}\int_{t_k+\frac{\tau}{2}}^{t_{k+1}}(t_{k+1} - \theta)^2 v_i'''(\theta)d\theta + \frac{1}{4\tau}\int_{t_k}^{t_k+\frac{\tau}{2}}(t_k - \theta)^2 v_i'''(\theta)d\theta.
\end{aligned}$$

From the Cauchy-Schwartz inequality

$$\begin{aligned}
\left(\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta)^2 v_i'''(\theta)\, d\theta\right)^2 &\leq \int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta)^4\, d\theta \int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2\, d\theta \\
&= \frac{\tau^5}{160}\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2\, d\theta \qquad \text{and} \\
\left(\int_{t_k}^{t_k+\frac{\tau}{2}} (t_k-\theta)^2 v_i'''(\theta)\, d\theta\right)^2 &\leq \frac{\tau^5}{160}\int_{t_k}^{t_k+\frac{\tau}{2}} (v_i'''(\theta))^2\, d\theta.
\end{aligned}$$

Using the estimate above and assuming that $\{\phi_j\}$ is orthonormal, we have

$$\begin{aligned}
\|\rho_A\|_W^2 &= \left\|\tau^{-1}[v_h(t_{k+1}) - v_h(t_k)] - v_h'(t_k+\frac{\tau}{2})\right\|_W^2 \\
&\leq \sum_{i=1}^{n} 2\left(\frac{1}{4\tau}\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta)^2 v_i'''(\theta) d\theta\right)^2 + \sum_{i=1}^{n} 2\left(\frac{1}{4\tau}\int_{t_k}^{t_k+\frac{\tau}{2}} (t_k-\theta)^2 v_i'''(\theta) d\theta\right)^2 \\
&\leq \frac{\tau^3}{1280}\left[\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} \|v_h'''(\theta)\|_W^2\, d\theta + \int_{t_k}^{t_k+\frac{\tau}{2}} \|v_h'''(\theta))\|_W^2\, d\theta\right] \\
&\leq \frac{\tau^4}{1280}\max \|v_h'''\|_W^2,
\end{aligned}$$

where $\max_{\theta\in[0,T]}$ is denoted by max.

To find an estimate for ρ_B, the same steps are followed.

From Taylor expansions

$$\begin{aligned}
&\frac{1}{2}[v_i'(t_{k+1}) + v_i'(t_k)] - v_i'(t_k+\frac{\tau}{2}) \\
&\quad = \frac{1}{2}\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta) v_i'''(\theta) d\theta - \frac{1}{2}\int_{t_k}^{t_k+\frac{\tau}{2}} (t_k-\theta) v_i'''(\theta) d\theta.
\end{aligned}$$

Using the Cauchy-Schwartz inequality again, we obtain

$$\begin{aligned}
\left(\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta) v_i'''(\theta) d\theta\right)^2 &\leq \int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (t_{k+1}-\theta)^2\, d\theta \int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2\, d\theta \\
&= \frac{\tau^3}{24}\int_{t_k+\frac{\tau}{2}}^{t_{k+1}} (v_i'''(\theta))^2\, d\theta \qquad \text{and} \\
\left(\int_{t_k}^{t_k+\frac{\tau}{2}} (t_k-\theta)^2 v_i^{(4)}(\theta) d\theta\right)^2 &\leq \frac{\tau^3}{24}\int_{t_k}^{t_k+\frac{\tau}{2}} (v_i'''(\theta))^2\, d\theta.
\end{aligned}$$

Therefore

$$\|\rho_B\|_W^2 \leq \frac{\tau^4}{48} \max \|v_h'''\|_W^2.$$

Combining the estimates for ρ_A and ρ_B yields

$$\|\rho_k\|_W^2 \leq \tau^4 \max \|v_h'''\|_W^2. \tag{6.20}$$

Now consider $\sigma_k = \tau^{-1}[u_h(t_{k+1}) - u_h(t_k)] - \frac{1}{2}[u_h'(t_{k+1}) + u_h'(t_k)]$. Following the same method as for ρ_k, we obtain

$$\|\sigma_k\|_W^2 \leq \tau^4 \max \|u_h'''\|_W^2. \tag{6.21}$$

The following estimate for ϵ_n (see (6.14)) is obtained from (6.20) and (6.21).

$$\|\epsilon_n\|_W^2 \leq 5T^2\tau^4 \max \|v_h'''\|_W^2 + 4\tau^4 \max \|u_h'''\|_W^2.$$

Using the result in Lemma 6.1 yields

$$\begin{aligned} \max \|e_n\|_W^2 \leq\ & 40T^4\tau^4 \max \|v_h'''\|_W^2 + 32T^2\tau^4 \max \|u_h'''\|_W^2 \\ & +2\tau^8 \max \|v_h'''\|_W^2 + 8\tau^6 \max \|u_h'''\|_W^2 + 2\tau^8 K_a \max \|u_h'''\|_W^2. \end{aligned}$$

Note that if $f \in C^2([0,T], X)$, then $u_h \in C^4([0,T], W)$ and consequently the following error estimate is obtained.

Theorem 6.1. *Error estimate.*
If $f \in C^2([0,T], X)$, then

$$\begin{aligned} \|u_h(t_k) - u_k^h\|_W \leq\ & 7T^2\tau^2 \max \|u_h^{(4)}\|_W + 7T\tau^2 \max \|u_h'''\|_W \\ & +\sqrt{2K_a}\,\tau^4 \max \|u_h'''\|_W \end{aligned} \tag{6.22}$$

for each $t \in (0,T)$.

Finally, error estimates for the fully discrete approximation of the solution of Problem G are obtained by combining results from Section 5 with Theorem 6.1.

7 Applications

The multi-dimensional wave equation

As an elementary application we consider the wave equation in a two-dimensional bounded domain denoted by Ω. Let Σ be a part of the boundary $\partial\Omega$. Given functions f, u_0 and u_1, find u defined on $\bar{\Omega} \times [0, T]$ such that

$$\begin{aligned} \rho\partial_t^2 u &= \nabla \cdot (A\nabla u) - k\partial_t u + f \quad \text{in } \Omega \times (0, T), \\ u &= 0 \qquad \text{on } \partial\Omega - \Sigma, \\ -A\ \nabla u \cdot n &= 0 \quad \text{on } \Sigma, \end{aligned}$$

while $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$.
The unit outer normal on the boundary is denoted by n. The given parameters in the problem are the matrix of functions $A = (a_{ij})$ and the functions k and ρ. We assume that the parameters are bounded above and below by positive numbers and the matrix A is uniformly positive definite. For the problem in [1], Σ is empty and $k = 0$.

Next consider the variational form. The test functions $T(\Omega)$ are functions in $C^1(\bar{\Omega})$ which vanish on $\partial\Omega - \Sigma$. Let (f, g) denote the usual innerproduct for $\mathcal{L}^2(\Omega)$. The bilinear forms a, b and c are defined as follows (using weak derivatives for b)

$$\begin{aligned} b(u, v) &= \iint_\Omega A\nabla u \cdot \nabla v \ dV \\ c(u, v) &= \iint_\Omega \rho uv \ dV \\ a(u, v) &= \iint_\Omega kuv \ dV. \end{aligned}$$

Let V denote the closure of $T(\Omega)$ in $H^1(\Omega)$. The weak variational form of the problem is to find u such that for each $t > 0$, $u(t) \in V$, $u''(t) \in \mathcal{L}^2(\Omega)$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v), \text{ for each } v \in V$$

while $u(0) = u_0$ and $u'(0) = u_1$.

Note that the restriction to two dimensions is not necessary. Next we show that Assumptions E1, E2, E3 and E4 are satisfied.

E1: V is dense in $\mathcal{L}^2(\Omega)$ since $C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$.

E2: Assuming that $\partial\Omega - \Sigma$ has positive length, the Poincaré-Friedrichs inequality can be proved and it follows that b is positive definite. The norm corresponding to the inner product b is $\|v\|_V = \sqrt{b(v,v)}$. It is equivalent to the norm of $H^1(\Omega)$ on V.

E3: The bilinear form c is clearly an inner product for $X = \mathcal{L}^2(\Omega)$ and the corresponding norm is equivalent to the norm of $\mathcal{L}^2(\Omega)$. The space $\mathcal{L}^2(\Omega)$ with inner product c is the space W (with norm $\|\cdot\|_W$).

E4: Clearly a is defined on $\mathcal{L}^2(\Omega)$ and is bounded.

Existence of a unique solution for the weak variational problem follows from Theorem 2.1, provided that $f \in C^1([0,T], \mathcal{L}^2(\Omega))$.

Now consider the finite element approximation of the problem. For simplicity we consider the two-dimensional case and use triangle elements. A finite dimensional subspace S^h of V is constructed, using piecewise linear basis functions.

The Galerkin approximation is to find a function $u_h \in C^2(0,T) \cap C^1[0,T]$ such that

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (f(\cdot, t), v)_X,$$

for each $v \in S^h$, while $u_h(0) = u_0^h$ and $u_h'(0) = u_1^h$.

Depending on the properties of the boundary of Ω, the parameters, initial values u_0 and u_1 and the function f, it can be proved that A1 and A2 hold. (It is beyond the scope of this article.) For the problem under discussion we have that $H = H^k(\Omega) \cap V$ (see Section 5). Instead of Assumption A2 we have the following result for piecewise linear basis functions: If $u \in H^k(\Omega)$ for $k \geq 2$, then

$$\|\Pi u - u\|_V \leq \widehat{C}_T h |u|_2,$$

where $|\cdot|_2$ denotes the usual semi-norm.

Consider application of Theorem 5.2. For the present problem we have

$$\|u - Pu\|_W \leq C_b \|u - Pu\|_V \leq \widehat{C}_b C_T h |u|_2. \tag{7.1}$$

Error estimate 7.1
Suppose $u_0^h = \Pi u_0$ and $u_1^h = \Pi u_1$. If the solution u of Problem G satisfies

Assumption A1, $u(t) \in H^2(\Omega)$ and $u'(t) \in H^2(\Omega)$, then

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq C_b \widehat{C}_T h |u(t)|_2 + \sqrt{2} C_b \widehat{C}_T h \big(3T \max |u'(t)|_2 \\ &\quad + 3K_a T \max |u(t)|_2 + (2 + 3K_a T)|u_0|_2 + 3T|u_1|_2\big), \end{aligned}$$

for each $t \in [0, T]$.

If the solution is sufficiently smooth, i.e. $E_b \subset H^2(\Omega)$, the Aubin-Nitche trick may be used. Instead of (7.1) we have $\|u - Pu\|_W \leq \widehat{K} h^2 |u|_2$ where $\widehat{K}$ depends on C_b and $\widehat{C}_T$. Using this estimate in Theorem 5.1, we obtain the following result.

Error estimate 7.2
Let $u_0^h = Pu_0$ and $u_1^h = Pu_1$. Assume that $u(t)$ and $u'(t)$ are in $H^2(\Omega)$. Then,

$$\begin{aligned} \|u(t) - u_h(t)\|_W &\leq \widehat{K} h^2 |u(t)|_2 + \sqrt{2} \widehat{K} h^2 \big(3T \max |u'(t)|_2 \\ &\quad + 3K_a T \max |u(t)|_2 + (2 + 3K_a T)|u_0|_2 + 3T|u_1|_2\big), \end{aligned}$$

for each $t \in [0, T]$.

From the definition of the spaces X, W and V and the bilinear forms c, a and b, it follows that the theory of Section 6 is applicable. Combining either Estimate 7.1 or 7.2 with Theorem 6.1 yields the error estimate for the fully discrete approximation.

Elasto-dynamics, hybrid models and structures

The theory in this article can also be applied to the Timoshenko beam model (with viscous type damping) [5], the Reissner-Mindlin plate model (undamped) [17] and linear elasto-dynamics (undamped) [7].

The theory of this article is really intended for systems where elastic bodies interact. Numerous examples can be found in [18], for example networks of linked beams.

Another example where the theory may be applied, is in article [19] where the importance of vibration analysis of structures in engineering is stressed. The authors consider a system consisting of two plates.

A hybrid model where a beam has complex boundary conditions due to a damping tip body attached to one end is considered in [20]. The damping is not of viscous type and the theory in this article does not apply, but it does for the undamped hybrid model.

Perhaps the most convenient example for the present discussion, is in a plate-beam model from [21]. The model problem referred to as Problem RMT, is of the same type as Problem G. If the damping in a joint is replaced by viscous damping on the plate, then Problem RMT is a special case of Problem G. It is proved in [21] that assumptions E1, E2 and E3 are satisfied. For viscous damping on the plate, Assumption E4 is easy to verify.

Finally, we mention the rather obvious fact that other types of damping need to be considered. Considering the existence results for different types of damping in [11], we believe that mere adjustments to the present article will not yield the desired results.

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