Appendix The smoothness condition for a Vold-Kalman filter

Howell (2001) defines the smoothness condition as follows:

To be smooth over an interval \((\alpha, \beta)\), a function \(x(t)\) must satisfy two conditions:

- \(x(t)\) must be differentiable (and hence continuous) everywhere on \((\alpha, \beta)\), and
- \(x'(t)\) must also be a continuous function on \((\alpha, \beta)\)

This requires further definition of the terms continuous and differentiable (Howell, 2001):

A function \(x(t)\) is continuous at a point \(t_0\) if \(x(t_0)\) and \(\lim_{t \to t_0} x(t)\) both exist and \(\lim_{t \to t_0} x(t) = x(t_0)\).

A function \(x(t)\) is differentiable at a point \(t\) if and only if \(\lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}\) exist. If \(x(t)\) is differentiable at every point in a given interval \((\alpha, \beta)\), then \(x(t)\) is said to be differentiable on the interval \((\alpha, \beta)\).

These concepts can now be applied to the structural equation for a 2-pole filter as given by Tůma at equation (2.3) in chapter 2,

\[
x(n) - 2x(n+1) + x(n+2) = \varepsilon(n)
\]

Illustration:

For a 2-pole Vold-Kalman filter, the sequence as filtered according to the structural equation may be written as \(x(n) = 2x(n+1) - x(n+2) + \varepsilon(n)\) (equation
(2.4) in chapter 2). If adjusting it to be continuous form, it becomes \(x(t) = 2x(t + \Delta t) - x(t + 2\Delta t) + \varepsilon(t)\). Since \(\varepsilon(t)\) is the error which is minimized by global solution of the data and structural equations (Tůma, 2005), it may be neglected for the purposes of the illustration, and the equation becomes \(x(t) = 2x(t + \Delta t) - x(t + 2\Delta t)\).

From the above, smoothness now requires that \(x(t)\) must be differentiable (and hence continuous) everywhere on certain interval \((\alpha, \beta)\) and \(x'(t)\) must also be a continuous function on \((\alpha, \beta)\).

**Differentiability**

\[
\lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{2x(t + \Delta t + \Delta t) - x(t + \Delta t + 2\Delta t) - 2x(t + \Delta t) + x(t + 2\Delta t)}{\Delta t}
\]

\[
\lim_{\Delta t \to 0} \frac{0}{\Delta t} = 1
\]

The limit will always exist. Thus, this function is differentiable (and hence continuous).

**Continuity of \(x'(t)\)**

Since \(x(t)\) is differentiable, then \(x'(t) = 2x'(t + \Delta t) - x'(t + 2\Delta t)\), thus

\[
\lim_{t \to t_0} x(t) = \lim_{t \to t_0} 2x(t + \Delta t) - x(t + 2\Delta t) = \lim_{t \to t_0} 2x(t + \Delta t) - \lim_{t \to t_0} x(t + 2\Delta t)
\]

\[
\Rightarrow \lim_{t \to t_0} x(t) = \lim_{t \to t_0} \left[ \lim_{\Delta t \to 0} \frac{2x(t + \Delta t + \Delta t) - x(t + \Delta t)}{\Delta t} \right] - \lim_{t \to t_0} \left[ \lim_{\Delta t \to 0} \frac{x(t + 2\Delta t + \Delta t) - x(t + 2\Delta t)}{\Delta t} \right]
\]

Swap the limits and notice that \(x(t)\) is differentiable and hence continuous, then \(\lim_{t \to t_0} x(t) = x(t_0)\), thus,
\[ \lim_{t \to t_0} x'(t) = \lim_{\Delta t \to 0} \left[ \lim_{t \to t_0} \frac{x(t + \Delta t + \Delta t) - x(t + \Delta t)}{\Delta t} \right] = \lim_{\Delta t \to 0} \left[ \lim_{t \to t_0} \frac{x(t + 2\Delta t + \Delta t) - x(t + 2\Delta t)}{\Delta t} \right] \]

\[ \Rightarrow \]

\[ \lim_{t \to t_0} x'(t) = \lim_{\Delta t \to 0} \frac{x(t_0 + \Delta t + \Delta t) - x(t_0 + \Delta t)}{\Delta t} - \lim_{\Delta t \to 0} \frac{x(t_0 + 2\Delta t + \Delta t) - x(t_0 + 2\Delta t)}{\Delta t} \]

\[ \Rightarrow \]

\[ \lim_{t \to t_0} x'(t) = x(t_0 + \Delta t) - x(t_0 + 2\Delta t) = x'(t_0) \]

So \( x'(t) \) is continuous.  

From the above the filtered signal from 2-pole Vold-Kalman filter is continuous and smooth. Similar procedures may be applied to 1 and 3 pole filters.
References


Tůma J. (undated study notes), Vold-Kalman Order Tracking Filtration. Faculty of Mechanical Engineering, Department of Control Systems and Instrumentation.

Vibratools (2005), MATLAB toolbox. Axiom EduTech, Sweden,


