Zeros of polynomials embedded in an orthogonal sequence

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Abstract

Let $\{x_{k,n}\}_{k=1}^{n}$ and $\{x_{k,n+1}\}_{k=1}^{n+1}$, $n \in \mathbb{N}$, be two given sets of real distinct points with $x_{1,n+1} < x_{1,n} < x_{2,n+1} < \cdots < x_{n,n} < x_{n+1,n+1}$. Wendroff (cf. [3]) proved that if $p_n(x) = \prod_{k=1}^{n}(x - x_{k,n})$ and $p_{n+1}(x) = \prod_{k=1}^{n+1}(x - x_{k,n+1})$ then p_n and p_{n+1} can be embedded in a non-unique infinite monic orthogonal sequence $\{p_n\}_{n=0}^{\infty}$. We investigate the connection between the zeros of p_{n+2} and the two coefficients $b_{n+1} \in \mathbb{R}$ and $\lambda_{n+1} > 0$, which are chosen arbitrarily, that define p_{n+2} via the three term recurrence relation

$$p_{n+2}(x) = (x - b_{n+1})p_{n+1}(x) - \lambda_{n+1}p_n(x).$$

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1 Introduction

In 1961, Burton Wendroff (cf. [3]) proved that given any n real points $x_{1,n} < x_{2,n} < \ldots < x_{n,n}$ and any $n+1$ real points $x_{1,n+1} < x_{2,n+1} < \ldots < x_{n+1,n+1}$, satisfying

$$x_{1,n+1} < x_{1,n} < x_{2,n+1} < x_{2,n} < \cdots < x_{n,n+1} < x_{n,n} < x_{n+1,n+1} \tag{1}$$

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then if

$$p_n(x) = \prod_{k=1}^{n}(x - x_{k,n}) \text{ and } p_{n+1}(x) = \prod_{k=1}^{n+1}(x - x_{k,n+1}), \qquad (2)$$

the polynomials p_n and p_{n+1} can always be embedded in an infinite sequence of monic polynomials that is orthogonal with respect to some positive Borel measure on $\mathbb{R}$. His proof shows that, given (1) and (2), all the polynomials of lower degree, namely $p_1,\ p_2, \ldots, p_{n-1}$, in any monic orthogonal sequence that contains p_n and p_{n+1}, are completely and uniquely determined by p_n and p_{n+1}. This is most easily seen by observing that, since any monic orthogonal sequence must satisfy a three term recurrence relation of the form (cf. [2])

$$p_{n+1}(x) \quad = \quad (x - b_n)p_n(x) - \lambda_n p_{n-1}(x),\ n \in \mathbb{N} \qquad (3)$$

where $p_0(x) = 1$, $p_{-1}(x) = 0$, $\lambda_n > 0$ and $b_n \in \mathbb{R}$, we have

$$b_n = \sum_{k=1}^{n+1} x_{k,n+1} - \sum_{k=1}^{n} x_{k,n} \qquad (4)$$

and λ_n is clearly also determined by the original configuration of $\{x_{k,n}\}_{k=1}^{n}$ and $\{x_{k,n+1}\}_{k=1}^{n+1}$ satisfying (1).

In contrast, the polynomials $p_{k+1},\ k \geq n+1$, in any monic orthogonal sequence $\{p_n\}_{n=0}^{\infty}$ containing p_n and p_{n+1}, are constructed successively and are defined by using the three term recurrence relation (3) and choosing constants $b_k \in \mathbb{R}$ and $\lambda_k > 0$ for $k = n+1, n+2, \ldots$. In [3], Wendroff states that if $a < x_{1,n+1} < \cdots < x_{n+1,n+1} < b$, in order to retain (a, b) as the interval of orthogonality, the coefficients b_{n+j} and $\lambda_{n+j} > 0$ should be chosen in such a way that the zeros of $p_{n+j+1},\ j \geq 1$ lie in (a, b) but he gives no indication of the connection between $b_{n+j},\ \lambda_{n+j}$ and the zeros of p_{n+j+1}.

In this paper, we discuss how the choices of λ_{n+1} and b_{n+1} influence the location of the zeros of p_{n+2}. Since each polynomial $p_k,\ k > n+1$, in an infinite monic orthogonal sequence $\{p_n\}_{n=0}^{\infty}$ that includes p_n and p_{n+1} is constructed iteratively using the three term recurrence relation, one can apply the results we prove here for p_{n+2} recursively to the polynomials p_{n+3}, $p_{n+4}, \cdots$

2 The coefficient b_{n+1}

We begin with a general lemma whose proof is an adaptation of the familiar proof that the zeros of a polynomial are continuous functions of its coefficients.

Lemma 1 *Let p and q be complex, monic polynomials of degrees n and $n+1$, respectively, and let*

$$r(z) = (z-\beta)q(z) - \lambda p(z),$$

where β and λ are complex numbers. Let $\zeta_1, \ldots, \zeta_t$ be the distinct zeros of q with multiplicities $m_1, \ldots, m_t$, respectively. For fixed λ, given any positive ε, there is a positive R such that if $|\beta| > R$ then there are m_j zeros of r within a distance ε of ζ_j.

Proof. Let $C_1, \ldots, C_t$ be circles centered at $\zeta_1, \ldots, \zeta_t$, each of radius δ, where $0 < \delta < \varepsilon$, and where δ is sufficiently small so that the C_j are exterior to each other. Since $q \neq 0$ on each C_j, we can find R such that if $|\beta| > R$ then $|(z-\beta)q(z)| > |\lambda\, p(z)|$ on each C_j. Thus, by Rouché's Theorem, $(z-\beta)q(z)$ and $r(z)$ have the same number of zeros inside each C_j. ∎

Given p_n and p_{n+1} defined by (1) and (2), the first polynomial in the (non-unique) orthogonal sequence that we construct is given by

$$p_{n+2}(x) = (x - b_{n+1})p_{n+1}(x) - \lambda_{n+1}p_n(x),\ \lambda_{n+1} > 0,\ b_{n+1} \in \mathbb{R}. \qquad (5)$$

We exclude the choice $b_{n+1} = x_{k,n}$ for any $k \in \{1, 2, \ldots, n\}$ where $\{x_{k,n}\}_{k=1}^{n}$ are the zeros of $p_n(x)$ which ensures that p_{n+2} and p_n have no common zeros. Our first result considers the zeros of p_{n+2} as functions of b_{n+1} with $\lambda_{n+1} > 0$ fixed.

Theorem 2 *Let (1), (2) and (5) hold with $b_{n+1} \neq x_{k,n}$ for any $k \in \{1, \ldots, n\}$ and suppose $\lambda_{n+1} > 0$ is fixed. Then, for each n,*

(i) $x_{1,n+2} < b_{n+1} < x_{n+2,n+2}$;

(ii) each zero of p_{n+2} is an increasing function of b_{n+1};

(iii) $\lim_{b_{n+1}\to\infty}(x_{k,n+2} - x_{k,n+1}) = 0$ *for each* $k \in \{1, 2, \ldots, n+1\}$;

(iv) $\lim_{b_{n+1}\to\infty}(x_{n+2,n+2} - b_{n+1}) = 0$.

Proof. It is clear that (iii) follows immediately from Lemma 1 (as does a similar result as $b_{n+1} \to -\infty$). Also, (iv) follows from (4) and (iii). (4) may be written as

$$b_{n+1} = (x_{1,n+2} + \cdots + x_{n+2,n+2}) - (x_{1,n+1} + \cdots + x_{n+1,n+1}),$$

and since

$$x_{1,n+2} < x_{1,n+1} < x_{2,n+1} < \cdots < x_{n+1,n+1} < x_{n+2,n+2},$$

(i) follows immediately.

Finally, we prove (ii). Suppose that $B_{n+1} > b_{n+1}$, and define

$$\begin{aligned} P_{n+2}(x) &= (x - B_{n+1})p_{n+1}(x) - \lambda_{n+1}p_n(x) \\ &= p_{n+2}(x) - (B_{n+1} - b_{n+1})p_{n+1}(x). \end{aligned} \tag{6}$$

By Wendroff's result (cf. [3]) the polynomials $p_0, p_1, \ldots, p_{n+1}$ are orthogonal to P_{n+2} for some Borel measure on $\mathbb{R}$ so we can conclude that P_{n+2} has $n+2$ real, distinct zeros which we denote by $X_1 < \ldots < X_{n+2}$. We need to show that $x_{k,n+2} < X_k$ for each $k = 1, \ldots, n+2$. It follows from (6) that $P_{n+2}(x_{k,n+2})p_{n+1}(x_{k,n+2}) < 0$. Since $x_{n+2,n+2} > x_{n+1,n+1}$ (the largest zero of p_{n+1}), and p_{n+1} is monic, we see that $p_{n+1}(x_{n+2,n+2}) > 0$, and hence that $P_{n+2}(x_{n+2,n+2}) < 0$. Since P_{n+2} is monic, this implies that P_{n+2} has a zero in $(x_{n+2,n+2}, +\infty)$. A similar argument (which we omit) shows that P_{n+2} has a zero in each of the intervals $(x_{k,n+2}, x_{k+1,n+2})$ and this implies that $X_k \in (x_{k,n+2}, x_{k+1,n+2})$ and $X_{n+2} \in (x_{n+2,n+2}, +\infty)$ which completes our proof of Theorem 2.

3 The coefficient λ_{n+1}

In this section we consider the zeros of p_{n+2} as we vary $\lambda_{n+1} > 0$.

Theorem 3 *Let (1), (2) and (5) hold. Then*

$$0 < \lambda_{n+1} \leq (x_{n+2,n+2} - x_{1,n+2})^2.$$

Thus if $x_{k,n} \in (a, b)$ *for all* k, $n \in \mathbb{N}$, *then* $0 < \lambda_{n+1} \leq (b-a)^2$ *for all* n.

Proof. Since the zeros of p_n and p_{n+1} are interlacing, it is clear that if $t > x_{n+1,n+1}$ then

$$\begin{aligned} p_{n+1}(t) &= (t - x_{1,n+1})(t - x_{2,n+1}) \cdots (t - x_{n+1,n+1}) \\ &\leq (t - x_{1,n+1})(t - x_{1,n}) \cdots (t - x_{n,n}) \\ &= (t - x_{1,n+1})p_n(t). \end{aligned}$$

In particular, this inequality holds with $t = x_{n+2,n+2}$. Since $p_{n+2}(x_{n+2,n+2}) = 0$, we see from (5) that

$$(x_{n+2,n+2} - b_{n+1})p_{n+1}(x_{n+2,n+2}) = \lambda_{n+1}p_n(x_{n+2,n+2}).$$

This, together with the inequality just established and Theorem 2(i) leads to the result since $p_n(x_{n+2,n+2}) > 0$. ■

4 The zeros of p_n and p_{n+2}

We now consider the role of b_{n+1} in determining the relative positions of the zeros of p_n and p_{n+2}. First, there is an alternative argument which yields more detailed information than Theorem 2(i). Suppose that u and v are consecutive zeros of p_{n+2} with $u < v$. Then, from (3), we see that

$$(u - b_{n+1})(v - b_{n+1})p_{n+1}(u)p_{n+1}(v) = \lambda_{n+1}^2 p_n(u)p_n(v).$$

It follows that $b_{n+1} \in (u, v)$ *if and only if*

$$\left(\frac{p_n(u)}{p_{n+1}(u)}\right)\left(\frac{p_n(v)}{p_{n+1}(v)}\right) < 0.$$

Now, by interlacing, there is exactly one zero of p_{n+1} in (u, v), and the function $p_n(x)/p_{n+1}(x)$ changes sign as x passes through this zero of p_{n+1}. Since $b_{n+1} \in (u, v)$ for exactly one choice of consecutive zeros u and v, we now see that each of the $n + 1$ intervals $(x_{i,n+2}, x_{i+1,n+2})$ contains either (i) exactly one zero of p_n but not b_{n+1}, or (ii) b_{n+1} and no zeros of p_n. This result is related to Stieltjes Theorem [2, p.46], and is discussed further in [1].

Next, each interval $(x_{k,n+1}, x_{k+1,n+1})$, $k = 1, \ldots, n$, contains exactly one zero of p_{n+2} (namely $x_{k+1,n+2}$), and exactly one zero of p_n (namely $x_{k,n}$); this follows directly from the interlacing property. The ordering of these two zeros within $(x_{k,n+1}, x_{k+1,n+1})$ is not immediately clear but, as we shall now show, it is completely determined by b_{n+1}.

Theorem 4 *In the notation given above, for $k = 1, \ldots, n$,*

$$x_{k,n+1} < x_{k+1,n+2} < x_{k,n} < x_{k+1,n+1} \quad \textit{if and only if} \quad b_{n+1} < x_{k,n}; \quad (7)$$

$$x_{k,n+1} < x_{k,n} < x_{k+1,n+2} < x_{k+1,n+1} \quad \textit{if and only if} \quad x_{k,n} < b_{n+1}. \quad (8)$$

Proof. We begin with the observation that $p_n(x_{k,n+1})$ and $p_{n+1}(x_{k,n})$, $k \in \{1, 2 \cdots, n\}$ have the same sign. Since $\lambda_{n+1} > 0$, and

$$\begin{aligned} p_{n+2}(x_{k,n+1}) &= -\lambda_{n+1} p_n(x_{k,n+1}), \\ p_{n+2}(x_{k,n}) &= (x_{k,n} - b_{n+1}) p_{n+1}(x_{k,n}), \end{aligned}$$

it follows that p_{n+2} has opposite signs at $x_{k,n+1}$ and $x_{k,n}$ if and only if $b_{n+1} < x_{k,n}$. Since $x_{k+1,n+2}$ is the only zero of p_{n+2} that lies between $x_{k,n+1}$ and $x_{k,n}$, this implies (7). Finally, (8) is logically equivalent to (7). ■

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