Exploring pre-service mathematics teachers' knowledge of learner thinking within the Lesson Study context

by

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Declaration

I, Victor Chakawanei (17261890), declare that the dissertation titled *Exploring preservice mathematics teachers' knowledge of learner thinking within the context of Lesson Study* which I hereby submit for the degree in Magister Educationis at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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The author, whose name appears on the title page of this dissertation has obtained for research described in this work the applicable research ethics approval.

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Dedication

This dissertation is dedicated to my loving wife Ziyanda Chakawanei, whose unwavering support, patience, and understanding sustained me throughout this challenging journey. Your boundless encouragement and sacrifices were the cornerstone of my perseverance, and I am forever grateful for your enduring love.

To my three precious daughters, Anotida, Anopa, and Atida, who often missed their father due to my relentless studies, your resilience and understanding have been my motivation to persevere. I hope this milestone serves as a testament to the importance of education and determination.

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In conclusion, this dissertation is a culmination of the collective efforts and support of these individuals and organisations. Your contributions have been indispensable, and I am truly honoured to have had the privilege of working with you all.

Abstract

One of the outcomes that newly qualified teachers seek to achieve is to know their learners and how they learn and think. Knowledge of student thinking (KoST) gives teachers the ability to assess how well learners understand mathematical concepts and read trends in misconceptions, and therefore, develop insightful strategies to address the misconceptions accordingly. In the current interpretivist qualitative case study, I used Lesson Study, which is an approach that foregrounds collaboration among a group of teachers, to explore mathematics preservice teachers' KoST during their work integrated learning. I explored the KoST phenomenon during specific Lesson Study stages, namely the lesson planning stage, the lesson presentation stage, and the post-lesson reflection stage. I answered the following research question: How do mathematics preservice teachers integrate KoST into teaching and learning within the Lesson Study cycle? A purposively drawn sample involved a team of four preservice teachers from one independent initial teacher education institution in South Africa. Data were collected through document analysis, participant and non-participant observation, and unstructured interviews. Data analysis was done deductively and inductively. The findings revealed that preservice teachers to a certain degree consider the elements of KoST during the lesson planning, lesson presentation, and post-lesson reflection stages. However, they prioritised and implemented these elements to a greater extent during the actual lesson presentation and observation of the Lesson Study cycle. In other words, preservice teachers place more emphasis on incorporating learner thinking during the active teaching process than during the planning and reflection stages.

Keywords: Lesson Study, preservice teachers, knowledge of learner thinking, mathematical thinking, Elements of learner thinking

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6 October 2023

To Whom It May Concern:

This letter is to confirm that *Exploring Preservice Mathematics Teachers' Knowledge of Learner Thinking Within the Context of Lesson Study* by Victor Chakawanei was edited by a professional language practitioner. It requires further work by the author in response to my suggested edits. I cannot be held responsible for what the author does from this point onward.

Regards,

Karien Hurter

List of Abbreviations

CCK	Common Content Knowledge
DHET	Department of Higher Education and Training
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
KoST	Knowledge of Student Thinking
MKT	Mathematics Knowledge for Teaching
NCTM	National Council of Teachers of Mathematics
PCK	Pedagogical Content Knowledge
PST	Preservice Teachers
SCK	Specialised Content Knowledge
SLT	Situated Learning Theory
WIL	Work Integrated Learning

Description of Key Terms

Learner: A term used in the South African context to refer to young children in formal education settings. In other contexts, learners are referred to as students.

Preservice teachers: Individuals who are preparing to become teachers but have not yet completed their formal teacher training and certification requirements. In the context of my study, this term specifically refers to student teachers who are in their 2nd year of study.

Mathematical thinking: This is a cognitive process that involves reasoning, problem-solving, and critical thinking. It is a way of thinking that involves the use of logic and quantitative reasoning to solve problems and make decisions. Mathematical thinking is not just about doing mathematics but also about applying mathematical concepts to real-world problems (Goos & Kaya, 2020).

Lesson Study: A teacher development model characterised by teachers collaboratively formulating long-term learning goals (Louws et al., 2017).

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CHAPTER 1: INTRODUCTION

1.1 Introduction

Teacher training programmes seek to produce teachers who can learn continuously, take responsibility, adapt to different situations, use various resources, and reflect on their practices. Teacher education programmes strive “to help novice teachers develop a lifelong ability to learn to teach and learn from teaching” (Sims & Walsh, 2009, p. 725). By so doing, they ensure newly qualified teachers have the potential to make a contribution to improving the quality of education. Murata and Pothen (2011) maintained that newly qualified teachers take time to execute in practice what they learnt from teacher education programmes. The long time taken by teachers to execute what they learnt is attributed to preservice teachers’ (PSTs’) lack of exposure to learners, which poses a limitation on how they understand how content and pedagogy connect to learning (Murata & Pothen, 2011)

Teacher educators found the notion of authentic learning through research-based lessons helpful to address the gap between theory and practice (Darling-Hammond & Lieberman, 2013). Susiani et al. (2018, p.2) defined research-based lessons as “a learning model which is associated with such activities as analysing, synthesising, and evaluating, and enables learners and lecturers to improve their assimilation and application of knowledge”. The use of research-based lessons facilitate analysis and understanding of what transpires in the classroom (Angelini & Álvarez, 2018). Research-based lessons are central to Lesson Study, which is the context that underpinned this study (Angelini & Álvarez, 2018).

Çelik and Güzel (2018) suggested that Lesson Study can be effectively used to improve mathematics teachers’ knowledge of student thinking (KoST). KoST gives teachers the ability to assess how well learners understand mathematical concepts and read trends in misconceptions, and therefore, develop insightful strategies to address the misconceptions. In South Africa, the term ‘learner’ is used in the schooling system instead of the term ‘student’, and therefore, the concept of KoST, universally used in literature, is synonymously used with knowledge of learner thinking in the current study. Çelik and Güzel (2018) conducted a Lesson Study with three high school mathematics teachers in Turkey to develop their KoST. The designed Lesson

Study increased the teachers' awareness of KoST. Every Lesson Study element engenders specific PSTs learning, and the interactive and collaborative nature of Lesson Study enables PSTs to share experiences and learn from each other and the knowledgeable other. The iterative and collaborative nature of the Lesson Study cycle enables PSTs' to develop more realistic lesson plans and redefine their focus on learner thinking (Dyer & Sherin, 2016). Lesson Study allows PSTs to be observers, which allows them to develop the skills to capture learning moments and interpret learner responses (Meiliasari, 2013). The reflection session enables PSTs to see the lesson deeper and focus on learner thinking (Meiliasari, 2013).

1.2 Problem Statement

The Minimum Requirements for Teacher Education Qualifications (MRTEQ) is a South African policy that guides initial teacher education programme design. According to the Department of Higher Education and Training (DHET, 2015, p.64), newly qualified teachers are expected to "have sound subject knowledge", "know who their learners are and how they learn", and to "be able to reflect critically on their own practice, in theoretically informed ways and in conjunction with their professional community of colleagues in order to constantly improve and adapt to evolving circumstances".

JET Education Services in collaboration with the DHET initiated the Initial Teacher Education Research Project that aimed to reflect on initial teacher education programmes across South Africa, focusing on Mathematics, English and teaching practice (Deacon, 2016). Looking at five universities, the Initial Teacher Education Research Project study found that "the overall quality of initial teacher education remains questionable" (Deacon, 2016, p.18). Initial teacher education institutions, and especially individual lecturers, have great autonomy to determine how to teach the content (Deacon, 2016). Bertram (2011) revealed several and complex reasons for the little change in the quality of South African education following government professional development initiatives. Bertram (2011)singled out ignorance about the kinds of knowledge, which include KoST, needed by teachers and the different ways of acquiring these knowledge types.

There is a discrepancy between the set outcomes by the DHET and the end product from initial teacher education institutions. Even though KoST is one of the competencies set out by the DHET, during practice PSTs tend to focus more on classroom management and learners' prior knowledge (Çelik & Güzel, 2017, 2018). The initial teacher education programmes provide theoretical background for teaching, yet teaching practice in a classroom is a complex experience during which one teacher must process a deluge of information, which makes it difficult for PSTs to apply theories in practice (Meiliasari, 2013). Murata and Pothen (2011) maintained that newly qualified teachers take time to execute in practice what they learnt from teacher education programmes. Other scholars (Korthagen, 2010; Meiliasari, 2013; Sims & Walsh, 2009) posited that they take time because of the difference between courses in universities and the reality of applying knowledge in the classroom.

1.3 Purpose and Rationale of the Study

The main purpose of the study was to investigate mathematics PSTs' KoST during their work integrated learning (WIL) using Lesson Study. In pursuit of the main purpose, the study sought to investigate PSTs' KoST during lesson planning, lesson presentation, and post-lesson reflection. There is a common agreement that effective teachers have distinct knowledge of learner's mathematical thinking and ideas (Hill et al., 2008). Therefore, focusing PSTs' attention on learner thinking through Lesson Study may help produce effective, lifelong, transformative, and influential teachers, thereby improving the quality of education.

An understanding of learner thinking has great benefits for both teachers and learners in the classroom. It is analogous to giving directions to a stranger to get to where you are, because in order to give directions, you must know where the stranger is. The developmental process of KoST ideas results in changes in teachers' instructional practices and beliefs, and increases student achievement (Wilson et al., 2013). It is of paramount importance that teachers be in a position to know and understand learner thinking for them to adequately help learners understand mathematics concepts. Even though KoST is one of the competencies set out by the DHET, An and Wu (2012) discovered that KoST gives teachers the ability to judge how well learners understand mathematical concepts and read trends in misconceptions, and therefore, develop insightful strategies to help them accordingly. For this reason, establishing best practices for teacher KoST will broadly assist the education fraternity.

1.4 Research Questions

The study sought to address the following primary research question: How do mathematics PSTs infuse KoST into teaching and learning within the Lesson Study cycle?

The primary research question was addressed using the following secondary research questions:

- Which features/components of KoST are reflected in the PSTs' lesson plans?
- How do PSTs apply the features/components of KoST during the teaching of a mathematics lesson?
- To what extent do PSTs consider the features/components of KoST when reflecting on the lesson?

The secondary research questions sought to address the primary research question and each question addressed the unique stages of the Lesson Study cycle. Through the first secondary research question, I sought to determine the features of KoST that found expression in the collaboratively planned lesson, which is the research lesson during Stage 2 of the Lesson Study cycle. Through the second secondary research question, I sought to investigate how the features/components of KoST permeate the teaching of the research lesson during Stage 3 of the Lesson Study cycle. Finally, through the third secondary research question, I sought to explore the extent to which PSTs use KoST features in their reflection session during Stage 4 of the Lesson Study cycle. Essentially the three secondary research question are sequential because they are located within the consecutive stages of the Lesson Study cycle. The stages of the Lesson Study cycle are presented in Figure 2.2.

1.5 Literature Review

The literature review provided context for my study by summarising existing knowledge, theories, and concepts related to the topic of study. It also helped substantiate my arguments and demonstrate the need for my study. The literature review also helped me formulate the research questions based on existing gaps and findings revealed by previous research. During the literature review, as a way of getting insights into the title of my study as well as the research questions, I identified the following headings that I address in detail in Chapter 2:

- Lesson Study as a teacher development model
- Lesson Study in PST education
- Learner mathematics thinking
- PSTs' KoST during lesson planning
- PSTs' KoST during the teaching of the research lesson
- PSTs' KoST during research lesson reflection

I also delved into the theoretical framework and selected two lenses to examine my study, namely the situated learning theory (SLT) and mathematics knowledge for teaching (MKT).

1.6 Research Methodology

The structure of my methodology was informed by the research onion proposed by Saunders and Tosey (2007). The research onion is a metaphorical representation of the layers involved in conducting research, and each layer represents a different methodological component. The components of the research onion are research philosophy, research approach, methodological choice, research strategy, time horizon, and techniques (sampling) and procedures (data collection and analysis). I deal with all these methodological components constituting the research onion in detail in Chapter 3. In addition, I address the issues of quality criteria or trustworthiness such as credibility, transferability, dependability, and confirmability. To uphold high ethical standards, I also considered sound ethical practices that include informed consent, confidentiality, anonymity, safety, and the right to withdraw from the study at any time. I discussed the quality criteria and ethical considerations in detail in Chapter 3.

1.7 Chapter Outline

The dissertation was structured into five chapters as follows:

- *Chapter 1: Introduction:* I introduce the research topic, context, and objectives. I outline the research problem, questions, and significance, and provide an overview of what the study aimed to achieve.
- *Chapter 2: Literature review:* In this chapter, I review existing literature related to the research topic. I summarise key theories, concepts, and empirical studies relevant to my research to provide a foundation for my study.

- *Chapter 3: Research methodology:* In this chapter, I explain the research philosophy, the research approach, the methodological choice, research strategies, time horizon, and the techniques and procedures I used to conduct the study. I also justify the chosen methodology, discuss issues related to the trustworthiness of the data and findings, and discuss the ethical considerations.
- *Chapter 4: Presentation of research findings:* I present the actual research findings based on the collected data in this chapter.
- *Chapter 5: Discussion of Results:* In this chapter, I discuss the research findings, offer interpretations, contextualise these within the broader literature, and answer the research questions. Furthermore, I explore the implications of the findings, their significance, and how they contribute to the field.

CHAPTER 2: LITERATURE REVIEW

2.1 Introduction

The purpose of this study was to investigate mathematics PSTs' KoST during their WIL conducted using Lesson Study. This chapter includes a review of the relevant literature on research relating to the infusion of KoST by PSTs during lesson planning and the research lesson, and how PSTs consider KoST during reflection session. This chapter is organised according to the structure depicted in Figure 2.1.

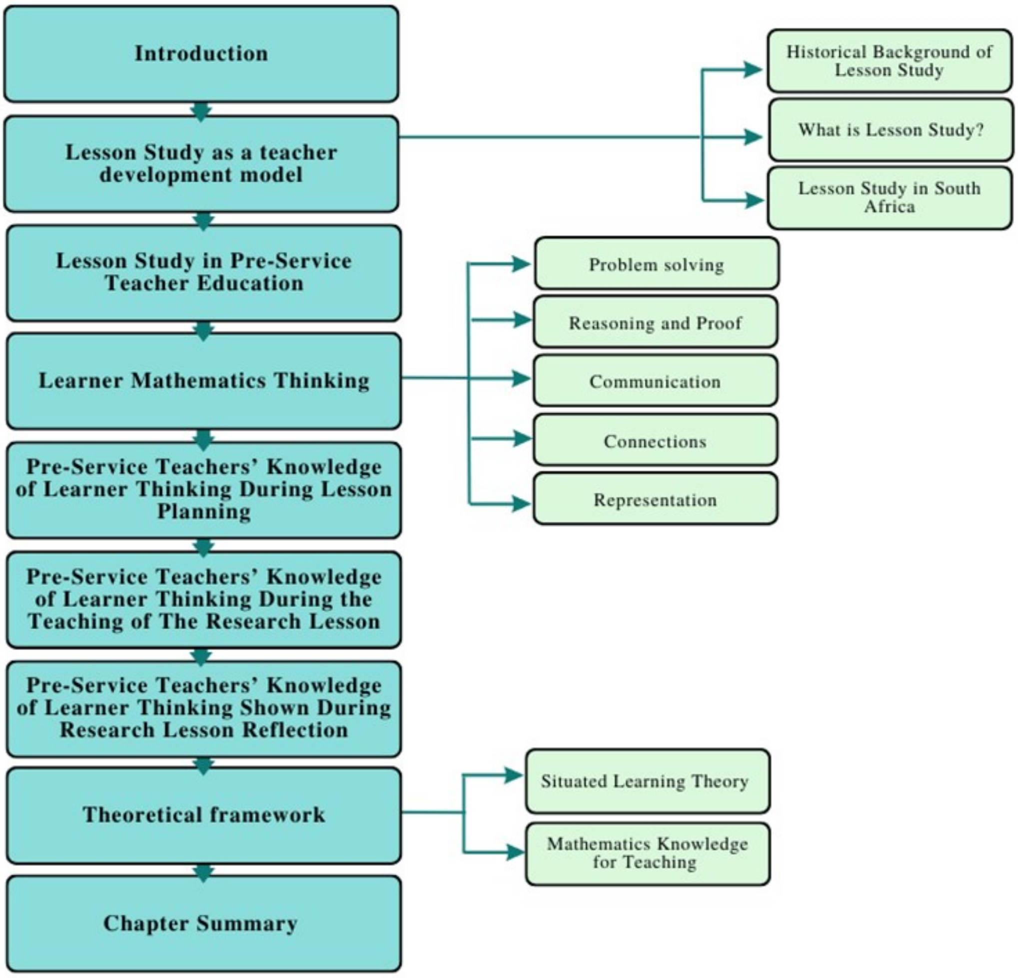

Figure 2.1 Outline for Chapter 2

2.2 Lesson Study As a Teacher Development Model

2.2.1 Historical background of Lesson Study

Japan's economy transitioned from one focused primarily on agriculture to one centred on industrial production from the 1860s to the early 1900s (Larssen, 2019). This period, called the Meiji period, witnessed the modernisation of the of the country's resources, including the education system, and a universal public education system was introduced in Japan (Makinae, 2019). The Meiji government introduced a new teaching approach called "the object lesson approach" which was based on the Pestalozzian theory that stated that "all cognition is based on one's intuition, and intuition is absolutely essential for human cognition" (Makinae, 2019, p.172). The Pestalozzian theory believes that humans intuitively recognise things, and therefore, build concepts. Intuition begins with the image humans receive through their senses. Teachers, therefore, expect that their students will learn intuitively (Makinae, 2019). Teaching children with concrete, authentic objects and occurrences from their immediate world helps them learn more efficiently, spontaneously, and reflectively, rather than through the application of generalisations and abstract concepts (Larssen, 2019). This is based on the notion that children already have the capacity to learn and develop. Teaching must be changed by using natural things to emphasise children's internal cognition (Makinae, 2019).

During the Meiji period in Japan, teacher training colleges known as normal schools, assumed the responsibility of spreading innovative teaching methods. This included training teachers, as well as the editing and publishing of textbooks and instructional guides. During this time, they introduced a practice known as the 'criticism lesson' in colleges. In a criticism lesson, college students would present a lesson to their peers, who would then watch and engage in discussions about the lesson. This observation and discussion process focused on four critical aspects namely method, matter teacher and children. Remarkably, these four points of criticism continue to be relevant in today's Lesson Study approach.

Through regular school supplies like instruction manuals, new textbooks, and newly graduated student teachers, the object lesson technique and criticism lesson were disseminated throughout the country. Over time, the use of the criticism lesson evolved from being used for PST preparation to being used for in-service professional development. This transition marked the genesis of Lesson Study in Japan.

2.2.2 What is Lesson Study?

Since its inception over a century ago, Lesson Study has been recommended as a suitable model for teacher professional growth and learning (Takahashi & Yoshida, 2004). It is a teacher developmental model that improves learner achievement and provides a framework for teacher professional development (Chokshi & Fernandez, 2004; da Ponte et al., 2022, 2023; Takahashi & Yoshida, 2004). Lesson Study is a process-based model characterised by teachers who collaboratively formulate long-term learning goals for learners (Louws et al., 2017). Amador and Carter (2018) defined Lesson Study as a cyclical process during which a group of teachers look into an issue, set a goal, design a lesson to address the problem, evaluate how the lesson was taught, and then meet afterwards to reflect on how the lesson can be improved with respect to learners' thinking. Sekao and Engelbrecht (2021) defined Lesson Study as a teacher-driven research model in which a group of teachers collaboratively work together honing a specific subject area to improve learning. All these definitions converge on the primary characteristic of Lesson Study, which is teachers seeking to improve teaching and learning by carefully observing each other as they present research lessons and then engage in critical debate (Makinae, 2019).

Due to different classroom contexts throughout the world, Lesson Study is being adopted and adapted in order to meet the different challenges. In order to achieve developmental influence, group members should include a number of crucial elements when carrying out research using Lesson Study. The following critical elements form the typical Japanese Lesson Study (Larssen (2019):

- Research goal(s) that address issues with teaching or learning.
- The *Kyouzai Kenyuu* (literally, "studying the teaching materials") component of Lesson Study.
- After finishing their *Kyouzai Kenyuu*, the group incorporates this information into the research lesson planning.
- A volunteer group member instructs the research lesson while the other members observe as the lesson is being taught.

- The role of the knowledgeable other: This participant is a guest to the Lesson Study group. The guest is a critical friend and provides support by giving advice during *Kyouzai Kenyuu*, offering a critical view of the group's proposed course of study, observing the presentation of the research lesson, and giving guidance during post-lesson discussion/reflection.
- Dissemination of the results: This can be accomplished easily with a presentation of the findings from each group or by sharing reports.

These key elements pointed out by Larssen (2019) are also echoed and highlighted by Takahashi and McDougal (2019). Drawing from the vast experience in Lesson Study as a teacher, Takahashi warned against omitting important Japanese Lesson Study steps as this affects the effectiveness of Lesson Study as a teacher development practice (Takahashi & McDougal, 2019).

Lesson Study has proved to be the backbone for improvement in teaching and improvement in the Japanese curriculum (Takahashi, 2014a). It has since been viewed as an integral part of teaching and is perceived to be "like the air" by Japanese teachers (Fujii, 2014). In instances where the national curriculum in Japan is revised, its effective implementation is facilitated using Lesson Study across the country (Takahashi, 2014b).

The Trends in International Mathematics and Science study video study and the ensuing book by Stigler and Hiebert (1999) brought Lesson Study to the attention of the international educational community (Larssen, 2019). Ever since that publication, educators around the world have recognised its power to help teachers improve their instruction and have resolved to adopt Lesson Study in their own countries (Takahashi, 2014a). Diverse adaptations of Lesson Study worldwide are primarily influenced by different theoretical traditions, which in turn influence the different educational cultures and policy contexts (Norwich, 2018).

Despite the tendency of some literature to portray Lesson Study as being a static shape and procedure, there are various versions that try to use Japanese Lesson Study to meet a range of classroom difficulties (Larssen, 2019). The interest in Lesson Study has now extended among non-Japanese educators and scholars from the United States to the Asian Pacific Economic Cooperation countries, the Middle East, and Europe as a result of the growing number of published articles and books written in English (Fujii, 2014). Lesson Study has spread to other countries, including the Philippines, Cambodia, Indonesia, Laos, Thailand, Ghana, Egypt, and South Africa (Angelini & Álvarez, 2018).

Vermunt et al. (2019) stated that Lesson Study provides a comprehensive set of guidelines and processes that assist teachers' professional development. It comprises several parts, namely identifying improvement goals/diagnostic assessment; collaborative research lesson preparation; teaching and observing research lessons; joint post-lesson discussion/reflection; and lesson improvement. In their quest to find the impact of Lesson Study on teacher learning with 214 teachers, Vermunt et al. found that Lesson Study had positive benefits on meaning-oriented and application-oriented teacher learning but that they struggled with educational innovation. Lewis et al. (2013) revealed that teachers in Lesson Study teams increased their understanding of mathematical content, improved their ability to elicit and analyse learners' thinking, developed a greater curiosity for mathematics and learners' thinking, emphasised learners' independent problem-solving, and increasingly used multiple representations to solve mathematical problems.

Çelik and Güzel (2018) conducted a Lesson Study with three high school mathematics teachers in Turkey to develop their KoST. The designed Lesson Study increased the teachers' awareness of KoST. Carter et al. (2016) performed a modified model of Lesson Study that was used to explore its potential to develop PSTs' ability to notice learners' science thinking. The six PSTs in their study demonstrated some aspects of professional noticing throughout the five Lesson Study cycles of early teaching practice. The PSTs noticed patterns in learner thinking and were able to respond to learner thinking. I provide some insight into what constitutes learner thinking in section 2.4.

Despite its global prominence and adaptation, there are some challenges associated with the implementation of LS which some researchers have documented. This is due to the fact that the cultural or educational contexts of other countries differ significantly to the Japanese context where LS originated and thrived. Adapting LS to local contexts while maintaining its core principles can, therefore, be a complex task (Sekao & Engelbrecht, 2021; Takahashi & McDougal, 2016; Widjaja et al., 2017). For instance, LS may be susceptible to *free-riding effect,* which refers to a situation where some participants may benefit from the process without contributing significantly, thereby undermining the collaborative nature of LS (Sekao, 2023). The *free-riding effect* is not a prevalent phenomenon among the Japanese teachers because for them LS is an entrenched practice, i.e. “for Japanese [teachers] LS is like air, part of everyday school life.” (Fujii, 2016, p. 411).

Furthermore, some teachers tend to feel reluctant to teach the lesson while being observed by their colleagues, which can undermine the effectiveness of LS (Sekao & Engelbrecht, 2021), especially as it pertains to observing learners’ thinking and documenting information that ought to inform post-lesson reflection. In addition, the effectiveness of LS can vary based on how it is implemented in different contexts. Factors such as school leadership support, teacher engagement, and alignment with school goals can influence the outcomes of LS initiatives (Sekao, 2023; Takahashi & McDougal, 2016).

LS is a collaborative process that involves planning, observing and refining lessons among teachers. When schools are geographically isolated, it can be challenging to establish and maintain effective lesson study groups that include teachers from different schools. Isolated schools may have limited resources to support the logistical implications involved (Halvorsen et al., 2021; Sekao, 2023).

2.2.3 Lesson Study in South Africa

In South Africa, Ono and Ferreira (2010) successfully applied Lesson Study to Mathematics and Science lessons to introduce Lesson Study in South Africa. Between 2013 and 2016, Adler and Alshwaikh (2019) conducted a case study of Lesson Study in South Africa through the Wits Maths Connect Secondary Project. Their primary emphasis was on exemplifying effective mathematics teaching practices. However, the limited timeframe posed significant constraints on Adler and Alshwaikh's implementation of Lesson Study in South Africa, which was confined to face-to-face interactions lasting six hours after school, spread across three consecutive weeks. There is not much literature about Lesson Study in South Africa since it is still a fairly new practice in the country, and there is even less literature on its use in PST education.

I used a five stage Lesson Study model adopted from Sekao and Engelbrecht (2021) to suit the South African theoretical traditions and context, which is depicted in Figure 2.2.

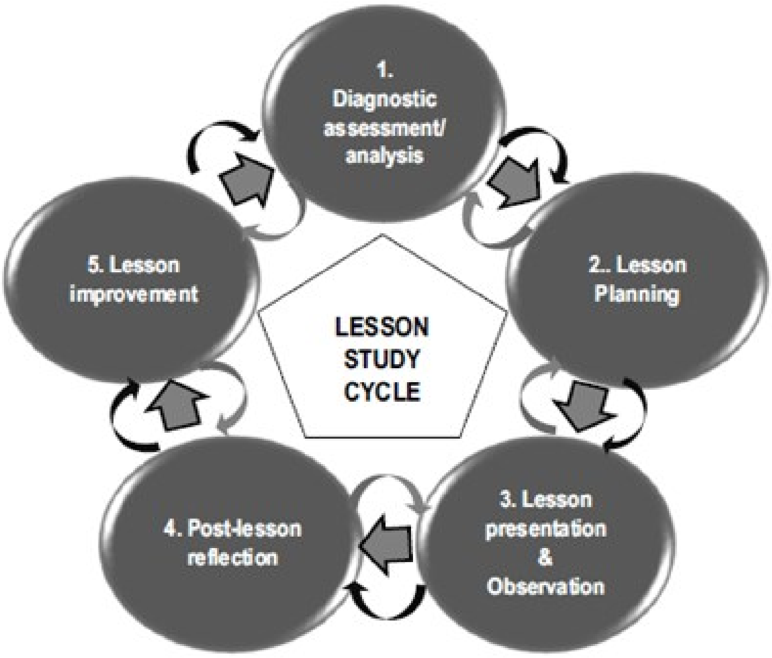

Figure 2.2 The Lesson Study cycle (Source: Sekao & Engelbrecht, 2022).

Every stage of the Lesson Study cycle has distinctive features that afford different experiences to teachers (Sekao & Engelbrecht, 2021). The first stage of diagnostic assessment/analysis allows teachers to analyse learner responses from prior assessments to find common misconceptions that emerge or they can identify challenging topics when teaching and use it to guide the Lesson Study session (Sekao & Engelbrecht, 2021). This implies that members of the Lesson Study group (teachers) come up with the overall goal of their Lesson Study.

During the second stage, teachers collaboratively plan a research lesson as a group. The critical aspect called *Kyouzai Kenyuu*, which translates to the study of materials for teaching, are useful here (Larssen, 2019). Teachers take the goals set during the first stage into account and look at them from a variety of viewpoints, such as the curriculum and the methods or assignments given to the students for them to learn (Larssen, 2019). This stage requires brainstorming methods and strategies to teach the identified topic (Sekao & Engelbrecht, 2021). During this period of designing a research lesson, teachers predict learner thinking and how learners will respond to the lesson activities and teaching methods that will be used (Myers, 2012). During the lesson preparation, members of the Lesson Study group go into greater depths in the research lesson, such as the phrasing of activities and how the teacher will explain some concepts (Myers, 2012).

At Stage 3, one participant-teacher teaches the lesson in front of their colleagues, who observe the lesson proceedings and carefully observe learner engagement and learning (Ustuk & Çomoglu, 2019). At this stage, learners' mathematical thinking is the main unit of observation. A knowledgeable other may be invited to observe and serves as a critical friend to support the Lesson Study group (Larssen, 2019).

After the observation is Stage 4, which is the post-lesson reflection during which members of the group converge to reflect on the lesson and discuss what was observed in light of the predicted outcomes (Lee, 2019). Teachers engage in reflective practices to investigate and understand their teaching practices and the learning experiences of their learners. Reflection is a powerful concept for teachers' professional development, allowing them to look back on their practices and make meaning of their experiences (Collet & Nakawa, 2022).

Stage 5 is lesson improvement and entails modifying the original lesson by considering what the Lesson Study group members discussed. The revised lesson is re-taught by a different member of the same group to different learners (Lee, 2019) . However, re-teaching the lesson is not mandatory, and it depends on what a particular Lesson Study group intends to achieve (Sekao, 2023). Finally, the improved lesson and reflection report is shared among with stakeholders, including the knowledgeable other and group members (Sekao, 2023).

The iterative nature of Lesson Study affords group members a chance to introspect on individual strengths and weaknesses and to adjust their practice to improve (Myers, 2012).

2.3 Lesson Study in Preservice Teacher Education

While there is documented history of the use of Lesson Study by in-service teachers, it has recently become popular to use with PSTs despite a continuum of hindrances for its implementation with PSTs (Botes et al., 2022; Burroughs & Luebeck, 2011; Chikiwa & Graven, 2019; Lamb, 2015a; Leavy & Hourigan, 2018; Lewis, 2019; Meiliasari, 2013; Mostofo, 2014; Murtafiah & Lukitasari, 2019; Myers, 2012; Saran, 2018; Süral, 2019). Lesson Study is typically practiced by experienced teachers in an in-service setting because it is meant to be collaboratively led and requires voluntary participation (Lewis, 2019). Student teachers' learning may be significantly influenced by their relationship with their mentor teachers, yet negotiating collaborative relationships can be challenging for them due to differences in Lesson Study experiences and power issues (Ni Shuilleabhain & Bjuland, 2019). The logistical constraints of accommodating PSTs' college schedules into the mix multiply the complications of using the Lesson Study model with PSTs (Burroughs & Luebeck, 2011; Mostofo, 2014; Ni Shuilleabhain & Bjuland, 2019). Furthermore, PSTs do not stay long enough at practice schools to form part of a stable community of practice that aims to build knowledge and experience (Lewis, 2019). PSTs must locate a willing classroom of learners, make travel arrangements, and learn enough about the educational needs of the class. PSTs' relative lack of experience of learners and partial knowledge of curriculum pose challenges when implementing Lesson Study with PSTs (Lewis, 2019). Making time for Lesson Study in a course that already lacks enough time to cover content is a challenge for PSTs. The strength of Lesson Study is often realised in its ongoing conduct through numerous cycles, yet time is an issue for PSTs (Lewis, 2019). In order to produce worthwhile gains for PSTs, Lesson Study requires extensive adaptations.

Burroughs and Luebeck (2011) conducted a qualitative study aimed at examining the results of involving PSTs with in-service teachers in a collaborative Lesson Study. They sought to understand the experiences of 24 PSTs (eight secondary and 16 elementary) and 10 in-service teachers. The Lesson Study process was a novel experience for both PSTs and the in-service teachers. The participants were grouped into groups of 4 to 7, and they spend 20 hours working collaboratively to develop an overarching lesson theme, plan and teach research lessons, observe, reflect, and ultimately, revise and re-teach the revised lessons to learners. Including PSTs in Lesson Study gave them opportunities to be critical thinkers in the context of mathematics education. The PSTs acquired ideas about how lessons are developed and enacted, realised the need to attend to learner understanding throughout the lesson, and demonstrated their ability to reflect. PSTs had difficulties anticipating learner responses that did not play out in the classroom.

Lamb (2015a) explored a Lesson Study model that constituted only PSTs aimed at determining its efficiency in facilitating pedagogical learning among the PSTs. The participants were 17 PSTs who were in their final year of completing a post graduate certificate in Secondary Physical Education. During school placements, PSTs adapted action research methodology as they participated in Lesson Study with peers. The findings revealed that the PSTs had more confidence in their understanding of learners' needs, incorporated reflective practices, acquired content and pedagogical knowledge, and learnt how to plan lessons. Lesson Study created a conducive environment for peer learning that augmented a better understanding of learners' needs and PSTs understanding of how to teach.

Mostofo (2014) investigated the effects of Lesson Study on mathematics undergraduate PSTs. The PSTs were first introduced to Lesson Study as they moved from the methods classroom to WIL. The study was conducted with PSTs from a private university specialising in mathematics. The class had a total of 8 PSTs and only six participated in the study. These PSTs were introduced to the Lesson Study process in their methods classroom. The six PSTs collaboratively crafted research lessons and individually taught the lessons during the methods classes and during field work. They all participated in the joint weekly reflections and surveys. The findings indicated that Lesson Study effectively improved and enhanced the efficacy of PSTs because of the opportunity they had to observe others as they teach, the chance to enact and teach the research lesson, and the iterative and collaborative nature of Lesson Study. Mostofo (2014)found that Lesson Study is an effective framework to bridge PSTs from a methods class to the field experience classroom, which in turn affords PSTs a chance to gain confidence before the actual teaching experience.

There is not much literature about Lesson Study in PST education in South Africa since it is still a fairly novice practice in the country. Botes et al. (2022) conducted a study to explore the perceptions of Lesson Study support intervention among a group of six PSTs within a South African university's school of education. Their research revealed that several facets of the Lesson Study approach had a lasting impact on the PSTs. They also found that PSTs had feelings of commitment, dedication, and motivation, which are elements of collaborative practice, knowledge sharing, and reflective practice.

2.4 Learners' Mathematics Thinking

Mathematics educators who become conscious of learner thinking are in a better position to transform learning and teaching to be significant for learners and for themselves (Çelik & Güzel, 2018). Teacher responses that incorporate knowledge of learner mathematical thinking and engage all learners in mathematical understanding increase learner participation and mathematical achievement (Ing et al., 2015). The American mathematics education reform document, the National Council of Teachers of Mathematics (NCTM, 2014) advised teachers to regard and position learners as authentic mathematical thinkers, and therefore, create activities that engage learners in mathematical thinking and working cooperatively. But what exactly is learners' mathematics thinking?

Several studies (An et al., 2004; An & Wu, 2012; Ferrini-Mundy, 2000; Lee, 2006; Mainali, 2021; Özaltun, 2014; Wilson et al., 2013) delved into KoST and listed its different components. For example, An et al. (2004) explained KoST with four components, namely considering learners' misconceptions and errors, building on learners' mathematical ideas, engaging learners in mathematics learning, promoting learners' thinking mathematically. Follow-up research by Lee (2006) added four more components, namely triggering different ideas, motivating students' learning, evaluating students' understanding, and using prior knowledge. Özaltun (2014) devised a framework that included nine components of KoST. These components encompassed aspects, including building upon learners' mathematical concepts, encouraging their mathematical thinking, stimulating and acknowledging divergent thoughts, involving learners in mathematical learning, assessing their comprehension and leveraging prior knowledge, fostering their motivation for learning, addressing misconceptions and errors, considering their difficulties, and anticipating their potential ideas and approaches.

Notwithstanding the various views pertaining to the elements of student/learners mathematical thinking, the NCTM (2000) proposed some principles and ambitious standards for school mathematics and identified five standards for learner thinking which I subscribe to in my study. Scusa (2008) and Sekao (2023) echoed these standards by illuminating the indicators of mathematical thinking of which teachers must be cognisant. The following are the five standards (NCTM, 2000):

- Problem-solving;
- Reasoning and proof;
- Communication;
- Connections; and
- Representation.

These standards are discussed in the following subsections.

2.4.1 Problem-solving

Ferrini-Mundy (2000) defined problem-solving as engaging in a task for which there is no predetermined solution. Dindlyal (2009) considered problem-solving as a task that elicits activity from learners and by so doing, they learn mathematics. Problem-solving is a tool to deepen learners' understanding and knowledge of mathematics (Vistro-Yu, 2009). Since the world is new to young children, naturally they exhibit problem-solving abilities, demonstrating curiosity, intelligence, and adaptability when encountering unfamiliar situations. (Ferrini-Mundy, 2000). It calls for teachers to possess this knowledge in order to nurture and harness learners' inherent problem-solving tendencies while fostering a mindset that appreciates problem-solving. Teachers can motivate learners to utilise the new mathematical concepts they acquire to create complex problems, cultivate diverse problem-solving strategies, and acquire the skills to reflect and monitor their own solutions.

By engaging in problem-solving activities, learners develop thought processes and develop habits of persistence, curiosity, and confidence when faced with new situations (Ferrini-Mundy, 2000). Carefully designed problems, which Scusa (2008, p.11) called "good problems" can give learners the chance to reinforce and expand what they already know and encourage mathematics learning. The aim of problem-solving is for learners to be able to reason and prove (Vistro-Yu, 2009). Reasoning and proving are two of the highest levels of skill in the hierarchy of mathematics skills (Vistro-Yu, 2009). A major duty of teachers is to choose and adapt worthwhile problems and mathematical tasks that may lead to development of reasoning and proving and the extension of mathematical ideas.

(An et al., 2004; Çelik & Güzel, 2017) identified the components of student thinking, and among them are building on learners' mathematical ideas and promoting learners' thinking of mathematics. Since problem-solving is an innate ability in children, it is the duty of teachers to build learners' ideas to enhance learning. Good problem-solving questions chosen by the teacher build on learner knowledge and promotes and extend mathematical thinking and knowledge. By continuously exploring a problem and solutions, learners are capacitated to generate more sophisticated ideas and achieve a deeper level of mathematical knowledge and understanding (Vistro-Yu, 2009).

Teachers should create an environment that treasures the learners' work in class, promote problem-solving, help make learners' strategies explicit, give ample time for learners to think, believe that learners are capable of solving problems, and listen attentively to learners' explanations (Ferrini-Mundy, 2000). It is imperative that teachers gather information through conversations and learners' work and use the information to assist individual learners in the class. Teachers may routinely encourage learners to reflect upon, explain and justify their answers. The practice is essential, as problem-solving leads to and confirms learner understanding (Ferrini-Mundy, 2000).

Bailey (2022) conducted a study aimed at investigating the development of teaching mathematics through problem-solving among beginning teachers in New Zealand. This qualitative research utilised observations, interviews, and focus groups to gather data from three novice teachers over a two-year period. The findings revealed that solely participating in professional development centred on problem-solving did not significantly shift these teachers toward a problem-solving-based pedagogy. However, they identified a specific lesson structure that was effective in supporting novice teachers to teach mathematics through problem-solving. They found that attending subject-specific complementary professional development initiatives alongside colleagues from their school, and having an in-school colleague who also teaches mathematics through problem-solving were important factors in sustaining the use of problem-solving in teaching mathematics. Additionally, the study suggested that a promising path for enhancing the pedagogy of teaching mathematics through problem-solving involves both novice and experienced teachers learning together. This approach supports learners in developing profound conceptual understandings and is considered an effective means of accommodating diverse learner needs. Therefore, neglecting problem-solving in the classroom may restrict learners' opportunities to cultivate advanced thinking, reasoning, and problem-solving skills, which are crucial for their future success in mathematics and other academic subjects.

Leavy and Hourigan (2020) conducted a study to assess the impact of a mathematics education course on the problem-posing skills of 415 prospective primary teachers in Ireland. They employed a pre- and post-test approach to gauge the participants' comprehension of what defines a mathematical problem and their capacity to formulate mathematical problems suitable for primary learners in Grades 1–4. The results showed that the mathematics education course had a significant positive effect on the problem-posing skills of the prospective primary teachers. They showed significant improvements in their ability to pose good problems, including a greater awareness of the openness of problem-solving situations, the use of various strategies, and the possibility of multiple solutions. The study highlighted the importance of developing skills to identify mathematically worthwhile problems and recommended enhancing problem-posing experiences to support prospective teachers in developing the knowledge and understanding required to pose mathematically worthwhile problems. The participants' initial conceptions of a 'good' problem included attention to context, children's abilities and prior knowledge, and the challenge set by the problem.

2.4.2 Reasoning and proof

Mata-Pereira and da Ponte (2017) asserted that reasoning is synonymous with thinking. The mathematical reasoning process includes formulating questions and solving strategies, and creating and testing generalisations and other conjectures (Mata-Pereira & da Ponte, 2017). People who reason and think analytically often observe patterns, structure, or regularities, and they often question whether these patterns are accidental or if they have a purpose, and they theorise and provide evidence to support their theories (Ferrini-Mundy, 2000). A mathematical proof is a formal technique of articulating specific types of explanation and reasoning. Scusa (2008) summarised and condensed the process of reasoning and proving as involving the formulation of conjectures, testing these conjectures, and subsequently providing proofs or reasoned explanations. Reasoning and proving are two of the highest level skills in the hierarchy of mathematics skills (Vistro-Yu, 2009). Even though young learners work from a small knowledge base, their logical reasoning start before school and constantly transforms as they encounter new experiences. Teachers should therefore uphold an environment that values, nurtures, and inspires students so they maintain the knowledge that the world should make sense.

Learners might lack the correct mathematical discourse, have their own ways of finding mathematical results, and convince themselves that they are true. They employ a variety of strategies to support their answers, which include perception, empirical evidence, and brief logical chains of deductive reasoning based on previously accepted facts. Learners make conjectures and reach logical conclusions that are defensible from their perspectives (Ferrini-Mundy, 2000). Usually learners naturally generalise from examples, and teachers should guide them to use examples to prove their generalisations. Being able to explain one's reasoning is an important skill that teachers should nurture in their learners.

Challenging tasks that have multiple solving processes are fertile grounds for disagreements, which fosters mathematical reasoning (Mata-Pereira & da Ponte, 2017). Learners must be given an environment that allows them to make conjectures, time to prove or disapprove these, and be able to explain and justify their ideas. Proposing a number of challenging tasks within a limited timeframe could potentially demotivate and weary many learners (Mata-Pereira & da Ponte, 2017). Teachers can also extend their learners' reasoning by probing beyond learners' observations and asking for arguments to support their answers. Classrooms should be rich with concrete materials to afford learners opportunities to manipulate objects to discover and demonstrate general mathematical truths using authentic examples (Ferrini-Mundy, 2000).

Mata-Pereira and da Ponte (2017) presented design principles for an intervention aimed at enhancing learners' mathematical reasoning in the classroom. The intervention focused on promoting generalisation and justification in mathematical reasoning processes, particularly in Grade 7 lessons on linear equations and functions. The paper identified four types of teacher actions in whole-class discussions, namely inviting actions, informing/suggesting actions, supporting/guiding actions, and challenging actions. They suggested that a combination of design principles is useful to promote generalisation and justification in mathematical reasoning processes. Teacher actions, such as inviting learners to participate, challenging or guiding them to participate, encouraging learners to share their ideas, and asking them to go beyond the task, can lead learners to generalise and justify their responses. Teacher actions play a crucial role in enhancing learners' mathematical reasoning, particularly in promoting generalisation and justification in mathematical reasoning processes.

2.4.3 Communication

Communication is a way of clarifying understanding and sharing ideas (Ferrini-Mundy, 2000). Communicating about mathematical ideas is a way for learners to express, explain, arrange, and consolidate their thinking. A learner who struggles to communicate cannot explain their thinking (Scusa, 2008). The communication process helps build meaning and cement ideas. In class, communication happen orally, with gestures, or with pictures, objects, symbols, letters, and drawings. Through communication, teachers are able to elicit learners' mathematical thinking and to give them instant and consolidated feedback (Sekao, 2023). Communication supports learning and allows teachers to identify misconceptions. An et al. (2004) and Çelik and Güzel (2017) mentioned the component, considering learners' misconceptions and errors as one of the components for learner thinking. Communication requires learners to reflect on their work and articulate their beliefs about the concepts introduced in the lesson, and therefore, communication exposes learners' misconceptions and errors. (Sekao, 2023) emphasised the need to encourage learners to communicate freely using correct mathematical language to explain their solutions, whether correct or wrong. Through communication, learners are better able to comprehend the meanings of different mathematical concepts and are more likely to remember them (Sill & Smith, 2017).

Communication makes mathematical thinking observable and it can be developed (Ferrini-Mundy, 2000). Effective classroom discourse support requires teachers to provide a space where learners feel comfortable sharing their opinions. Teachers should examine and talk about both excellent and problematic pieces of mathematical writing on a regular basis. Hence learners should be given the opportunity to talk and write about mathematics on a daily basis. Opportunities to communicate can be expanded through the use of learning resources, enriched use of mathematical language, and interactions between classmates through group work (Ferrini-Mundy, 2000). A learner who is able to communicate mathematically is able to seek clarification, has the ability to explain their thinking concisely, is conscious that it is acceptable to struggle and make mistakes in Mathematics and is capable of asking others to explain when they come up with new ideas (Scusa 2008) It is the duty of teachers to help learners to talk about mathematics and explain their strategies and answers. By so doing, teachers should be diligent in giving opportunities that allow varied forms of communication.

Learners' ability to follow, analyse, and reflect on what others say in class are essential for the learning process and understanding of content. Therefore, when learners encounter challenges in comprehending their peers' communications in the classroom, teachers can assist by prompting them to express their reasoning in simpler terms, fostering clarity for both themselves and their peers. Teachers should aim to cultivate a collaborative learning environment where learners not only communicate their mathematical concepts with the teacher but also with one another. It is the responsibility of the teacher to actively comprehend the messages learners are conveying and promote both individual and collective intellectual growth within the class. (Ferrini-Mundy, 2000).

Shan et al. (2014) conducted an empirical analysis to examine the influence of effective communication, sharing of achievements, and fostering positive classroom environments on learning performance. They aimed to reveal how these factors affect learning performance in school learners and fill the gap in limited research that examines the effectiveness of interactions between learners and teachers, achievement sharing, and the classroom environment. Shan et al. emphasised that when teachers create a sense of community, quickly respond to the needs of the learners, and foster positive relationships, learners are more engaged and enthusiastic about learning and their academic achievement tends to improve. Shan et al. (2014)suggested that teachers can use several strategies to improve communication and create a positive classroom environment, including offering better opportunities and more convenient channels for interaction between the institution and learners; understanding what learners need; praising them when they perform well; encouraging them when they succeed; avoiding criticism; being cheerful, enthusiastic, and warm; and giving learners more opportunities to interact with each other.

Ambreen (2021) conducted research to investigate children's perceptions of ability-based group work within a primary school setting. The study involved 27 primary school children, aged 9 to 10, representing diverse backgrounds including British, Asian, Eastern European, and African heritage. The classroom was organised into five distinct ability groups, with children assigned to these groups based on assessed academic levels. The findings indicated that children held mixed opinions regarding ability-based group work and that they wanted more opportunities to independently participate in both group and individual work. They emphasised the importance of taking children's perspectives into account when designing group work activities and offering chances for them to participate in both collaborative and individual learning experiences.

2.4.4 Connections

When learners show connections in mathematical ideas, it reveals their deep understanding of mathematics. Even though teachers teach mathematics as a subject with separate strands of topics, it is an integrated field of study. Understanding of mathematics involves making connections. For young learners, the most important connection is between the intuitive, informal mathematics they learn through their experience and the mathematics they learn in school (Ferrini-Mundy, 2000). The other connections, such as between concepts, topics, mathematics and other subjects, and mathematics and everyday life are supported by the link between learners' informal experiences and formal mathematics experiences. Learners' abilities to experience mathematics as a meaningful endeavour rest on these connections.

These linkages can be facilitated by teachers in a number of ways, for example, highlighting the many situations in which learners experience mathematics both in and outside school. They should bring out connections between and among mathematical ideas that learners are gaining. Teachers should plan lessons so skills and concepts are taught in a way that connects learners' experiences. Young children usually connect new mathematical ideas to old ideas through the use of concrete objects (Ferrini-Mundy, 2000). Teachers should embrace and encourage learners' strategies to make connections among mathematical ideas and the way these ideas are represented. Teachers should constantly link relevant mathematical ideas during routine school activities by asking questions related to the current incident. When planning tasks for new concepts, teachers should incorporate topics previously taught to enable learners to forge connections with previously learnt mathematical concepts and procedures.

Connections are usually best made when learners are challenged to apply what they have learnt in projects and investigations (Ferrini-Mundy, 2000). By so doing, they formulate questions and inquiries and decide on best methods to gather, record and represent data. Teachers have the chance to interpret learners' understanding and interpretation of mathematical situations, but they must listen to learners to identify the connections that learners bring and they should use this information to further learners' mathematical skills and knowledge, thereby establishing new connections.

Selvianiresa and Prabawanto (2017) investigated the effectiveness of the contextual teaching and learning approach in improving the mathematical connection ability of primary school learners. The contextual teaching and learning approach in primary school mathematics involves presenting lessons in a way that is relevant to learners' real-life experiences and problems. The study was conducted in one primary school in the city of Kuningan, Indonesia, with two groups of fourth-grade learners. The first group consisted of 20 learners who received contextual teaching and learning instruction, and the second group consisted of 20 learners who received direct instruction. They found that the contextual teaching and learning approach is more effective in improving the mathematical connection ability of primary school learners compared to direct instruction. They also found that learners who received contextual teaching and learning instruction showed more improvement in their mathematical connection ability compared to those who received direct instruction. This was demonstrated by their ability to recognise and link various problems presented in the study and answer them in a structured and clear manner. They further found that learners who received direct instruction tended to memorise the steps of workmanship but were unable to relate the various problems that exist in the matter.

Da Ponte (2007) explored the increasing use of mathematical investigations and explorations in Portugal since the 1990s, and the impact it has had on learning, teachers' classroom practices, and teacher education. Da Ponte (2007)discussed the theoretical and practical foundations of investigations, the types of investigations that can be used in the classroom, and the challenges and benefits of implementing investigations. Investigations can help learners develop a deeper understanding of mathematical concepts and processes and can make mathematics more engaging and enjoyable for both learners and teachers. Regularly doing investigations with discussions of and reflections on the results and the processes used can influence learners' conceptions in a significant way. Da Ponte (2007)found that investigations can be a valuable tool for promoting learning and engagement in mathematics.

2.4.5 Representation

The term representation refers to the act of capturing mathematical ideas and the product itself (Ferrini-Mundy, 2000). The way mathematical ideas are represented determines how people understand and use the ideas (Ferrini-Mundy, 2000). In other words, representation refers to the processes and to the products noticeable externally and internally in the minds of the people doing mathematics. Representations should be considered crucial components to help learners understand mathematical relationships and concepts.

By examining learners' representations, teachers can acquire important insights into how learners perceive and think about mathematics. Teacher should develop a habit of creating links from learners' individual representations to more formal ones and give learner opportunities to construct, refine, and use their own personal representations as tools to support learning (Ferrini-Mundy, 2000). Representations help learners organise their thinking. Teachers should carefully examine learners' representations and listen to their conversations to get insights into the growth of mathematical thinking and to be able to support learners as they connect their languages to the formal language of mathematics. By studying, challenging, and interpreting learners' representations, teachers can learn more about how they think and how well they understand mathematical ideas.

One major responsibility for teachers is to create an environment that supports, accepts, and encourages the use of multiple representations. Teachers should help learners efficiently create and use a variety of representations. They can help learners consider other views and methods of expressing their thinking by encouraging learners to share their different representations.

Mainali (2021) emphasised the significance of utilising various forms of representation in the teaching and learning of mathematics. The adoption of multiple modes of representation is suggested to enhance the effectiveness of mathematics instruction. They discussed the different types of representation and their effectiveness in teaching mathematics. The appropriate use of representations in teaching mathematics is an important factor in student learning. Different types of representation, such as visual, symbolic, and concrete, can be used to teach mathematics effectively. Teachers should carefully select the most suitable representations and determine their optimal use based on the specific nature of the mathematical content being taught. The use of multiple modes of representation in teaching mathematics can help learners to develop a deeper understanding of mathematical concepts.

Selmer et al. (2021) explored how PSTs notice aspects of learner thinking in written work and which aspects they focus on. The research was conducted in a university mathematics method classroom where PSTs were asked to analyse written work from their future learners. They found that PSTs engaged in noticing aspects of learner thinking and focused on both disciplinary and non-disciplinary aspects. They also found that explicit questioning allowed PSTs to engage in describing, evaluating, interpreting, and responding about learner thinking without shifting to noticing more general aspects.

2.5 Preservice Teachers' KoST During Lesson Planning

According to Hill et al. (2008, p. 373), "in order to increase teachers' effectiveness and learners' outcomes, teacher education programs should focus on preparing teachers who have profound knowledge of reform-based mathematics content and knowledge of learners' mathematical thinking and learning". There are limited studies demonstrating that PSTs possess a knowledge domain of PCK, which Hill et al. (2008) termed "Knowledge of Content and Students" (KCS) or "teacher's knowledge of learner mathematical thinking and learning", which is widely believed to be an important component of teacher knowledge.

Despite the challenges faced by PSTs such as difficulty to anticipate potential learning challenges that school children may face during lessons, less knowledgeable about learner psychomotor and cognitive abilities, less learner focused and little knowledge about activities which facilitate learning, Lamb (2015a) witnessed a positive outcome in lesson planning by PSTs. In a quest to find the effectiveness of Lesson Study that is owned and managed by PSTs to facilitate mutual spaces of pedagogical learning between peers, Lamb (2015a) engaged 17 PSTs who were in their final year of completing a post graduate certificate in Secondary Physical Education. They found that PSTs had increased confidence in understanding the learners' needs, anticipatory reflections, ability to visualise content, ability to think through what to teach and teaching strategies.

Guner and Akyuz (2020) investigated how preservice middle school mathematics teachers noticed learners' mathematical thinking in the context of Lesson Study that was taken as a teaching practice component in a teacher education program. They aimed to reveal what PSTs focused on and how they interacted with things that caught their attention during the three major phases of Lesson Study (i.e. planning, teaching, and reflecting on the research lesson). They completed four Lesson Study cycles. They found a gradual improvement in noticing learner thinking in the PSTs' planning. Towards the end of the four Lesson Study cycles, the PSTs demonstrated their KoST by pointing out learners' potential mistakes and confusions that may arise, and guessing learner approaches and strategies during the planning phase. The PSTs went to the extent of justifying the reasoning behind possible learner strategies during planning. The PSTs considered a couple of elements of learner thinking in their planning.

Lee (2019) investigated the how PSTs noticed learner thinking expertise development during their teaching practice through Lesson Study over a period of six weeks. They selected a focal group of six elementary PSTs, and the PSTs were involved in planning and reviewing the lessons. An adapted Lesson Study approach was used and comprised four stages, namely exploration of learners' thinking, the initial research lesson planning, the enactment of the research lesson, and Lesson Study reflection sessions. They found that during the early stages of the Lesson Study cycles, PSTs were mainly focused on group organisation rather than learner thinking or lesson and instructional goals. It was evident that in the later Lesson Study cycles, PSTs' planning focused on ideas of learner thinking and identifying key mathematical ideas that influence instructional strategies adopted in the lesson. The PSTs established connections between different representations, and therefore, reinforced such ideas with relevant activities. They also catered for diversity by planning differentiated instruction and activities.

Sahin-Taskin (2017) conducted research aimed at finding what PSTs think about lesson planning and how they believe their plans help them teach effectively. Their study involved primary PSTs who were taking a teaching practice course at a faculty of education. There were 18 PSTs in the study, and the researchers collected data through semi-structured interviews. Their findings indicated that the PSTs found it difficult to find appropriate activities for learners' learning levels, interests, and needs. They also found it difficult to determine whether their planned activities met the learners' levels.

Ndlovu and Chiromo (2019) explored the development process of preservice mathematics teachers' conception of using concrete material to improve learners' conceptual understanding of mathematical. The study was conducted at the Department of Mathematics Education at the University of KwaZulu-Natal in Durban, South Africa. The research involved 60 foundation phase preservice mathematics teachers who were enrolled in a mathematics education module. They used a mixed methods methodology, encompassing pre- and post-tests, observations, and interviews, to gather data. Their findings suggested that PSTs' conception of using manipulatives gradually evolved over time, and their reasoning improved as they engaged in structured co-operative learning and effectively engaged in mathematics talk.

2.6 Preservice Teachers' KoST During the Teaching of the Research Lesson

Research lessons are a critical component of teacher preparation programs. They involve PSTs designing and conducting lessons with learners while being observed and evaluated. Effective research lessons require a comprehensive understanding of how learners think and learn, as well as the ability to adapt instructions accordingly (Cajkler, 2019).

Guner and Akyuz (2020) conducted Lesson Study as a component of teaching practice with PSTs to improve their noticing learner thinking. The teaching practice course was done in the final semester of a four-year teaching curriculum. They used the four Lesson Study cycles with a group of four PSTs. Guner and Akyuz (2020)chose to focus on one PST case in all the phases of the Lesson Study cycles. As witnessed in the planning phases of the Lesson Study cycles, the case PST did not notice learner thinking while teaching. During the first cycles of the Lesson Study, the case PST neither attended nor responded to learner ideas in ways that could reveal learner thinking. The PST ignored wrong answers from learners as she proceeded to solve the problems and did not ask for reasons behind learner strategies. The PST directly communicated the correct answers whenever she was correcting learner mistakes instead of prompting them to reason it out on their own. During the final Lesson Study cycles, the case PST showed some improvement in noticing leaner thinking as she attended to learner responses and encouraged them to reveal their thinking. The PST tried to prompt learners about the solution instead of telling them.

Morrissey et al. (2020) explored the nature of elementary PSTs' probing questions in relation to implementing problem-solving tasks. They aimed to understand the ways in which PSTs attempt to ask probing questions and to help redesign the explicit teaching of the questioning module based on the emerging types. The sample consisted of 115 PSTs in five sections of the course. The researchers designed and implemented an explicit questioning module involving a series of core activities to foster PSTs' ability to use purposeful probing questions. They found that while PSTs were able to ask probing questions that focus on context, meanings, or representations, some of their asked questions focused solely on information and procedures and did not probe into learner thinking. The majority of PSTs believed that asking questions related to information and procedures is equivalent to asking probing thinking questions, which was worrisome. This is because questions related to information and procedures do not consider the underlying conceptual understanding of learners and do not probe into learner thinking.

Ramaligela et al. (2019) conducted a study in South Africa that aimed to investigate the knowledge of instructional strategies and PCK of PSTs in mathematics and technology. They found that a majority of PSTs in both mathematics and technology exhibited limited PCK. In both subjects, the classroom practice of most PSTs did not reflect a comprehensive understanding of the subject matter or the needs of the learners. In the elaboration phase, four technology PSTs were able to execute the elaboration phase but only one mathematics PST was able to accomplish it. However, the PSTs were unable to coherently introduce new concepts and to connect concepts in a procedural manner.

2.7 Preservice Teachers' KoST During Research Lesson Reflection

Effective lesson reflection requires PSTs to demonstrate their understanding of learner thinking by identifying, among other things, cognitive challenges and misconceptions learners encountered during the lesson, the effectiveness of instructional strategies in promoting critical thinking, the strengths and weaknesses of the lesson, the evidence of learning, and the implications for future practice.

Guner and Akyuz (2020) conducted Lesson Study as a component of teaching practice with PSTs to improve their noticing of learner thinking. The teaching practice course was done in the final semester of a 4-year teaching curriculum. They conducted the four Lesson Study cycles with a group of four PSTs. As witnessed in the planning phases of the Lesson Study cycles, PSTs were focused on classroom management, time management, lesson implementation, learner behaviour, and teacher pedagogy during the reflection phases of the first cycles of Lesson Study. There were no reference to learner thinking during the first cycles of Lesson Study. The case PST attended to her own teaching approaches by addressing the challenges she faced. She referred to the entire class's mathematical understanding as opposed to individual leaners. There was a gradual improvement in the last cycles of Lesson Study during reflection as the PSTs were making inferences that considered KoST. The case PST referred to specific learner reactions and gave insights about the possible reasoning behind the reactions.

A similar study was conducted by Lee (2019) to investigate PSTs noticing learner thinking development during their teaching practice through Lesson Study over a period of six weeks. The PSTs were involved in planning and reviewing lessons. A focal group of six elementary PSTs were selected to conduct the research. An adapted Lesson Study approach was used that comprised four stages, namely exploration of learners' thinking, the initial research lesson planning, the enactment of the research lesson, and Lesson Study reflection session. They found that PSTs exhibited professional growth during the later Lesson Study reflections as they focused more on learners' thinking and responses than in the earlier Lesson Study reflections where they discussed learners' dispositions. In the later Lesson Study reflections, the PSTs spoke about learners' interesting ideas by mentioning specific events and activities. The PSTs referred in their reflections to individual learners' specific new approaches as evidence of mathematical concept understanding and they probed learners with question that triggered reasoning.

Amador and Weiland (2015) examined PSTs' noticing s during Lesson Study in order to describe their deductions on learner reasoning, their considerations of learners' mathematical thinking, and the consequent decisions the PSTs make. The study consisted of 24 PSTs who were clustered into four groups of six and the groups worked collaboratively to engage in an adapted Lesson Study process. Their findings indicated that most PSTs generated general statements and that generally descriptive statements were common when reflecting on lesson enactment. The PSTs spoke about the learners' focus on completing their work rather than about learner mathematical thinking. In rare cases when the PSTs made interpretive comments about classroom events, they would dilute these by referencing classroom management or specific actions from the lesson.

2.8 Theoretical Framework

I used two theoretical frameworks, namely the SLT and Shulman's MKT. The former framed the Lesson Study context and while the latter framed the KoST. The two theoretical lenses are briefly discussed in the following subsections.

2.8.1 Situated learning theory

Situated learning theory is a theory of knowledge acquisition. According to Besar (2018, p. 50), "situated learning theory, or at least elements of it, is emerging as a possible vehicle for revitalising the understanding of, and prescriptions for, how knowledge is developed and organised within workplaces". The theory assumes that knowledge must be explored in an authentic context, making social interaction and collaboration essential. Therefore, Lave and Wenger (1991) located learning in social interactions, or instances of co-participation and co-construction, as opposed to solely regarding it as the acquisition of certain forms of knowledge. By so doing, learning entails involvement and participation in 'communities of practice'.

Wenger (2011, p.1) explained the concept of communities of practice as "groups of people who share a concern or a passion for something they do and learn how to do it better as they interact regularly". Community members share, collaborate, and learn from one another face-to-face and electronically. New members gradually drift from the periphery of the community of practice to the centre and become more accustomed and active within the community, and therefore, becomes an expert. This transition usually occurs unintentionally. Lave and Wenger (1991) termed this transition the legitimate peripheral participation. Wenger (1998) stated that learning is unavoidable since failure to learn one thing means learning a different thing.

Lesson Study as a teacher development model best provides an authentic context for teachers to learn collaboratively in a community of practice where they freely explore and apply the KoST in their practice. The community of practice made possible by Lesson Study allows teachers to make sense of educational ideas such as KoST, and therefore, they learn to reflect on their own practice from the learners' perspective. SLT helped me respond to the "… the Lesson Study cycle?" aspect of the primary research question which states: How do mathematics PSTs use KoST during the Lesson Study cycle? SLT in this case helped me unpack Lesson Study as the context in which my study took place.

2.8.2 Mathematics knowledge for teaching

In his seminal work, Shulman (1986) made a distinction between two kinds of content knowledge, namely subject matter knowledge and PCK. Subject matter knowledge is obtained by participating in formal course work in a discipline (Kang, 2016). PCK is the combined form of knowledge in which a teacher actively integrates and synthesises different knowledge bases, topic-specific knowledge, teacher and student beliefs, and student learning outcomes (Coenders & Verhoef, 2019). Shulman (1986) further gave two dimensions for PCK, namely the conceptions and preconceptions brought by different learners to the classroom and the way in which subject matter is formulated and represented that makes it comprehensible to others. The notion of PCK was further refined and elaborated on by Ball et al. (2008) in their seminal work where they introduced the concept of Mathematics Knowledge for Teaching (MKT) (see Figure 2.3).

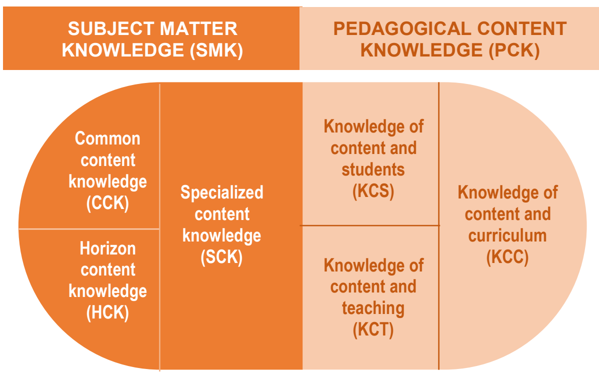

Figure 2.3: Domains of MKT (Source: Ball et al., 2008)

MKT consists of two domains, namely Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). SMK represents a pure form of content knowledge which is typically acquired through formal education and participating in a structured course work in a specific field. PCK represents a specific type of knowledge that encompasses the fundamental aspects essential for effective teaching. Each of the two domains is further constituted by three subdomains which I have clarified in the next paragraphs.

SMK comprises of Common Content Knowledge (CCK), Specialised Content Knowledge (SCK) and Horizon Content Knowledge (HCK).

Common content knowledge refers to the shared understanding of essential concepts, facts and axioms that are considered fundamental and commonly taught and widely accepted and recognised within mathematics. It is the mathematical knowledge and skills that find application beyond the teaching field (Ball et al., 2008).

Specialised content knowledge is knowledge and skills that are exclusive to teaching. Teachers unpack mathematical knowledge making features of specific content visible and accessible for learners (Ball et al., 2008). SCK is acquired through extensive study, practical experience and ongoing engagement with the subject matter. Teachers must however be able to choose, make and use mathematical representations effectively. For instance, they should be able to discern the benefits and drawbacks of utilising rectangles or circles for comparing fractions (Ball et al., 2008).

The other domain under SMK is *horizon content knowledge* which goes beyond the content knowledge as it encompasses an understanding of the interconnectedness of mathematical topics across the entire curriculum (Ball et al., 2008). It encompasses knowledge about current and future directions, evolving theories and innovative practices within mathematics.

PCK consists of Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of content and Curriculum (KCC). *Knowledge of content and students* is knowledge that amalgamates knowledge of learners and knowledge of mathematics (Ball et al., 2008). In other words, KCS entails understanding of both the subject matter (content knowledge) and the characteristics, needs, and abilities of the learners they are teaching (student knowledge). The KCS subdomain entails anticipating learner thinking, envisaging the kind of strategies that motivate learners and what makes a topic easy or hard for learners, and interpreting learners' incomplete thinking (Kang, 2016).

Knowledge of content and teaching is a combination of knowledge about content and knowledge about teaching (Ball et al., 2008). This knowledge enables teachers to assess the instructional benefits and drawbacks of various representations utilised for teaching a particular concept and determine the instructional opportunities provided by different methods and approaches (Ball et al., 2008). KCT provides teachers with the insights necessary to plan their teaching in a way that confronts misconceptions (Leavy, 2015).

The final subdomain under PCK is *knowledge of content and curriculum* which entails familiarity with the sequencing and organization of topics to be taught within mathematics. It involves having a comprehensive grasp of the intended learning outcomes, scope and sequence of topics, and the progression of skills and knowledge across different grade levels or educational stages.

As the framework in Figure 2.3 depicts, the six subdomains are categorised into two knowledge bases, namely SMK and PCK. Notwithstanding my focus on KoST, which is primarily situated within the KCS subdomain, the other knowledge subdomains cannot be disregarded because they form part of the overall model to understand KoST.

KCS features quite prominently in the framework called The Minimum Requirements for Teacher Education Qualifications (MRTEQ). In MRTEQ the "minimum set of competences" expected of a newly qualified teacher and the knowledge mix outlined for all initial teacher education institutions and programmes in South Africa (DHET, 2015) are outlined. This document stipulates KCS as one of the crucial knowledge domains that newly qualified teachers should possess. It states that newly qualified teachers are envisaged to "have sound subject knowledge", "know who their learners are and how they learn", and "be able to reflect critically on their own practice in theoretically informed ways and in conjunction with their professional community of colleagues in order to constantly improve and adapt to evolving circumstances" (DHET, 2015, p.64). The Initial Teacher Education Research Project in five South African universities by Deacon (2016) aimed to evaluate initial teacher education programmes all over South Africa, specifically focusing on mathematics, English and teaching practice, and they found that these institutions have great autonomy to determine how to teach the content. Even though the country has settled on the new Curriculum and Assessment Policy Statement, which together with government policies guide universities in designing their initial teacher education programmes, there is no basic national initial teacher education curriculum (Deacon, 2016). Bertram (2011) revealed several complex reasons for the little change in the quality of South African education following government professional development initiatives. Bertram singled out ignorance of the kinds of knowledge, including KoST, needed by teachers and the different ways of acquiring these knowledge types.

2.9 The relevance of the two theoretical lenses for my study

Although my study is primarily about exploring KoST, the context in which KoST is explored is the LS setting. It was therefore necessary for me to explore my study through two theoretical lenses. MKT directly addresses the issues of KoST that are reflected in my study's research questions, whereas SLT provides a theoretical base for LS. Therefore, using the two theoretical frameworks, i.e. SLT and MKT enabled me to fully respond to the research questions. Had I adopted single theoretical lens, either SLT or MKT, I would have experienced some limitations in exploring my study fully. In fact, if my study did not involve collaborative work among teachers, MKT alone would have provided a solid theoretical base for my study.

2.10 Chapter Summary

This chapter commenced by unpacking the background of Lesson Study with its roots in Japan and delved in great detail into its architecture. I then conducted a review of Lesson Study literature as enacted by PSTs in their planning, lesson presentation, and post-lesson reflection in light of KoST. Subsequently, I explained what learner thinking is and outlined its indicators of which the representative teacher teaching the research lesson should be cognisant of during the presentation. The chapter concluded with a description of the two theoretical frameworks adopted in this study, namely SLT and MKT.

CHAPTER 3: RESEARCH METHODOLOGY

3.1 Introduction

Nieuwenhuis (2016) asserted that methodology encompasses all the procedures a researcher use to collect and analyse data and describe and explain phenomena. Creswell and Poth (2018, p. 18) defined methodology as "the procedures used in the study". In other words, a research methodology explains how the researcher intends to answer the research questions and the practical steps of the research (Creswell & Poth, 2018). Therefore, this chapter answers the questions of how data were collected, from whom or where it was collected, and how it was analysed.

The research methodology adopted in this qualitative study was guided by the research onion framework coined by Saunders and Tosey (2007). The metaphor of a research onion, shown in Figure 3.1, illustrates the techniques used to collect data along with the procedures to analyse it.

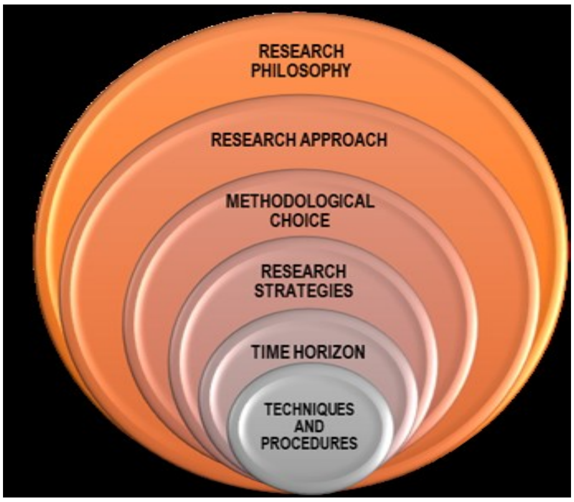

Figure 3.1 The research onion (adapted from Saunders et al., 2019)

The context and parameters within which data collecting techniques and analysis procedures were chosen were determined by my understanding of and decisions related to the outer layers of the research onion (Saunders & Tosey, 2007). The sections in this chapter describes how the layers or stages of the research onion were adapted in this study. This chapter addressed the methodology adopted to answer how mathematics PSTs infuse KoST into teaching and learning within the Lesson

Study cycle. The current chapter is organised according to the structure depicted in Figure 3.2.

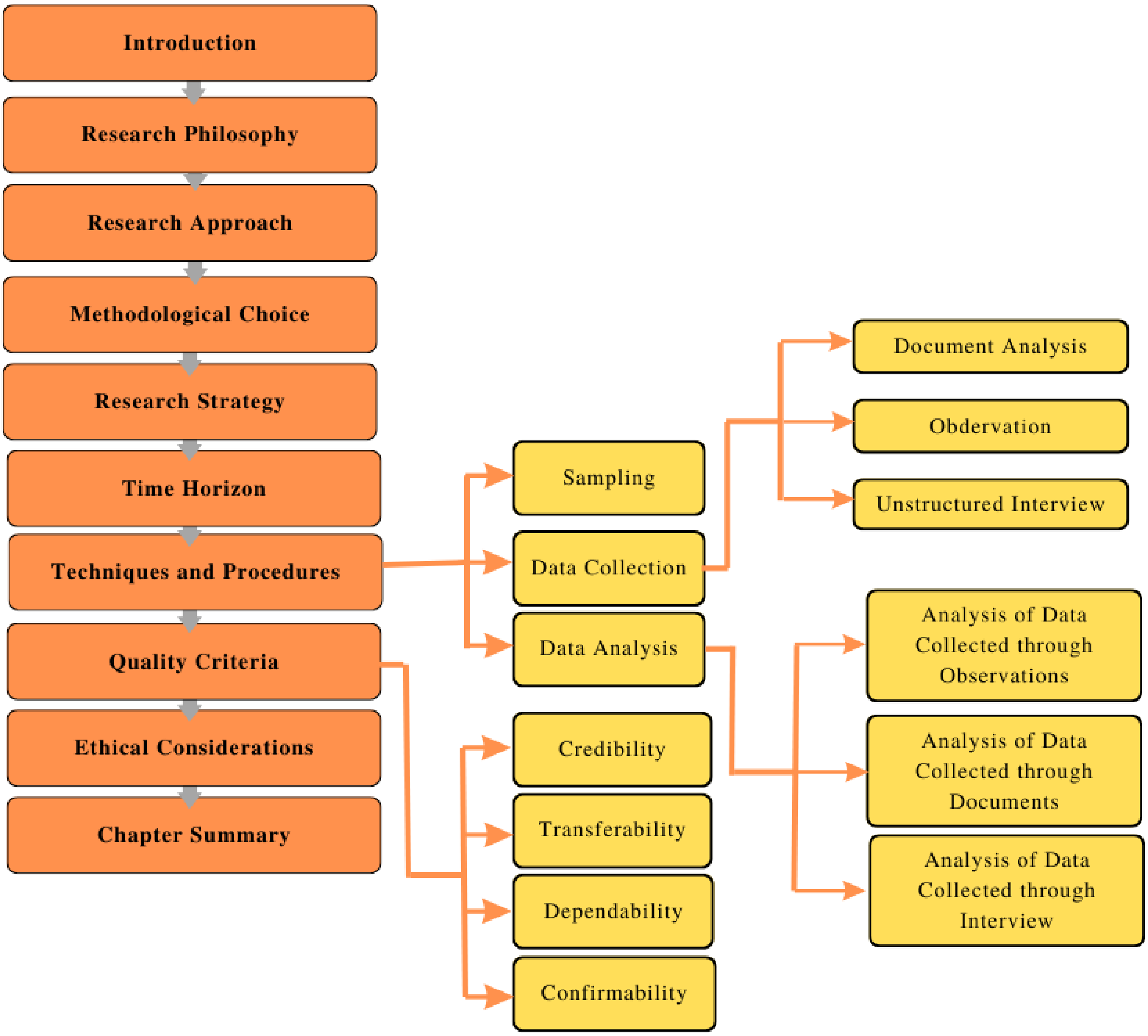

Figure 3.2 Outline for Chapter 3

3.2 Research Philosophy

The “taken for granted assumptions” and how a researcher views the world inevitably shape how they understand research questions and the associated research design (Saunders & Tosey, 2007). Whether researchers are aware of it or not, they always bring certain beliefs and philosophical assumptions to their research (Creswell & Poth, 2018). Saunders and Tosey (2007) maintained that a researcher's philosophy refers to their individual comprehension of what defines acceptable knowledge and the process through which it is created. These philosophies can be deep-rooted views about the nature of problems that need to be explored, what research questions to ask, or how data are gathered (Creswell & Poth, 2018). Philosophy is defined as "the use of abstract ideas and beliefs that inform our research" (Creswell & Poth, 2018, p.16).

The social world that social science scholars seek to investigate has two extreme and broad perspectives (Durdella, 2019). For some scholars, the social world is real, and they posit the existence of an objective world that is external to individuals, and therefore, the truth arises from a correspondence between a hypothesis and empirically observed facts. For other scholars, every individual creates their own social world, which means the social world is subjectively constructed, and therefore, a product of human cognition (Durdella, 2019).

I adopted an interpretivist perspective since it is particularly valuable when studying complex social phenomena, human behaviour, and subjective experiences. There is a greater chance of understanding people's perceptions of their activities when they are studied in their social contexts (Nieuwenhuis, 2016; Pulla & Carter, 2018). Interpretivism allowed me to delve into the depth and richness of Lesson Study where PSTs' interpretations and meanings of KoST were central to understanding the social phenomenon. Nieuwenhuis (2016) maintained that interpretivism proposes the existence of multiple explanations of phenomena, and that these realities differ across time and place. I chose to adopt interpretivism since it was well-suited for my research that aimed to explore multiple perspectives from PSTs and learners. Perspectives and perceptions are subjective in nature and are best captured by interpretive qualitative methods. Furthermore, the interpretivist paradigm adopted in this study assumes socially constructed multiple realities that are qualitative in nature. Interpretivism embraces the importance of cultural and contextual variations and does not conform to one-size-fits-all explanations (Durdella, 2019). The Lesson Study setting in this study was unique and represented a distinct context. Adopting an interpretivist approach positioned me to explore the complex interconnected system of human life to provide a deeper understanding of mathematics teaching and learning by PSTs.

The social world can only be understood from within, and therefore, it must be studied intersubjectively (Durdella, 2019). I immersed myself as a participant observer in some stages of Lesson Study (lesson planning, lesson presentation, and post-lesson reflection) in order to observe and experience the social interaction from the participants' (PSTs') perspectives and experience the subjective reality. Through observing lesson planning and lesson presentation, I came to an understanding of the authentic context (Lesson Study) to get the meanings that PSTs attach to different learner actions. In other words, I empirically observed the elements of KoST on the part of the PSTs at play from lesson planning, research lesson presentation, and post-lesson reflection.

Different societies have different beliefs and cultures that are directly determined by their environment, the inhabitants of the society, and how they interact among themselves (Durdella, 2019). Interpretivism posits that truth and reality are products of creation rather than outcomes of discovery (Rehman & Alharthi, 2016). It is unattainable to gain an understanding of reality as it is since it involves the mediation of our senses (Rehman & Alharthi, 2016). Nieuwenhuis (2016) echoed this assumption, stating that the human mind is the source of meaning, and therefore, researchers should develop a sense of understanding of the meaning revealed by people to phenomena in their contexts. In order to see the world as the PSTs saw it, I employed unstructured interviews during which I used multiple follow-up questions to get more clarity and meaning. I used multiple data collection methods, including document analysis of a lesson plans, observation, and unstructured interviews, for triangulation and acquiring meaning from multiple angles.

Another assumption of interpretivism is that the social world does not exist independently of human knowledge. The knowledge and understanding of researchers are limited to experiences they have encountered, prior knowledge, beliefs, and values, and therefore, meaning making is always a personal enactment (Nieuwenhuis, 2016).

Avenier and Thomas (2015) postulated that our comprehension of reality is exclusively acquired through social constructs like language, consciousness, shared interpretations, documents, tools, and various artefacts. Therefore, I used document analysis to gain insight about the KoST that PSTs pre-planned for their lesson. I used unstructured interviews during the post-lesson reflection stage to see the world as PSTs saw it.

3.3 Research Approach

The design of a research project is mainly determined by the extent to which it is concerned with theory testing or theory building (Saunders et al., 2019). Two contrasting reasoning approaches, deductive and inductive reasoning, are usually adapted. Deductive reasoning is a top-down approach that starts with a general theory or hypothesis and then narrows down to specific observations or predictions. Researchers begin with a theoretical premise and then make specific predictions based on that theory (Azungah, 2018; Creswell, 2014). Inductive reasoning is a bottom-up approach that starts with specific observations or data from participant experiences and then generalises to form a theory or hypothesis (Azungah, 2018; Creswell, 2014).

In this research, I adopted a sequential approach of deductive-INDUCTIVE where I initially applied a deductive approach. The lowercase and capital letters in the approach implies that this research is more inductively inclined than deductive. Since qualitative researchers operate within theoretical frameworks, pure induction is impossible (Azungah, 2018; Taylor et al., 2015). When conducting research, there are assumptions about the world that we bring into the research, and we approach the research with goals and questions in mind. I used the deductive approach to derive the initial codes from a combination of existing literature, and to derive the research aim and research questions. The inductive approach was used extensively in this study to read raw data and derive concepts and themes.

This study was guided by interpretivism as a philosophical approach, and therefore, using an inductive approach was inevitable as I was concerned with the meaning participants attach to events in their everyday contexts, and I therefore recorded finer details of their experiences that could not be reduced to statistical equations. By threading the two approaches together, I aimed to forge a robust and holistic research approach that aligned with my overarching research philosophy and was tailored to the unique contours of my study. While these two distinct approaches may seem to inhabit opposite ends of the research spectrum, they each offered unique strengths and capabilities.

3.4 Methodological Choice

This layer of the research onion highlights the important choice all researchers face when designing their research where they determine the use of quantitative, qualitative, or mixed methods approaches (Saunders & Tosey, 2007). These are the overarching strategies that influence which data collection and data analysis techniques can be adapted in a research study.

According to Maree and Pietersen (2016. p.162), "quantitative research is a process that is systematic and objective in its ways of using numerical data from only a selected subgroup of a universe (or population) to generalise the findings of the universe that is being studied". This research method is used to investigate numbers and statistics and is often used to measure differences between groups or relationships between variables as well as to testing hypotheses (Queirós et al., 2017). In a quantitative research, the data can be quantified (Queirós et al., 2017).

On the other hand, a qualitative research design focuses on aspects of reality that cannot be quantified, such as words, concepts, perceptions, or ideas. These are often used to gain an understanding and explanation of the dynamics of social relations (Queirós et al., 2017). This type of research deals with non-numerical data and deduce meaning from the data about a phenomenon (Creswell & Poth, 2016). In other words, qualitative research is any type of research that produces results not arrived at by statistical procedures. Creswell and Poth (2018) articulated the following common characteristics of qualitative research:

- Natural setting: Qualitative researchers gather data directly at the location where participants encounter the issue being investigated. Participants are not taken to labs or handed instruments to complete.
- Researcher as key instrument: Researchers in this case collect data themselves through document analysis, observing behaviour, and interacting with participants.
- Multiple methods: Researchers rely on multiple data source, and they typically gather various forms of data, such as interviews, observations, and document analysis.

- Participants' multiple perspectives and meanings: Qualitative researchers focus on learning the meaning that the participants hold about a problem. The participants' meanings further suggest multiple perspectives and diverse views.
- Context dependent: The research is situated within the context or setting of participants or sites.
- Emergent design: The original research plan cannot be overly rigid, as every stage of the process may undergo alterations once the researcher enters the field and commences data collection.

I used a qualitative research approach since it is a way of approaching the empirical world (Taylor et al., 2015). When studying people in their contexts, we tend to discover their intricate but finer experiences that can be lost when using other methodological choices. Therefore, reducing peoples' words and experiences to numbers, the human side of people's life gets lost. A qualitative research approach was appropriate for this study because of the detail required by the research questions that drove the study. I needed a complex detailed understanding of the PSTs' KoST; hence, I used the qualitative research approach that takes account of their thoughts. These details were best discovered through direct engagement with the PSTs in the classroom as they teach in a Lesson Study setup and analysed qualitatively. According to Creswell and Poth (2018, p. 46), "we cannot separate what people say from the place where they say it—whether this context is their home, family or work". This research involved observing elements of learner thinking during the research lesson presentation stage of Lesson Study, which created unquantifiable data that required subjective interpretation, and therefore a qualitative design was necessary. Moreover, during the post-lesson reflection session, the participants expressed their perceptions and perspectives in their own words.

3.5 Research Strategy

The label for this layer suggests that researchers can use one or more strategies within their research design as they plan how to answer the research questions (Saunders & Tosey, 2007). The research strategy adopted in this study was a case study focused on investigating four PSTs' KoST during their WIL conducted through Lesson Study. Lewis (2015) defined a case study as a research design in which cases, such as a program, event, action, or process, of one or more individuals are thoroughly analysed by the researcher. In simpler terms, Rule and John (2011) defined a case study as a systematic and in-depth investigation of a particular instance in its context in order to generate knowledge. A case study allows the researcher to closely examine a particular instance in greater detail due to the close collaboration between the researcher and the participant (case; (Nieuwenhuis, 2016). This research strategy offers flexibility and a focus on a specific system (case; (Durdella, 2019). It is most often preferred when answering 'how' and 'why' questions (Nieuwenhuis, 2016). The case study method, with its use of multiple data collection and analysis methods, provides the researcher with opportunities to triangulate data, thereby strengthening the research findings and conclusions (Nieuwenhuis, 2016).

The case in this research involved four PSTs, all of whom were in their 2nd year of pursuing a Bachelor of Education (Intermediate Phase) degree at a private higher education institution. These four student teachers experienced Lesson Study as a part of their academic program. They completed their first year of foundational coursework and were progressing to more specialised training in the field of education. The Intermediate Phase means they were preparing to become teachers who will teach learners in Grades 4–6. In their first year of studies, the PSTs had the opportunity to observe in-service teachers in action before they began their own teaching practice, also called WIL. Notably, all four of these student teachers were placed in the same school for their WIL experiences. The shared placement was significant as it allowed for a comparative analysis of their Lesson Study experiences within a consistent school context.

The school where they were place is situated in an urban area, indicating that it serves a community with diverse social and economic backgrounds. One significant feature of this school is its substantial access to educational resources. It is well-equipped with various facilities, materials, and educational tools that support the teaching and learning process. The school has a substantial enrolment of 1 500 learners, which is a relatively large learner population, and the school employs 62 teachers, which is a significant teaching workforce. The size of the teaching staff can influence the learner-teacher ratio, and subsequently, the quality of education. This primary school operates as a fee-paying institution, which means that parents or guardians are required to pay fees for their children's education. The physical infrastructure of the school is noteworthy. It includes well-furnished classrooms, which impacts the learning environment and the comfort of both learners and teachers. The specific focus of the study in this case was a Grade 4 class in the school. This grade level typically includes learners around the ages of 9 and 10, who are at a crucial stage of their primary education. There were 20 learners in the class, which is a manageable class size for effective teaching and learning.

3.6 Time Horizon

This layer defines the time frame over which the researcher undertakes the research (Saunders & Tosey, 2007). When research is undertaken to answer a question involving data collection at a specific point in time, it is called cross-sectional; and longitudinal research involves the collection of data repeatedly over a long period of time in order to compare the data (Saunders & Tosey, 2007).

This study adopted a cross-sectional time horizon since data were collected over a short period, which was one school term during which the PSTs were doing WIL. A cross-sectional time horizon was adopted since the research questions needed a snapshot of time to be answered. Only one complete Lesson Study cycle was sufficient to obtain the desired results. Relevant data were collected during the different stages of the Lesson Study cycle, namely collaborative lesson planning, lesson presentation, and a reflection session. One complete Lesson Study cycle was accomplished during the WIL period. Saunders and Tosey (2007) claimed that the cross-sectional time horizon is most likely to be used for surveys or case studies.

3.7 Techniques and Procedures

3.7.1 Sampling

Madondo (2021) defined sampling as the selection of participants or elements from a population. A population is the total number of persons or elements in a research site (Madondo, 2021). The target population was all PSTs who participated in Lesson Study during WIL at a private higher education institution at that particular time. A sample of four participants in this study were selected using non-probability sampling techniques, namely convenience and purposive sampling. In non-probability sampling, randomisation is not important when selecting the sample, and subjective methods are used to decide which elements are included in the sample (Etikan, 2016). Non-probability sampling has the advantages of being less expensive and of often being quicker to implement than probability sampling (Etikan, 2016).

Convenience sampling falls under the category of non-probability sampling, wherein individuals from the target population who meet specific practical criteria such as accessibility, geographical proximity, availability at a given time, or willingness to participate are selected for the purpose of the study (Etikan, 2016). With convenience sampling, cases are selected based on availability (Schreier, 2018). Convenience sampling is also referred to as accidental sampling because members of the sample happen to be situated, spatially or administratively, near to where the researcher is collecting data (Etikan, 2016). Schreier (2018) argued that any individual who fits the requirements to have the experience under investigation would be a good participant.

With purposive sampling, also known as judgement sampling, the researcher has a plan and only includes people who are appropriate for the study's objectives (Etikan, 2016). Purposive sampling involves selecting participants specifically for their personal attributes (Etikan, 2016). In simpler terms, the researcher determines what information is necessary and looks for those who can provide the needed information derived from experience or knowledge.

The combined use of convenience sampling and purposive sampling in this study was carefully chosen and justified based on the unique circumstances. As a lecturer at the private institution where the participants were enrolled, I had immediate access to the target group of PSTs. This made convenience sampling a practical and efficient choice, given the proximity and availability of the PSTs. However, not all students in this group had prior exposure to Lesson Study. To ensure a focused and manageable sample that met the research criteria, purposive sampling was applied. The selection process involved identifying PSTs who were proficient with Lesson Study, which was a crucial characteristic that aligned with the study's objectives. Additionally, convenience sampling was used to reach a broader pool of potential participants. Invitations were extended to all 21 Bachelor of Education (Intermediate Phase) students. Out of these students, 10 voluntarily expressed interest in participating in Lesson Study.

Ultimately, six students embarked on the Lesson Study journey, and four of them happened to be practising at the same school. The decision to include these four PSTs in the study was based on their suitability and ensuring that the sample represented individuals well-versed in Lesson Study practices and practising in the same school. It is important to note that the study primarily focused on the PSTs themselves, and that the learners in the school served as indirect participants due to the study's central emphasis on the PSTs and their experiences with Lesson Study. This combined sampling strategy was thoughtfully designed to effectively address the research questions within the given context.

3.7.2 Data collection

As I sought to investigate PSTs' KoST during their WIL conducted through Lesson Study, I employed three qualitative data gathering techniques, namely interviews, participant observation, and document analysis, to help me yield cogent information pertaining to the phenomenon under study. The study followed a five stage Lesson Study model adopted from Sekao and Engelbrecht (2021), and three of the stages were of interest to answer the research questions. These stages are lesson planning, lesson presentation and observation, and post-lesson reflection. The three stages were the main sources of data. The three data collection techniques that I used are discussed in the following subsections.

3.7.2.1 Document analysis

Karppinen and Moe (2019) viewed a document as a social phenomenon since it was made in a particular setting, with a particular intent, by a particular group of individuals, at a particular time and location. In addition to the fact that documents can be falsified or tampered with, they can include mistakes, omissions, and distortion. Documents can be kept, retrieved, and copied, making them more stable than the majority of other data sources (Karppinen & Moe, 2019). I analysed the lesson plans and learners' work for this study. Since teachers would anticipate learner thinking during lesson planning phase, my focus was to identify all the elements of learner thinking that teachers articulated in their planning (Annexure A). The kind of activities and exercises that PSTs planned to do in their lesson was another major focus. I compared the learners' work with the activities that were planned by the PSTs and looked at how these enhanced learners' thinking.

3.7.2.2 Observation

I acted as a participant observer during the lesson planning phase of Lesson Study. The participant observer gets into the situation and works with the participants to design intervention strategies (Nieuwenhuis, 2016). The researcher takes part in the situation under observation and may influence its dynamics or even try to alter it (Nieuwenhuis, 2016). Therefore, in order to acquire an insider perspective on the setting, the researcher fully immerses themselves and acquires an emic perspective (Nieuwenhuis, 2016). During the lesson planning stage, my role as a researcher primarily involved providing guidance and not actually planning the lesson for them. For instance, the participants were student teachers who possessed limited experience and understanding of lesson planning as a result, my role was primarily to provide guidance on the principles of LS related to planning, rather than doing the actual planning work with/for them. Student teachers didn't have access to different textbooks, therefore I made textbooks available for them and observed how they utilised these resources to produce a lesson plan. The PSTs planned to teach fractions in their research lesson presentation, and they had to anticipate and infuse learner thinking in their planning.

I acted as a complete observer during the research lesson presentation. In cases where a researcher is a complete observer, they are a non-participant and look at the situation from a distance (Nieuwenhuis, 2016). This kind of observation is the least obstructive form of observation (Nieuwenhuis, 2016). The volunteer PST presented her lesson with no distractions from myself as observer or the other group members. I used an observation tool (Annexure B) to mainly focus on capturing moments of learner thinking and how the presenter applied KoST during the lesson. For example, the presenter used an example of bread to present fractions. This was not included in the lesson plan but she noticed an element of divergent thought from a learner and had to intervene by using an everyday example. I made video recordings throughout the lesson presentation.

3.7.2.3 Unstructured interview

Unstructured interviews, sometimes referred to as open-ended interviews, are often in the form of conversations and done with the aim of showing the researcher the participants' views, ideas, beliefs, and attitudes about certain events (Nieuwenhuis, 2016).

During the reflection stage, open-ended or unstructured interviews were conducted in the form of a conversation with the intention to explore the participants' views and ideas and what they noticed during the research lesson presentation on learner thinking. As the PSTs were reflecting on the proceedings of the research lesson presentation, I interjected by asking questions such as 'how did the presenter invoke divergent thought during the presentation lesson?' and 'what do you think she could have done to understand what the learner was thinking?' The participants gave their insights into the proceedings of the research lesson, mainly focusing on learner thinking. I used elaboration probes to get the full picture and to get the participants to tell more about what they observed during the research lesson presentation.

3.7.3 Data analysis

There are a variety of qualitative data analysis methods, and I resorted to thematic data analysis. Kiger and Varpio (2020) defined thematic analysis as an approach for evaluating qualitative data, which entails exploring a data set to identify, scrutinise, and report recurring themes. Thematic analysis can illustrate how certain social constructs develop and searches for more latent and deeper themes within the data (Kiger & Varpio, 2020). When attempting to understand a set of experiences or behaviours across a data set, thematic analysis is a suitable and effective technique to employ (Kiger & Varpio, 2020). This method examines patterns of meaning within a data set, organising bodies of data based on similarities or themes. This method was useful in this study because three sets of data were generated. Triangulation was easy because of the themes from the three data sets.

I kept to the principle of clustering different data sets (observation notes, interview data, and document analysis) separately as suggested by (Nieuwenhuis, 2016). All collected data were transcribed verbatim, taking into consideration the conversation gestures and impressions since these are important. The next important step was coding of the data. A code is a word or brief phrase that symbolically ascribes a summative, prominent, essence-capturing, and/or evocative attribute for a segment of language-based or visual data (Saldaña, 2021). In simpler terms, a code is a label that describes a piece of content. Coding is a way of making sure the data is valid and ascertaining that data analysis is undertaken systematically and enabling the research to be reviewed by other researchers. Coding helps researchers quickly retrieve data and enables the examination and comparison of different cases (Nieuwenhuis, 2016).

I used deductive coding during codification. I used predetermined codes (*a priori*) based on the elements of KoST; for example, "what fraction of bean bags are red?" was coded as problem-solving. The deductive approach allowed me to undertake analysis with tightly focused lenses that enabled me to quickly identify relevant data and to avoid destructions and decoys.

My coding consisted of two stages, namely initial coding and line-by-line coding. In the initial coding stage, I got the general overview of the data using the initial set of *a priori* codes and then delved deeper into the data during the line-by-line stage. I also used the descriptive coding during the initial coding stage. The coding process was followed by code categorisation, which entailed reviewing the different codes and bundling similar/related codes into categories to help organise the data effectively. For example, the code 'connection between mathematics and other subjects' and the code 'incorporate topics previously taught' were bundled into the category 'connections'. I noticed themes from the categories; for example, problem-solving, connections and communication are some of the themes that emerged. The main findings were deduced from the categories/themes formulated.

3.7.3.1 Analysis of data collected through observations

I used a constructionist analysis approach to analyse the observations. A constructionist approach aims to uncover meaning-making processes that people in the field use to make sense of their world (Marvasti, 2014). According to Marvasti (2014, p.371), "instead of mining the data for general and enduring concepts or patterns, constructionist analyses highlight particular and contextually meaningful processes". Constructionist analysis is concerned with how participants create their social worlds using spoken and written words (Marvasti, 2014). With constructionist analysis in mind, I focused on analysing the evidences of KoST during lesson planning, lesson presentation, learners' work, and post-lesson reflection from the field.

3.7.3.2 Analysis of data collected through documents

My starting point for the document analysis was analysing the form and content of documents to determine how they function, the purpose the author intended the document to serve, and what the document is doing (Coffey, 2014). According to Coffey (2014, p.373), "the analysis of documents can therefore examine those cultural and organisational features that are implicitly invoked when records and documents are produced and used". The documents I analysed in this study were mainly lesson plans prepared by the PSTs and learners' written work. I mainly looked at how these lesson plans were crafted; which elements of KoST, such as problem-solving and representation, were included; how much the planned teaching activities probe learner thinking; whether the planned activities involve problem-solving; and to what extend the learning and teaching material enhances learner thinking.

3.7.3.3 Analysis of data collected through interviews

The interview data were captured through videos. The first step was transcribing all the data. The data were organised and labelled without including participants' names for confidentiality but in a way that ensured the data were accessible for analysis. The transcription was re-checked for accuracy. Re-checking transcribed data is time consuming but necessary to build confidence in the research data analysis and the interpretation of the findings (Galletta & Cross, 2013). Reading, transcribing, and organising the data revealed some themes that addressed the research questions (Galletta & Cross, 2013). The next step was to identify and label thematic patterns that reflect ideas evident in the data. These ideas are called codes (Galletta & Cross, 2013); for example, 'probing learners to reflect and justify their answer' emerged as a code. As I moved along with data analysis, a network of relationships between thematic codes was revealed. Clustering the codes under broader themes or categories brought new views to the data interpretation (Galletta & Cross, 2013). After formulating the categories, it is important to assess the codes' viability to respond to the research questions (Galletta & Cross, 2013). Data interpretation took greater meaning as I invested additional efforts in clustering codes.

3.8 Quality Criteria

If data are measurable, it is possible to measure its validity; however, qualitative data cannot be measured on a scale (Noble & Smith, 2015). It is imperative to address how qualitative researchers establish that their research findings are credible, transferable, confirmable, and dependable (Noble & Smith, 2015). Qualitative researchers speak of trustworthiness to judge the quality of qualitative research (Korstjens & Moser, 2018). Trustworthiness simply poses the question 'can the findings be trusted?' (Korstjens & Moser, 2018).

I used the four categories of credibility, transferability, dependability and confirmability to understand the means of establishing validity/trustworthiness in this study. The following subsections discuss how these elements were established.

3.8.1 Credibility

Credibility seeks to answer questions of how close the findings are to reality and how the findings can be easily believed by the reader (Nieuwenhuis, 2016). I used three forms of credibility, namely prolonged engagement, triangulation, and member checking. I invested a prolonged time into transcribing my own data (research lesson, post-lesson reflection interviews) that were in video formats. The observation lesson and the post-lesson reflection were transcribed word for word. These took a considerable time but the process was worth the investment. I did not read the text only once to create the themes but I spend hours with the transcriptions to generate themes before I claimed I understood how the four PSTs permeated KoST during planning, research lesson presentation, and post-lesson reflection. My subjects and I spent a considerable time interacting on Lesson Study as the context of this study, and we started when the students experienced their first WIL in their first year at college. Lesson Study was not new to them and neither were the subjects to me as researcher. Therefore, the time I spent with the data and the participants adds credibility to the current study.

Data triangulation combines data drawn from different sources and at different times, in different places or from different people (Flick, 2017). Triangulation entails the use of multiple lines of sight (Nieuwenhuis, 2016). There are four methods of triangulation, namely triangulation by methodology/data collection, triangulation by theory, triangulation by different researchers/coders/observers, and triangulation by despondence (Flick, 2018). In this study, I used triangulation by methodology. I used document analysis to extract data from the lesson plan the PSTs compiled. The main focus was investigating the features of KoST the PSTs incorporated in the lesson plan. The PSTs presented a lesson in the research lesson presentation stage of Lesson Study, and I observed how they applied their KoST during the lesson. During the post-lesson reflection session when the PSTs reflected on the lesson, I conducted interviews to determine how they consider KoST when reflecting on the lesson. I used three different data collection methods during three different stages of Lesson Study, and therefore, triangulation was achieved to validate the data.

Member checking is one of the methods used to validate the study. Member checking means asking the participants to verify the data gathered or analysed to make sure the researcher is accurately interpreting their voices (Nieuwenhuis, 2016). I allowed my subjects to see how I quoted their interviews and that gave me a deeper insight of how they perceived my interpretations. This process gives importance to the subjects' voice.

3.8.2 Transferability

Transferability is how the researcher proves that the findings in their study are applicable to other contexts (Statistics Solutions, 2023). The phrase 'other contexts' means similar situations, similar populations, and similar phenomena (Statistics Solutions, 2023). Transferability "invites readers of research to make connections between elements of study and their own experience or research" (Nieuwenhuis, 2016, p.124). I took considerable time painting the context of the findings of my study to give readers an opportunity to determine whether the research is transferable to their contexts.

3.8.3 Dependability

Dependability is a measure of how a study can be replicated by other researchers and result in consistent findings (Statistics Solutions, 2023). If another person intends to repeat the study, they should obtain clearly articulated information that enables them to do so, and they should be in a position to obtain similar findings as the original study (Statistics Solutions, 2023). I kept a record of my decisions, especially data collection and data analysis processes, to ensure my reasoning and revisions are traceable. I documented the categories created or revised during data analysis to enable another person to see decisions made and how I arrived at interpretations.

3.8.4 Confirmability

Confirmability is the degree of neutrality in the research study's findings (Statistics Solutions, 2023). This means that the findings are based on participants' responses and not any potential bias or personal motivations of the researcher (Statistics Solutions, 2023). Confirmability refers to how close the influence by participants are to the findings as opposed to the influence/bias/motivation/interest of the researcher (Statistics Solutions, 2023). I made sure to create an audit trail that highlights every step of data analysis, which in turn provide a rationale for my decisions.

3.9 Ethical Considerations

An important ethical aspect is the protection of the participants' identities (Maree, 2016). Since this research involved humans, it was vital to consider ethical guidelines. Before data collection, I applied for ethics approval from the Ethics Committee of the University of Pretoria (Annexure C). I obtained permission from the college managing director to work with the PSTs (Annexure D), and subsequently got consent from participant PSTs (Annexure E) to take part in the research. Since minors were involved as participants and their written work were used, I obtained an assent from participating class (Annexure F) and consent from their parents (Annexure G). Prior to implementing the lesson, consent letters were sent to all parents. The signed forms were returned to the school through their children. Of the 16 learners in the class, only 11 participated in the lesson, as they had obtained the necessary permission from their parents. The remaining learners, who had not received approval from their parents, were moved to another class for the duration of the lesson. This arrangement was made in advance to ensure that they would not miss out on the concept of fractions, since it was being taught in the other Grade 4 classes which were not part of my study. The videos and participant information were kept private and confidential. I adhered to the following principles to safeguard participants from potential risks:

- The participants must not be exposed to any form of harm under any circumstances;

- Voluntary participation by participants;
- The anonymity of individuals involved in the research must be guaranteed;
- Dignity of the participants should be valued, respected, and prioritised;
- Full consent must be acquired from the participants before commencing the study;
- Privacy protection of the research participants should be ensured;
- Confidentiality of the research data should be ensured; and
- Adherence to the ethical guidelines specified in the Ethics and research Statements of the Faculty of Education of the University of Pretoria.

3.10 Chapter Summary

In this chapter, I discussed the procedures and methods used for data collection and data analysis and justified my choices. The chapter adopted the research onion to structure it. In a nutshell, the research adopted a qualitative research method. I employed primary data collection methods, including interviews, document analysis, and observations. I concluded with a discussion on the criteria for data quality and ethical considerations. In the next chapter I present and analyse the data gathered from interviews, document analysis, and observations.

CHAPTER 4: PRESENTATION OF RESEARCH FINDINGS

4.1 Introduction

The purpose of this study was to investigate mathematics PSTs' KoST during their WIL conducted through Lesson Study. The aim of the chapter is to provide a clear and concise summary of the research findings from the data collected during the Lesson Study cycle. As revealed in the previous chapter, I collected data from the Lesson Study cycle during which four PSTs collaborated to plan, teach, and reflect on a lesson. Consequently, data were collected from each of the three crucial stages.

Data for the study was gathered through interviews, observations, and document analysis. The findings are presented in a manner that integrates the insights obtained from the interviews and analysis of learners' work into the overall analysis. The data generated were analysed taking into consideration the literature review, theoretical framework, and the research questions. Five broad themes that were predetermined codes (*a priori*) based on the elements of KoST emerged and guided me as I went through the data. Therefore, the codes and themes that were derived from the literature were reasoning and proof, problem-solving, communication, connections, and representation. The data are presented here in accordance with the research questions. The headings used to structure this chapter were derived from the research questions; for example, the first research question asks which features/components of KoST are reflected in the preservice lesson plans, and subsequently, the heading became 'lesson plan'. Therefore, three broad categories emanated from the data, namely lesson plan, lesson presentation, and post-lesson reflection. The chapter is organised according to the structure depicted in Figure 4.1.

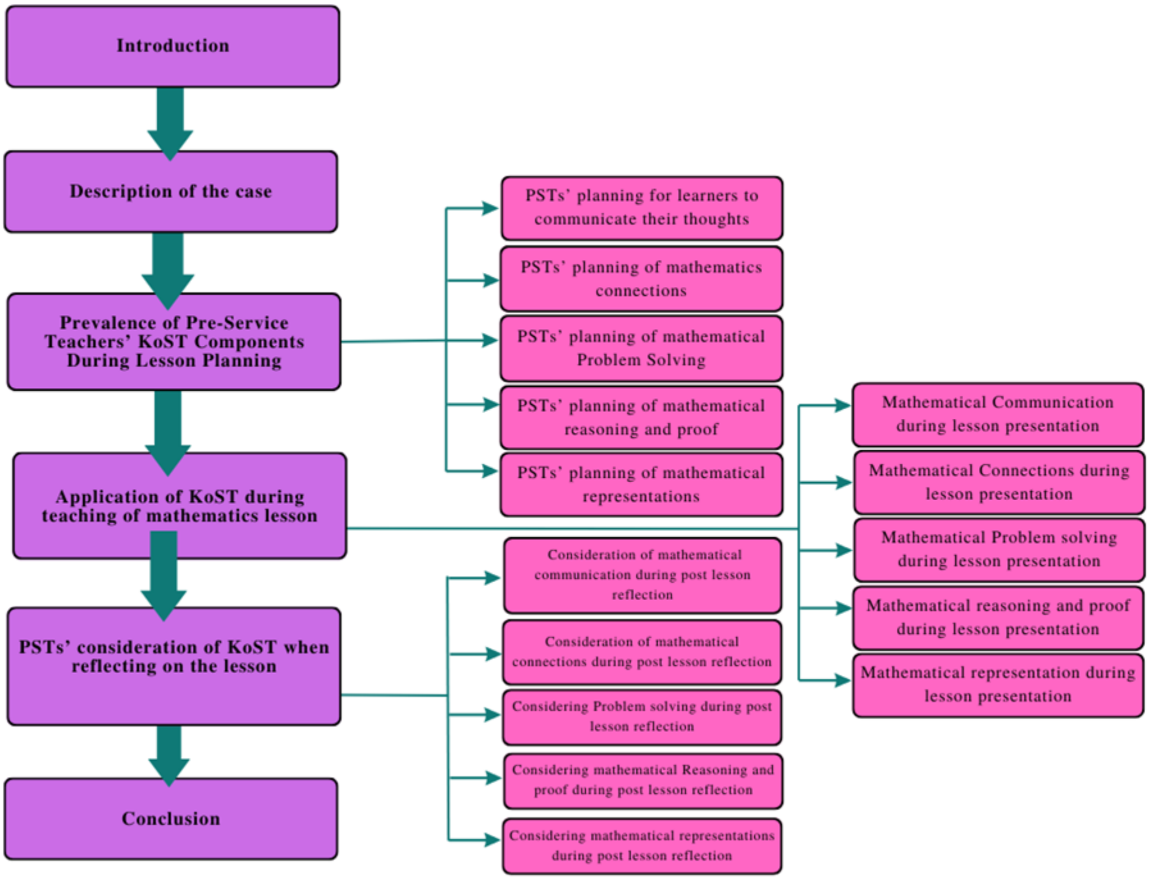

Figure 4.1 Outline for Chapter 4

4.2 Description of the Case

It was vital to protect the anonymity of the participants, and therefore, I developed codes to represent the participants. Four PSTs participated in the study and I used the following codes for them: PST S, PST M, PST MP, and PST L. PST S volunteered to teach during the research lesson presentation as per the prescriptive of Lesson Study. The Grade 4 learners who were present during the research lesson presentation were coded Learne1, Learner 2…. Learner 11.

4.3 Prevalence of Preservice Teachers' KoST Components During Lesson Planning

The lesson plan, which focused on teaching fractions, was created through a collaborative effort involving four PSTs. The lesson plan was for a Grade 4 class conducted during the third term of the South African school calendar. As discussed in the previous chapter, document analysis and interviews were employed to thoroughly examine the data. The findings are presented in a manner that integrates the insights obtained from the interviews into the overall analysis. The focus of my analysis was communication, connections, problem-solving, reasoning and proof, and representation as components of learner thinking. Some of these components of learner thinking were evident from the lesson plan. The analysis helped me gather relevant information to respond to the following research question: Which features/components of KoST are reflected in the PSTs' lesson plans?

4.3.1 PSTs' planning for learners to communicate their thoughts

Generally in the context of education, communication is a way to clarify understanding and exchange of information, ideas, and instructions between the teacher and learners (Ferrini-Mundy, 2000). Communication in lesson plans encompass a clear articulation of instructions, learning objectives, promoting dialogue, and active engagement to facilitate the teaching and learning process. Therefore, during my analysis of the lesson plan, I expected the lesson plan to provide freedom for learners to share opinions and present a safe space for effective classroom discourse support. I expected that the PSTs would create opportunities for varied forms of communication (verbal and non-verbal) and interaction between classmates through group work and by introducing interactive learning activities in their lesson plan articulation. An effective lesson plan that strives to enhance communication should detail an environment where learners share their mathematical ideas not just with the teacher but also with one another, and such opportunities for communication can be expanded by using learning and teaching materials (resources). Through communication, teachers are able to elicit learners' mathematical thinking and to give them instant and consolidated feedback (Sekao, 2023).

In their planning, the four PSTs articulated the procedures and activities they were going to use during the research lesson presentation. I use verbatim extracts from their lesson plan that spoke to the expectations of communication listed in the preceding paragraph. In their planning, the PSTs envisaged creating an environment that would probe the sharing of ideas as the lesson plan stated, "I will ask the class what fraction of the balls are white and write it with them according to their understanding of the numerator and denominator". Assessments aligned with learner thinking are accurate measures of students' understanding and progress, allowing

PSTs to adjust their teaching accordingly (Popham, 2009). The moment a teacher asks probing questions to the class, they seek to engage and solicit information from the audience. In the above statement, the PSTs seek to solicit learner thinking and engage with them according to their responses. This extract is a clear evidence that the PSTs planned to create an environment that allows learners to share their ideas freely as evidenced by the phrase "write it with them according to their understanding". This shows that learners were to be given a platform to share their understanding, thereby opening opportunities for interaction with the teacher (PST).

The PSTs maintained their stance to seek to understand what learners think through probing as the following extract shows: "I will ask the class what fraction of the balls are red and write it with them on the board = $\frac{1}{4}$". This statement and the subsequent extract reiterate a method espoused by the PSTs to create an interactive environment that entertains the learners' voices: "During the concept development phase I will probe learner thinking through questions and address these accordingly". This clearly reveals that the PSTs planned to use probing questions to engage learners, and therefore, consider learner thinking during the lesson presentation on the topic fractions.

In addition, the group of PSTs anticipated learners' misconceptions in their planning of the research lesson. They compiled a section in the lesson plan, called "anticipation of misconceptions in fractions" and "strategy to address misconceptions", where they planned anticipated misconceptions and strategized ways to address these. Misconceptions and divergent thoughts can be difficult to delineate but it is vital to cater for these and find ways to address them. Furthermore, the PSTs catered for divergent thoughts from their learners, as evident in the following statement: "In this I will allow for divergent thought from learners during the lesson and address them accordingly". This shows that the PSTs were aware of and anticipated learner deviations in their thinking; catering for these is vital for learning to take place. Through communication and engagement, they anticipated their learners to have different thoughts and that they would need ample time to address these thoughts and misconceptions.

The lesson plan articulates an engaging activity that included the use of resources among the learners, as shown in the following extract:

- In this part I will start with the loafs of bread.
- I will pick different students in different groups to share a loaf of bread. One person will get a loaf of bread for themselves.
- Each group will cut out their share of the wall chart (loaf of bread) and paste it on the board.

Opportunities to communicate are expanded by using interactive activities that involve concrete learning resources. This planned activity provokes learners to interact and share ideas about fractions. It is one opportunity that creates varied forms of communication as learners interact with resources.

PSTs predicted common misconceptions which learners often have. One of the misconceptions that they anticipated was that, "The denominator represents some parts in a whole". Their proposed solution, written in the lesson plan, to address the misconception was to use "visual representation of the numerator and denominator and where they need to be written". The actual lesson development stated that,

> I will continue to show them a visual representation of the numerator and denominator and where [how] they are written in the fraction. I will use a puzzle. The puzzle pieces will be complete below the line (denominator) and only 2 on top of the line (numerator)

In this context, when PSTs stated "...where they are written in the fraction," they were referring to the specific placement of the numbers in a fraction, i.e. the number that is written above the division line and the number that is written below the division line. The PSTs were referring to a picture puzzle, where the various pieces combined to form a complete image was regarded as a whole. They were referring to the situation where a specific number of pieces were chosen, and these pieces represented part of the whole picture. While using visuals can help address certain fraction misconceptions, this puzzle activity may cause more confusion than clarity. The fundamental concept of fractions is equal sharing of a whole into equal parts, but the use of a puzzle does not accurately portray this idea of equal parts. While the student teachers had a basic understanding of fractions, they failed to recognise that the different puzzle pieces were not equal in size or shape. This oversight meant that the puzzle could not be effectively used to teach the concept of fractions as equal sharing.

This realisation highlights the importance of selecting appropriate teaching materials and ensuring that they accurately represent the mathematical concepts being taught.

4.3.2 PSTs' planning of mathematics connections

Understanding of mathematics involves making connections. Connections come in different forms, such as connections between informal and formal mathematics, between concepts, between topics, between mathematics and other subjects, and between mathematics and everyday life. The teacher's ability to facilitate these connections provokes thinking and enables learners to learn and experience mathematics.

In my analysis of the lesson plan, I expected PSTs to incorporate topics previously covered to enable learners to forge connections with previously learnt mathematical concepts and procedures. I conjectured that the PSTs would spotlight many scenarios and activities where learners would experience mathematics both in and outside school. Creativity was another major expectation from the lesson plan that would encompass skills and concepts taught in ways that would connect with learners' intuitive mathematical experiences. Young children usually connect new mathematical ideas to old ideas through the use of concrete objects (Ferrini-Mundy, 2000). Projects and investigations force learners to make connections by challenging them to apply what they have learnt. In their planning, effective teachers select tasks that help learners explore new mathematical ideas in familiar contexts. Teachers ask questions that encourage and challenge learners to explain new ideas and develop new strategies based on mathematics already known.

The four PSTs realised that mathematics is not a collection of isolated topics but rather an interconnected web of ideas. They incorporated previously learnt concepts to introduce the new concept to be taught. This is evident from the lesson plan extract of the lesson outcomes as they included a concept in Outcome A that is taught from Grade 2 even though at that stage learners are expected to write fractions as 1 half (Department of Basic Education, 2011). The following are the outcomes as per the lesson plan (see Annexure H):

LESSON OUTCOMES:

A. Develop learners' ability to write fractions.

Learners must be able to differentiate between a numerator and denominator.

B. Learners must be able to compare fractions of different sizes and recognise which fractions are smaller or larger.

The concept covered in Outcome A was used as an introduction to the lesson since it is part of the learners' prior knowledge because it was covered in previous grades. The concept covered in Outcome B is the new concept for Grade 4 Term 3 (Department of Basic Education, 2011). The PSTs used this connection to provide a natural progression of ideas.

To gain insight into why PSTs referred to Grade 3 work in their planning process, I asked a question during an unstructured interview about the importance of connecting to prior knowledge. The PSTs emphasised the significance of establishing connections to learners' previous knowledge and experiences as a foundation for effective instruction as follows.

> *Interviewer*: Why did you refer to the Grade 3 work when you introduce fractions to Grade 4 class?
>
> *PST S*: Oh, why did I introduce that? For them to understand fractions more, especially when it comes to mathematics. They will have to have prior knowledge, pre-conceived knowledge. So in order for them to get to know what is a fraction, they will have to remember what they did in Grade 3. And so that is why I implemented that one.

PST S' response stipulates the importance of prior knowledge in order for learners to understand new knowledge.

As they progressed in their planning, the PSTs emphasised the importance of connections between the intuitive informal mathematics that learners learn through experience and the mathematics they learn in school by using authentic everyday examples to develop concepts. They planned to use activities that are familiar to learners using concrete material. The following extract from the lesson plan shows this:

> In this part I will start with the loafs of bread.
> I will pick different students in different groups to share a loaf of bread. One person will get a loaf of bread for themselves.
> Each group will cut out their share of the wall chart (loaf of bread) and paste it on the board.

> Here I will show them how if you have more people in your group your piece of bread is smaller. So even though the fraction looks bigger $\frac{1}{4}$ you still have less than someone who is $\frac{1}{2}$.
>
> I will give out the activity where students will be able to compare their fractions to see what is smaller and larger.

This clearly demonstrates that the PSTs were aware of the vital connections that learners need in order to engage with concepts in mathematics. Such connections create moments where learner thinking is revealed and PSTs' assistance would come handy in the learning process.

Throughout their planning, the PSTs frequently used the phrase "I will show", which concerned me. In the given example, they mentioned, "Here I will show them how if you have more people in your group your piece of bread is smaller". This statement suggests that learners will solely rely on being shown new knowledge, which can hinder their ability to independently discover and make connections between familiar contexts and the concept being taught. It risks positioning learners as passive recipients of knowledge rather than active participants in their own learning process. The following is the conversation and the rationale on why the PSTs need to show learners the new knowledge:

> *Interviewer*: If we can look at the third bullet there, where you said, "I continue to show…". Why is the statement 'show' recurring in your lesson, if we can also look at … let me check for you, "Here I will show them…". What is the rationale behind 'showing'?
>
> *PST S*: Ok, so you get your different types of learners where there is visual learners, kinaesthetic learners, and so on, but this visual representation is for all the learners because when it comes to fractions and mathematics, it's better to bring real life into the lesson so that they are able to relate to daily stuff like the pizza being cut into pieces, like glass that is half full or half empty, whatever, so that they know they can see with their own eyes and then implement it into mathematics. It will link together there. So, they will better understand it if we show it to them instead of just telling them what it is.

PST MP: I agree with her because in some instances for instance let's say maybe you are representing a quarter and then remember we asked them to share it some parts of the... some students would rather go share it the four parts and leave the one, not understanding they have to share it one out of the four. So, when you show them visual things, even the one that wasn't sure can be 100% sure that "oh, ok this is actually how it's being done".

Based on their responses, it appears that the PSTs did not fully grasp the significance of learners making their own discoveries instead of being shown new information. However, in their explanations, they referenced a different scenario where they highlighted the importance of providing new opportunities for learners.

4.3.3 PSTs' planning of mathematical problem-solving

Problem-solving is not merely an objective in the pursuit of learning mathematics but also a significant method for achieving it (Ferrini-Mundy, 2000). Instead of training children to be merely arithmetic and algebraic human calculators, the emphasis in the mathematics curriculum should be on nurturing learners' creativity, adaptability, and self-reliance, enabling them to confidently and flexibly tackle novel problems (Foster, 2023). The process of problem-solving offers a practical framework for the application of mathematics, and carefully selected mathematical problems offer learners the chance to reinforce and expand their existing knowledge, while also fostering their mathematical learning (Ferrini-Mundy, 2000). Foster (2023) defined problem-solving as how learners tackle and engage in non-routine tasks for which solution method is not known in advance. It is vital to cultivate a culture of problem-solving in learners. In school mathematics, the predominant practice is to encourage learners to simply mimic a predetermined method they have been taught (Foster, 2023).

During the analysis of the lesson plan, I expected a couple of problem-solving aspects that PSTs could focus on during planning, such as carefully selected problems/tasks coupled with clear problem-solving strategies for that specific topic. This stands in contrast to the common practice of attempting to teach general strategies that are claimed to be applicable across different mathematics topics (Foster, 2023). The tasks should be non-routine and the solution methods should not be known in advance. Routine tasks are exercises where learners are invited to imitate a prescribed method they have been taught. Other expectations from the lesson plan were creating an environment that values the work learners do in class, helping learners make their strategies explicit, giving ample time for learners to think, and believing that learners are capable of solving problems.

The PSTs lived up to expectations, as evidenced in their lesson plan, that learners are capable of solving problems. The following statement in parenthesis from the lesson plan is testimony to that: "I will ask the class what fraction of the balls are white and write it with them according to their understanding of the numerator and denominator. (REMEMBER: Do not tell them, they must conclude the answer themselves.) = $\frac{3}{4}$". This statement shows that the PSTs believed and conceded that learners are natural problem solvers and should be afforded a chance to work on a problem on their own and divulge their thinking.

In another instance in the lesson plan, the PSTs realised they needed to give ample time for learners to work through the problems given to them so they can apply their minds to the given problems. They wrote, "I will hand out activity one and allow 10 minutes for the learners to do the activity". This statement reveals that the PSTs realised the importance of time as it allows learners an opportunity to think deeply, explore various approaches, and develop a deeper understanding of the mathematical concepts involved.

Despite the fact that PSTs considered two aspects of time and that the learners can do mathematics on their own, they did not include non-routine problems in their planning. They only included routine problems that learners would learn during the lesson. Figure 4.2 shows the routine problems in the lesson plan the PSTs planned to give learners. Considering the objectives of the lesson and how they planned to execute the lesson, all the activities they planned were routine and in line with what they planned to teach. The PSTs did not include non-routine problems that would compel learners to think critically, analyse the problem from different angles, promote higher-order thinking skills, and help develop logical reasoning abilities.

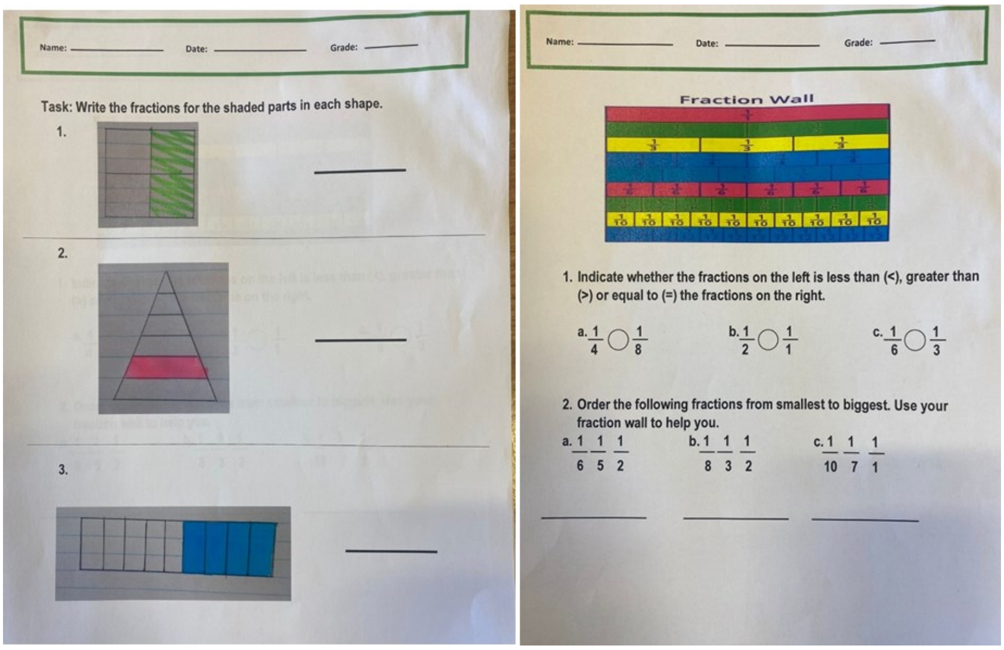

Figure 4.2 The PSTs' planned fraction activities

Analysing the activity that the PSTs designed for learners depicted in the first frame of Figure 4.2, it becomes apparent that they used a triangle in question 2 to assess the learners' understanding of fractions. This approach was problematic, as dividing a triangle into equal parts can be challenging and complex. The parts into which the triangle was divided into are visibly not equal.

It became clear that PSTs planned to assess learners using routine problems in their responses to interview questions about the objectives of the activity that they designed:

> *Interviewer*: Alright maybe if you can try and explain to me, what were you trying to achieve with Activity 2?
>
> *PST MP*: We were actually trying to meet the outcomes to see if learners did follow and understand what a numerator and a denominator represent.

Given that the concept of numerator and denominator was intended to be taught as prior knowledge before introducing the topic of comparing fractions, it can be concluded that Activity 2 was a routine exercise rather than a non-routine activity that stimulates higher-order thinking.

4.3.4 PSTs' planning of mathematical reasoning and proof

Stylianides et al. (2013) viewed reasoning and proofing as a range of activities that facilitate exploring and understanding why and how things work in various mathematical areas. Reasoning and proofing involve two major activities namely, generalising, such as forming educated guesses (conjectures) about potential mathematical relationships, and constructing arguments to determine the validity or falsehood of those generalisations, some of which may meet the criteria for proofs (Stylianides et al., 2013).

When planning lessons, teachers must take into consideration that learners can make assertions and conjectures and that these should always have reasons. Questions such as 'why do you think it is true?' should dominate lessons. Teachers should help learners see that assertions need to be supported or refuted by evidence. It is vital for teachers to encourage learners to make conjectures by asking questions such as, 'what do you think will happen next?' and 'what is the pattern?'. Teachers can support learners in reviewing conjectures that are valid in one situation to determine if they remain valid when applied to a different scenario. Learners tend to express their ideas and reasoning in their own language and frequently investigate them using tangible objects and real-life examples (Ferrini-Mundy, 2000). Therefore, teachers must give learners activities that encourage the use of concrete material and plan to introduce a variety of real-life examples applicable to the topic being taught. Authentic examples afford leaners an opportunity to discover mathematics. Teachers should strive to assign tasks that require learners to explain their thought processes and defend their viewpoints with evidence. PSTs should encourage respectful disagreement and help learners develop counterarguments. They are expected to incorporate opportunities for group work or discussions in which learners are free to articulate their reasoning and debate ideas.

Since learners tend to investigate their ideas and reasoning through concrete/tangible objects, the PSTs incorporated activities in their planning that use concrete materials. The following extract from the lesson plan shows how the PSTs intended to use concrete materials to give learners an opportunity to discover and investigate their thinking:

> I will ask 4 learners to stand in front and pick a ball from the bag. Three learners will pick white balls and one learner will pick a red ball. I will ask the class what fraction of the balls are white and write it with them according to their understanding of the numerator and denominator. (REMEMBER: Do not tell them, they must conclude the answer themselves.) = $\frac{3}{4}$

The mathematics reasoning process can be initiated by the teacher through activities given to the learners. The above quote unpacks how the PSTs intended to use concrete material to give learners a chance to investigate their mathematics thinking. Learners need to manipulate objects to discover and demonstrate general mathematical truths using authentic examples. The planned activity is teacher-directed through asking probing questions. Probing questions lead learners to think critically and reason through concrete activities.

4.3.5 PSTs' planning of mathematical representations

Representation is encompassed in SCK, which falls in the category of subject matter knowledge under the MKT framework. Hill et al. (2008) viewed SCK as mathematical knowledge that teachers use to engage in teaching specific tasks, such as how to accurately represent mathematical ideas and provide an explanation or justification of the mathematical idea. Representation refers to the processes and the products noticeable externally and internally in the minds of the people doing mathematics (Ferrini-Mundy, 2000). Representations should be considered crucial components in helping learners understand mathematical relationships and concepts.

A major responsibility of teachers is to create an environment that supports, accepts, and encourages the use of multiple representations. Teachers should help learners efficiently create and use a variety of representations. Learners need a variety of representations that supports their understanding for them to be deeply knowledgeable about fractions and to have a set of tools that significantly expand their capacity to think mathematically.

The PSTs' planning showed that they intended to use three main forms of representation in their lesson, namely physical displays of fraction bars, discrete balls, and fraction symbols. The following extract shows their planning:

> I will ask 4 learners to stand in front and pick a ball from the bag. Three learners will pick white balls and one learner will pick a red ball. I will ask the class what fraction of the balls are white and write it with them according to their understanding of the numerator and denominator. (REMEMBER: Do not tell them, they must conclude the answer themselves.) = $\frac{3}{4}$. I will ask the class what fraction of the balls are red and write it with them on the board. = $\frac{1}{4}$

The PSTs intended to use balls of different colours to represent fractions and introduce the fraction symbol written in numeric form. The coloured balls represent ratios of discrete elements, but this is not enough representations of fractions since different representations frequently shed light on different facets of a complex concept like fractions. Therefore, the PSTs included one more form of representation, namely fraction bars. The following is an extract from the lesson plan where they plan to represent fractions using fraction bars:

> In this part I will start with the loafs of bread.
> I will pick different students in different groups to share a loaf of bread. One person will get a loaf of bread for themselves. Each group will cut out their share of the wall chart (loaf of bread) and paste it on the board. Here I will show them how if you have more people in your group your piece of bread is smaller. So even though the fraction looks bigger $\frac{1}{4}$ you still have less than someone who is $\frac{1}{2}$.

The PSTs planned to use the familiar context of bread to cut out fractions that caters for the part-whole interpretation of fractions. This can help learners see fraction equivalence and the meaning of the addition of fractions. The PSTs also planned to use the fraction symbols to represent fractions.

In their plan, the PSTs intended to use puzzles to introduce fractions to Grade 4 learners. Puzzles by their nature often focus on solving a specific problem or completing a task without providing a deep conceptual understanding of fractions. Learners might develop procedural fluency in solving the puzzle but lack a solid foundation of what fractions represent or how they relate to other mathematical concepts. The following extract from the lesson plan depicts how they intended to teach using puzzles:

> I will continue to show them a visual representation of the numerator and denominator and where they are written in the fraction. I will use a puzzle. The puzzle pieces will be complete below the line (denominator) and only 2 on top of the line (numerator).

Depending on the complexity of the puzzle, learners may develop misconceptions or misunderstandings about fractions. If the puzzle presents fractions in a non-standard or overly simplified way, learners might struggle to connect their knowledge to real-world fraction concepts, leading to confusion or incorrect assumptions.

Learners often demonstrate the product of their thinking through written activities. The PSTs intended to study and interpret learners' work as follows:

> This part all the student teachers will walk through the class and see whether the students grasp the concepts and or whether they are forming new misconceptions that was not considered yet. At the end of the lesson we will collect the worksheets and reflect on learner thinking and how the lesson accommodated for possible misconceptions.

By examining learners' representations, teachers can acquire important insights into how learners perceive and think about mathematics. The PSTs planned to go around interpreting learners' understanding through the representations they were making. This shows that they value the KoST because that knowledge allows them to create ways of imparting knowledge to learners.

4.4 Application of KoST During Teaching of a Mathematics Lesson

Observation takes place during a Lesson Study stage called lesson presentation and observation. During this stage, most PST participants and myself (as the researcher) observed the proceedings of the lesson as one PST volunteer presented the research lesson. The data presented in this section was gathered through my observation as the researcher. The volunteer PST in this case represented the ideas contributed by all PST group members. The volunteer was PST S. The observation helped me gather relevant information to respond to the following research question: How do PSTs apply the features/components of KoST during the teaching of a mathematics lesson?

4.4.1 Mathematical communication during lesson presentation

Communication is a way of clarifying understanding and exchange information, ideas, and instructions between the teacher and the learners (Ferrini-Mundy, 2000). Communication played a major role in articulating and revealing learner thinking during the lesson presentation and was a two-way process from the teacher to the learners and the other way around that made mathematical thinking observable. Communication during lesson presentation involves the transmission and reception of messages and ideas through various modes, such as verbal language, non-verbal cues, visual aids, and gestures. During lesson presentation, effective communication is crucial to convey the intended content, facilitate understanding, and promote active engagement. It encompasses both the teacher's delivery of instructions, explanations, and demonstrations as well as the learners' active participation, questioning, and feedback.

Key aspects of communication during lesson presentation include the teacher striving to express ideas and instructions in a clear and concise manner and ensuring learners understand the content being taught. Effective communication involves active listening on the part of both the teacher and the learners. The teacher attentively listens to learners' questions, responses, and feedback, and the learners listen to the teacher's instructions and explanations. Visual aids such as charts, diagrams, slides, or props are used to enhance communication by providing visual representations that support understanding and retention of information. Non-verbal communication, such as facial expressions, body language, and gestures, plays a significant role in conveying emotions, enthusiasm, and engagement during lesson presentation. Communication during lesson presentation includes encouraging learners to ask questions, facilitating discussions, and promoting active participation. This fosters a two-way flow of information and encourages critical thinking and deeper understanding. Effective communication involves providing timely and constructive feedback to learners, acknowledging their efforts, and addressing any misconceptions or gaps in understanding.

In my observations during the lesson presentation, PST S fostered a two-way flow of information and encouraged critical thinking and deeper understanding. This was achieved by promoting active participation and asking probing questions. The following is an extract from the transcription of a video recording of the lesson presentation proceedings. The lesson was just starting, and PST S adopted an interactive teaching style with the learners.

PST S: Now, do you think fractions is fun? Who can tell me what a fraction is?

Learner 6: Mam, a fraction is a number of parts in a shape.

PST S: Yes, it's a part of something, like for an example, if we have 3 sweets and I give you 1 sweet, you will have 1 of 3 sweets. You have a part of the total of sweets that we had. Ok, there we go. Ok, so fractions can be complicated but if you know where to start and you know the vocabulary, you will be fine, ok. So let's start with the numerator and the denominator. Who can tell me what a numerator is? Without reading yes?

Learner 8: The number on the bottom.

PST S: Are you sure? Why do you say the number at the bottom?

Learner 8: The number of pizzas.

PST S: The number of pizzas in total, but why do you say that it's the numerator? Ok, think of that for a moment. Ok yes.

Learner 7: It's the top number of the fraction.

PST S: The numerator, yes, he is correct. It's the top one. Why do you say that?

Learner 7: Because it's the one that is selected.

PST S connected with the learners by asking probing questions that compelled critical thinking in the learners. She encouraged learners to self-reflect on their responses, giving them a chance to think analytically and respond again. This was achieved through questioning for an example, "Are you sure? Why do you say the number at the bottom?" When a teacher asks a learner 'why', they are trying to encourage critical thinking, reveal understanding or misconceptions, and enhance communication and mathematics language skills. Asking learners 'why', helps them develop metacognitive skills by making them aware of their own thinking processes. When learners are asked to explain the 'why' behind their response, it opens up opportunities for deeper discussion and elaboration.

It was not only PST S who showed the culture of probing learners during the lesson presentation but also the other observer PSTs. A written activity was given to assess learners' understanding of the concept taught, and every observer PST stood up to monitor the progress of the activity. All the PSTs seized that moment as an opportunity to assist and probe learner thinking. The following is an extract of a conversation between PST L and a learner during the written activity:

> *PST L*: You have to tell me, what is a numerator? What's a numerator? What is it?
>
> *Learner 11*: The numerator tells us how many parts we have.
>
> *PST L*: And what is a denominator?
>
> *Learner 11*: Denominator tells us how many parts there are altogether.
>
> *PST L*: Yes, and I need you to count correctly, neh?
>
> *Learner 11*: Mam, so 4 is at the bottom.
>
> *PST L*: You must know that. You have just told me now. I need you now to count properly.

This is clear evidence of an environment of communication created by the PSTs through probing questions. By so doing, the PSTs were able to elicit mathematical thinking from the learners. This environment affords learners the freedom to express and share their ideas and talk about mathematics. Probing learners let the teacher identify misconceptions and get sufficient time to address them. This particular learner made a mistake in determining the fraction of the shaded part in a given activity. After a careful step by step intervention by PST L the learner realised their mistake and rectified it. The learner's subsequent work demonstrating their revised understanding is shown in Figure 4.3.

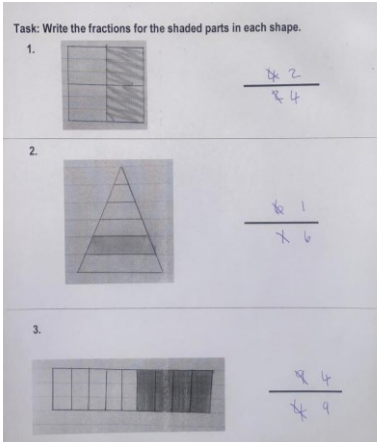

Figure 4.3: Learner's corrected work for activity 1

By employing probing questioning techniques, PST L was able to effectively address and resolve a misconception held by the learner regarding the correct representation of numerators and denominators. The learner had incorrectly inverted fractions due to an underlying conceptual misunderstanding. However, through a series of targeted probing questions, PST L facilitated the learner's identification and self-correction of the error. The probing dialogue induced metacognitive reflection that enabled the learner to independently rectify their previous flawed approach. For instance, in question 3 of Figure 4.3, Learner 10 rectified his original incorrect answer $\frac{9}{4}$ to the correct answer $\frac{4}{9}$.

As the lesson progressed, PST S reinforced the correct use of mathematical language to support understanding and retention of information. PST S used visual aids and concrete material in the process. This was after introducing the topic of the day to the class as she narrated and solved real-life fraction problems and it was time to get practical. The following extract shows how PST S used bean bags to reinforce correct mathematical language and support understanding and retention of information. She chose four learners to come to the front of the class, and she had four bean bags in two distinct colours placed in an open box. Three of the bean bags were red and one was green. The four learners were instructed to pick one bag each from the box. The following extract shows the questions the teacher asked to get the learners to understand the concept of fractions:

PST S: How many of you have red? So how many are they?

Learner 4: 3.

PST S: And which one has a green one? So, let's ask the question … What fraction of people have a red bean bag?

Learner 4: 3.

PST S: So how will we write that? We have our numerator, which will be?

Learner 11: 4.

PST S: Ok, why is our numerator 4?

Learner 11: I mean, 1?

PST S: 1. Perfect, just like that we have our numerator, which will be 1. Why is it only 1?

Learner 8: 3 [Learner 2 corrected the teacher].

PST S: No, no, no, sorry, sorry. 3. [the teacher acknowledges Learner 2's correction]. Yaah, we doing 3. Which fraction of the bean bag are red? So how many red bean bags are there?

Learner 4: 3.

PST S: 3. And that's the …?

Learner 11: Numerator.

PST S: Numerator, yes. And then what is the denominator?

Learner 4: 1.

PST S: Wait, let me ask her.

Learner 9: 1.

PST S: Why do you say 1?

Learner 9: Oh mam, its 4.

PST S: 4. Why do you say 4? Yes

Learner 11: Mem, because there is 3 out of 4.

PST S: Yaa, now there is … Yes.

Learner 11: Mam, there is 4 people.

PST S: 4 people in total. There is 4 bean bags, ok. No matter what colour they are, there is 4 bean bags. Ok. So and then what fraction are the green bean bags?

It is evident from the conversation that the teacher was not able to adequately demonstrate and model how to use the bean bags to represent and develop the concept of fractions. Learners appeared confused about the concept being developed and the purpose of the manipulatives. The use of different coloured bean bags created a perception of fractions as ratio and yet the intention was to present it as fractions as equal sharing. In Grade 4 Term 3, the curriculum recommends the making of fractions through grouping or sharing which links with understanding the relationship between division and fractions (Department of Basic Education, 2011). Learners who developed misconceptions (especially Learner 3) about numerator and denominator were unattended to by the presenter. Instead, she carried the question over to Learner 4 without addressing the difficulties that Learner 3 faced. PST S noticed the misconception but was overwhelmed and could not proactively address it to prevent misunderstandings from taking root.

The presenter demonstrated effective communication through active listening as she acknowledged divergent thoughts from learners. She attentively listened to learners' responses and feedback. At the peak of an activity, learners were constructing a fraction wall chart from paper strips that the presenter simulated as bread. The learners were instructed to equally share the strips among their groups. The group members varied in number for each group and ranged from a group of two to a group of eight learners. The following extract depicts what transpired when the group of eight learners shared their paper strips:

PST S: 8, now you guys are going to share a loaf of bread as well, so lets start with 1 person. Take your scissors, and you take your bread. Yeah, you can cut your piece. You cut yours and so on. So, while we cutting, lets see, what is the smallest fraction on the board now?

Learner 7: 1 over 8.

Learner 11: 1 over 6.

PST S: Yes, 1 over 6 because 1 over 8 is not there yet. Ok, and why do you say that? Because?

Learner 11: Because its 6.

PST S: Yah, but why is it smaller than the other ones?

Learner 7: Because the denominator is larger than the others.

PST S: Ok, that is also a way of seeing which fraction is smaller. It's going to be the other way around, you?

Learner 5: Because it is split into small sixes.

PST S: Yes, where one person had a whole loaf of bread by himself, the last people had to share a loaf of bread between 6 people. So, who is gonna get the biggest part?

The presenter taught the class a method of ordering fractions by looking at the sizes of the paper strips, and Learner 1 articulated a different and more abstract method of comparing fractions, i.e. comparing the cardinality of number symbols in the denominator. PST S identified the different method presented by Learner 1, and she acknowledged it for the benefit of other learners. However, there was lack of discipline-specific mathematical language in teaching fractions. There was a continuous misuse of terminology in naming fractions e.g. 1 over 8. Such type of terminology perpetuates the thinking that each fraction is a set of two different numbers (Department of Basic Education, 2011).

The presenter strived to express ideas and instructions in a clear and concise manner, ensuring that learners understood the content being taught. In most cases, PST S would seek to understand what learners were saying and advance learners' thinking. The following extract from the lesson presentation details how PST S advanced learner understanding and thinking by asking the question 'why' to a learner. This was during the construction of a fraction wall chart, and she wanted to have an insight into whether the learners understood the concept of comparing fractions.

> *PST S*: 2 out of 6. Ok, so let's see again what now will be the smallest fraction on the board. Yes?
>
> *Learner 8*: $\frac{1}{8}$.
>
> *PST S*: $\frac{1}{8}$. Why do you say that? Is what? You are correct just tell me why you saying that? Ok, you at the back?
>
> *Learner 11*: Mam, its 1 over 8
>
> *PST S*: Yah, ok what I'm asking is why is this [$\frac{1}{8}$] smaller than this one [$\frac{1}{6}$]?

By asking 'why', the learners were prompted to articulate the rationale behind their answer. This reinforced their conceptual understanding as they had to demonstrate their grasp of the topic and demonstrate how they arrived at the correct response. Asking 'why' prompts them to explain the reasoning behind their response. This encourages them to think more deeply about the concept, connecting it to prior knowledge and understanding. By so doing, learners reveal their thinking to the teacher, and therefore, the teacher acts accordingly.

4.4.2 Mathematical connections during lesson presentation

Connections in the context of lesson presentation encompasses the integration and recognition of relationships, similarities, and interdependencies between various mathematical concepts and procedures. When teachers make connections, they highlight the links and coherence within the subject, as well as make connections to real-life situations and other disciplines.

The volunteer PST was expected to incorporate topics previously covered to enable learners to forge connections with previously learnt mathematical concepts and procedures. Specifically, the PST should have related the concept of comparing fractions to prior knowledge, such as comparing whole numbers or comparing quantities. In the process, the PST should highlight the similarities and differences between comparing fractions and comparing whole numbers, reinforcing the connection between the two concepts. I expected to observe PST S spotlight many scenarios and activities where learners would experience mathematics both in and outside school. The presenter should have connected the concept being taught to practical everyday scenarios, which would have allowed learners to see how fractions are relevant to everyday situations. Young children usually connect new mathematical ideas to old ideas through the use of concrete objects (Ferrini-Mundy, 2000). Manipulatives help to connect informal approaches with formal approaches (Jones & Tiller, 2017). Projects and investigations force learners to make connections by challenging them to apply what they have learnt.

During the lesson on ordering and comparing fractions, the presenter provided a recap of the concept of numerators and denominators. This concept of numerators and denominators had been previously introduced to learners in Term 2 of Grade 4. Therefore, in Grade 4 Term 3, introducing the concept of 'ordering and comparing fractions' with a focus on numerators and denominators serves as a way to revisit and review their prior knowledge. The following extract is how PST S introduced the lesson:

> *PST S*: Who can tell me what a fraction is?
>
> *Learner 1*: A number of parts in a shape.
>
> *PST S*: Yes, it's a part of something like, for an example, if we have 3 sweets and I give you 1 sweet, you will have 1 of 3 sweets. You have a part of the total of sweets that we had, ok. There we go. Ok, so fractions can be complicated, but if you know where to start and you know the vocabulary, you will be fine, ok. So lets start with the numerator and the denominator, who can tell me what a numerator is? Without reading yes?
>
> *Learner 8*: The number on the bottom.

The teacher in this case had to revise what learners already encountered before introducing the new content. This helped learners make connections between what has already been taught with the new concept in the same subject and allow them to see how these topics and concepts are interlinked. I looked forward to PST S to relate the concept of comparing fractions to comparing whole numbers or comparing quantities and highlight the similarities and differences between comparing fractions and comparing whole numbers. As the lesson progressed, I realised that the learners needed a thorough recap of what fractions, numerators, and denominators are as they lacked that basic knowledge.

PST S used manipulatives during her presentation to connect the formal approaches to the informal that are familiar to the learners. When she was introducing the concept of fractions to learners, she used four bean bags, three red ones and one green. The idea was to equally share the four bean bags among four learners, and she probed the learners to give the fraction of red bean bags to the number of total bean bags. Equal sharing is not a new concept in learners' everyday life, and therefore, bringing this scenario into formal learning connected learners to their everyday life activities, leading to higher retention rates and more positive attitude towards mathematics. The excitement to do mathematics using manipulatives was witnessed when PST S introduced a scenario of equally sharing a loaf of bread (fraction strip) among the learners. Here she was introducing the concept of comparing fractions using a fraction wall chat. The learners developed the fraction wall chart through an exciting activity as follows:

> *PST S*: Yes, perfect. Ok, and now we were doing sweets, we were doing fractions. We are still doing fraction but we are doing it by giving loaves of bread. Who is hungry?
>
> *Learners*: Me, me, me!

Learners found this activity fun as they were involved in the process and they grasped the concept much easier and could easily relate this to what they normally do in their social contexts. It became easy for learners to engage and comprehend the new concept taught by using manipulatives.

4.4.3 Mathematical problem-solving during lesson presentation

Engaging in problem-solving tasks helps learners cultivate a profound comprehension of subjects, hone their capacity to apply knowledge across diverse scenarios, and foster self-regulated learning abilities. These tasks frequently require learners to dissect intricate situations, assess evidence critically, and adeptly convey their reasoning (Hmelo-Silver et al., 2007). In other words, problem-solving skills are essential for learners to thrive in the modern world, and incorporating problem-solving activities in lessons can enhance learners' critical thinking, decision-making, and collaboration abilities, leading to better academic and real-life outcomes. Problem-solving skills are essential as they foster higher-order thinking, promote deep learning, and prepare learners for the demands of the 21st century (Foster, 2023).

The PSTs were expected to design and present real-world or authentic problems that reflect the challenges learners may encounter beyond the classroom. These tasks should be relevant, engaging, and meaningful to learners to allow them to connect their learning to real-life situations; to reveal strategies for problem analysis, such as asking probing questions, considering multiple perspectives, and evaluating evidence; to probe learners to reflect on their problem-solving processes and evaluate their strategies and approaches; to provide opportunities for collaborative problem-solving where learners can work in groups or pairs to brainstorm ideas, share insights, and collectively arrive at solutions; to provide timely and constructive feedback to learners on their problem-solving efforts; to encourage them to learn from mistakes and iterate their solutions; and to emphasise the importance of perseverance, resilience, and a growth mindset in problem-solving.

It was evident that the PSTs made it a culture to ask learners to reflect, explain, and justify their answers. PST S probed learners to reflect on their problem-solving processes and evaluate their strategies and approaches. This was shown in how the presenter was conducting the lesson. In the following excerpt, PST S was working on an activity that involved the use of bean bags in two different colours to illustrate the concept of ratios in fractions. She posed a question to the learners about the ratio of red bean bags to the total number of bean bags in the box. During the process, she inquired about the specific quantity of red bean bags and then the overall number of bean bags to determine the fraction representing the red bean bags:

> *PST S*: Wait, let me ask her.
> *Learner 10*: 1.
> *PST S*: Why do you say 1?
> *Learner 10*: Oh, mam its 4.
> *PST S*: 4. Why do you say 4? Yes?

Learner 10: Mam, because there is 3 out of 4.

PST: Yaa, now there is … Yes.

Learner 9: Mam, there is 4 people.

PST S: 4 people in total. There is 4 bean bags ok. No matter what colour they are there is 4 bean bags. Ok. So, and then what fraction are the green bean bags? Let me try someone else. Yes?

Learner 2: 1.

PST S: 1. Ok, so that will be the numerator or the denominator? The 1 on top or the one below? If there is 1 …?

Learner 2: 1 on top.

PST S: 1 out of? Let Yes?

Learner 3: 1 out of 3.

In this scenario, the teacher guided the learners in solving the given problem and encouraged them to provide justifications for their answers. This was accomplished through a series of scaffolded questions. Similarly, in the following excerpt, PST L prompted the learner to explain and justify their reasoning. However, she overlooked the importance of using correct mathematical language and unintentionally reinforced a misconception. The issue of correctly naming fractions was not addressed. According to the Department of Basic Education (2011), learners should be able to accurately name fractions. It is advised to avoid using phrases like '3 over 4' as it may lead learners to perceive each fraction as two separate numbers, rather than understanding $\frac{3}{4}$ as a single number that falls between $\frac{1}{2}$ and 1. Following the lesson on ordering and comparing fractions, PST S assigned an activity to the learners to assess their comprehension of the concept. The other PSTs circulated around the classroom to observe and support the learners. During this time, PST L assisted a learner who held misconceptions regarding the comparison of fractions as follows.

PST L: So smallest to biggest, which one of these three fractions is the smallest fraction?

Learner 9: 1 over 10.

PST L: Yes, and then why do you say that? Why do you say it's 1 over 10?

Learner 9: 10 is the bigger number.

PST L: But why? How many people are sharing that loaf of bread?

Learner 9: 10.

PST L : And then, why is it the smallest than someone sharing for 7 people? Because what? What will be smaller?

Despite the teacher's lack of emphasis on the correct naming of fractions, she consistently encouraged the learners to explain and justify their reasoning. This approach compelled the learners to freely express their logical thinking, allowing the teacher to assess the class's level of understanding regarding the newly taught concept.

4.4.4 Mathematical reasoning and proof during lesson presentation

According to the NCTM (2014), reasoning in mathematics involves making logical connections, recognising patterns, and drawing conclusions based on mathematical relationships. It is the process of using logical thinking skills to analyse problems, make conjectures, and develop arguments (NCTM, 2014). Reasoning enables learners to understand and effectively apply mathematical concepts. Proof is the establishment of evidence or logical arguments that demonstrate the validity of mathematical statements (NCTM, 2014). It involves constructing logical chains of reasoning, using axioms, definitions, and previously established theorems (NCTM, 2014). Proofs provide certainty and reliability to mathematical knowledge, ensuring that mathematical statements hold true in all cases.

The PSTs should have been mindful that learners have the ability to make claims and propose hypotheses, and they should have consistently inquired about the reasoning behind these conjectures. Prominent questions such as 'why do you believe it to be true?' should be prevalent throughout the lessons. Teachers should help learners understand that assertions require evidence to either support or challenge them. It is crucial for teachers to foster an environment where learners feel encouraged to generate conjectures by asking questions such as 'what do you predict will occur next?' or 'what patterns do you observe?' Additionally, teachers can help learners evaluate conjectures that are valid in one context to determine if they remain valid when applied to a different scenario. Teachers should aim to provide assignments or tasks that prompt learners to articulate their reasoning and support their viewpoints using evidence. The PSTs should have fostered an environment that encourages respectful disagreement and helps learners develop counterarguments. It was expected that the PSTs would incorporate opportunities for group work or discussions where learners have the freedom to express their reasoning and engage in debates about ideas.

The approach to teaching used by the PSTs created an environment where the learners could make mathematical conjectures. Towards the end of the class during the fraction wall activity, a significant moment occurred when a learner made discoveries and brought them to the attention of PST S. The following extract shows this moment:

> *Learner 1*: Mam, it's starting to look like a fraction wall.
>
> *PST S*: Yaa, it's starting to look like this fraction we have down here.
>
> *Learner 7*: Mam, you can't see it.
>
> *PST S*: Yeah, it's close. We are working our own now …
>
> *Learners*: Yes.
>
> *PST S*: … so it's the same.

The teacher did not explicitly provide the fraction wall chart to the learners, and instead, they were left to discover and as they constructed the fraction wall themselves. In this particular case, the learner had no alternative but to make a conjecture due to the instructional approach used by the teacher.

The question 'why' was prevalent throughout the lesson whenever PST S interacted with the learners. She made extensive efforts to inquire about the reasons behind the learners' responses, regardless of whether the answers were right or wrong. The following is an example where a learner provided a correct response and was prompted to provide the rationale behind their answer.

> *PST S*: 2 out of 6. Ok, so let's see again what now will be the smallest fraction on the board. Yes?
>
> *Learner 8*: $\frac{1}{8}$
>
> *PST S*: $\frac{1}{8}$. Why do you say that? Is what? You are correct, just tell me why you saying that? Ok, you at the back?

The question 'why' dominated in the lesson, especially when learners were giving wrong responses. By so doing, PST S helped learners understand that assertions require evidence to either support or challenge them. By posing the question why, the teacher wanted to understand the reason that learners would provide. Although this particular concept is about reasoning and proof, the main focus in the context of the study is tilted towards reasoning than proof. Isolated cases of proofing were witnessed especially in the following conversation when the learners were engaged in an activity of sharing bread among themselves in different groups. The group sizes varied, with some groups consisting of two, three, or more learners, up to a maximum of eight learners in each group. The following conversation came up as learners were comparing the sizes of pieces of bread they got in their respective groups:

> *PST S*: Yah, but why is it smaller than the other ones?
> *Learner 7*: Because the denominator is larger than the others.
> *PST S*: Ok, that is also a way of seeing which fraction is smaller. It's going to be the other way around, you?
> *Learner 5*: Because it is split into small sixes.
> *PST S*: Yes, where one person had a whole loaf of bread by himself, the last people had to share a loaf of bread between 6 people. So, who is gonna get the biggest part?

The teacher asked the question 'why' a sixth is bigger than an eighth and learner 1's reason was synonymous to proofing why a sixth is larger than an eighth. Learner 2 responded to the teacher on why a sixth is larger than an eighth by bringing up the idea of comparing the sizes of bread and therefore proofing his assertion.

4.4.5 Mathematical representation during lesson presentation

Representation refers to the various ways in which mathematical concepts, ideas, and relationships can be visually or symbolically depicted or expressed. It involves using different tools, models, diagrams, graphs, symbols, and notations to represent mathematical information and communicate mathematical ideas. According to the NCTM, representation is an essential component of mathematical proficiency (Ferrini-Mundy, 2000). It allows learners to make connections between mathematical concepts and real-world situations, as well as between different mathematical ideas (Ferrini-Mundy, 2000).

It is essential for teachers to establish a supportive and encouraging environment that embraces the use of multiple representations in mathematics. Teachers play a crucial role in helping learners effectively create and employ diverse representations. To foster a deep understanding of fractions and to enhance learners' mathematical thinking skills, it is important to provide them with a range of representations that support their comprehension and expand their mathematical toolkit. A range of representations which included fractions in geometrical shapes, set models (bean bags) and rectangular strips were utilised throughout the lesson presentation in order to enhance understanding of fractions. The Grade 4 curriculum stipulates that learners should not only work with one kind of model as that limits their understanding of fractions (Department of Basic Education, 2011).

The PSTs mainly studied and interpreted learners' written work to gain an insight into how they perceived and thought about mathematics. The following extract shows how immediately following PST Ss' presentation of the lesson, she assigned work to the learners to evaluate their comprehension:

> *PST S*: Ah, let's see what we can do with the activity I am gonna give you, and then we are going to observe whether you grasped the concept. You guys can seat, so give me the bean bags. Thank you so much. Thank you. Ok, so the task is, write the fractions for the shaded parts in each shape.

The statement above clearly demonstrates what PST S intended to do. She intentionally gave the learners work to observe whether they grasped the concept or not. This allowed her to gain learner thinking from what they represented on paper. The other PSTs went around to observe what the learners were writing, and the following is an extract of how one of them helped a learner after having studied the learner's work:

> *PST L*: You have to tell me, what is a numerator. What's a numerator? What is it?
>
> *Learner 11*: The numerator tells us how many part we have
>
> *PST L*: And what is a denominator?
>
> *Learner 11*: Denominator tells us how many parts there are altogether.
>
> *PST L*: Yes, and I need you to count correctly, neh?
>
> *Learner 11*: Mam, so 4 is at the bottom.
>
> *PST L*: You must know that. You have just told me now. I need you now to count properly. The shaded part is the numerator and the denominator is …?
>
> *Learner 11*: Altogether.

PST L: Yes, I need you to write correctly, you can scratch. Then now count properly the numerator.

After studying what the learner wrote, PST L realised an error in the understanding of the concept, and she resorted to asking probing questions to determine where the error was that prohibited the learner from understanding. She succeeded since the learner got the correct answers after the incident as depicted in Figure 4.3.

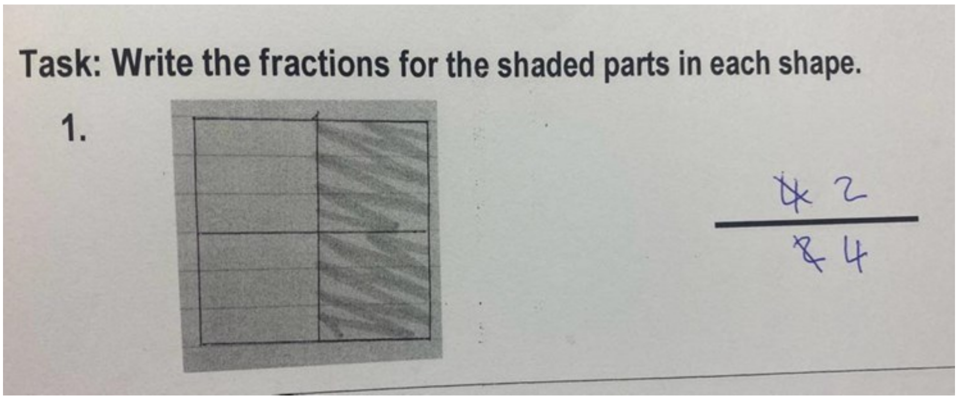

Figure 4.4 Learner's corrected work for Activity 1

In this particular case, learner 10 had difficulty distinguishing between a numerator and a denominator. Upon reviewing the learner's response, PST L took the opportunity to reinforce the definitions of these terms and clearly articulate the difference between them. This guidance proved helpful for the learner to arrive at the correct answer. It is important to note that PST L did not directly provide the answers but instead facilitated the learner's realisation of their mistakes.

In the same fashion, PST S analysed a learners' work in the following extract:

PST S: You have to show me which one is the correct one.

Learner 7: This one.

PST S: This one? So you need to correct yourself because you used the 2. You need to scratch 1. Show me the biggest here.

Learner 7: The biggest out of the all of them?

PST S: No, between the two. Which one is the biggest? Then this means?

Learner: Greater.

After carefully examining the learner's work, PST S discovered that the answers provided were incorrect. Through probing, she identified that the learner struggled with differentiating the greater-than and smaller-than signs. Figure 4.4 illustrates the learner's work, clearly showing that all the answers were incorrect because the learner did not understand the comparison signs. The use of representation in this case allowed PST S to gain insight into the learner's thought process and facilitate targeted assistance to address the issue.

PST S observed that the learner had difficulty distinguishing between the greater than and smaller than signs. The learner could identify which fraction was larger but struggled to determine the correct sign to use. It is possible that the learner did not carefully read the instructions as the signs were clearly displayed. Instead, they relied heavily on the instructions given by the teacher.

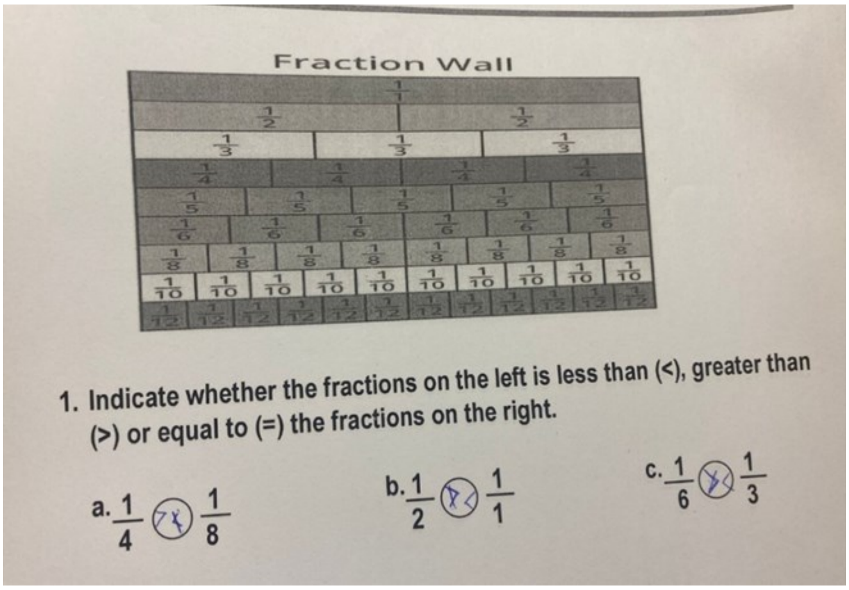

Figure 4.5 Learner's corrected work for Activity 2

As the Grade 4 curriculum suggests, the assessment on comparing fractions was accompanied by a fraction wall chart where learners can refer to as depicted in Figure 4.5.

4.5 PSTs' Consideration of KoST when Reflecting on the Lesson

Following the research presentation, a meeting was held with all four PSTs to discuss their findings. The post-reflection phase primarily centred around how the PSTs incorporated KoST during the lesson. Data were collected through research observations and unstructured interviews, which involved posing questions to the PSTs to gain clarity on specific aspects. The presentation of the findings incorporates the insights gained from the interviews, thus integrating them into the overall analysis.

The focus of my analysis was communication, connections, problem-solving, reasoning and proof, and representation as components of learner thinking. The analysis of the gathered information allowed me relevant insights to address the following research question: To what extent do PSTs consider the features/components of KoST when reflecting on the lesson?

4.5.1 Consideration of mathematical communication during the post-lesson reflection

During the reflection phase, I examined how the PSTs reflected on their use of communication to enhance learning in the classroom. Communication serves as a means to clarify understanding and facilitate the exchange of information, ideas, and instructions between teachers and learners. By focusing on this aspect, I gained insights into how the PSTs assessed their use of communication strategies to foster an optimal learning environment.

During the reflection session, the PSTs deliberated on the accurate and appropriate use of mathematical language. PST S, who was the volunteer presenter, gave the following account of how important the correct use of mathematical language is:

> *PST S*: I wanted to say the smaller and the bigger one, but then I am like, we are talking about vocabulary we trying to get the vocabulary there so I was like these are some of the questions that they are going to ask in the future, so rather then do it now. Lesser than greater than, they know what this is, how it looks, and they are not saying smaller, larger.

This is clear evidence that PST S intended to explain concepts in layman's terms but was guided by the mathematical discourse that should be maintained in class to correctly teach learners from the onset. With the correct mathematical discourse, learners are able to articulate their minds in a universal way that is accepted in the mathematics world. The following extract demonstrates how other PSTs reflected and maintained observation of the correct use of mathematical language.

> *PST M*: Then the vocabulary, I was so impressed because the learners were saying whole half, quarter like mathematical language was emphasised.
>
> *Interviewer*: Was it by PST S or they already had that vocabulary?

PST M: They already had that, and then PST S was just building on what they already know by not saying they already know the whole numbers. She kept on saying half, a quarter is a quarter, a third just like that the way we use it in Maths.

This articulates how the observer PSTs were immersed in the lesson and monitored the proceedings. PST M revealed how PST S was communicating with learners emphasising the correct use of mathematical language.

In other reflections, they expressed their awareness of what a conducive environment for learning should be. They strived to create a comfortable environment for learners to share their opinions:

PST S: So in terms of that but I know the No when the best asked, and I said no maybe I could have said, ah, why do you say that? Why do you say that instead of saying NO. I was just like saw his face was like, I was like …

PST M: Why you say that, or can we hear someone else.

PST MP: I think maybe hear somebody else is also not so nice. He needs to understand why he thought that this is the correct answer … because it's now what did I say wrong.

PST M: I saw that other guy when you said 'no', he was like … I was about to say that. I was about to say that like as…

PST MP: Said that if someone gives a wrong answer just … better challenge them to get to the correct answer instead of shutting them up.

PST S: Yaah, that's where you are mediator, in terms of that, like for example, with the aah what's that thing, growth mindset and fixed mind set. If you say no, they will have a fixed mind set. That will be like 'ok, I am not getting anything'. And then especially when I said no to this, like how many answers did give already. He is already fixed on how many or whatever. So when you say ok, uhm for example, why did you say that? Or in terms of that they will have a growth mindset they will be thinking, their thinking will start to … ok, but why did I do that? And then he will try to explain. There was one lady that can't tell me why at all, she couldn't tell me why. She was seated in front of PST L that one …

The conversation here came about after PST S responded to a learner who gave a wrong answer by saying "NO". The other PSTs observed this and the reaction of the learner soon after being shut out by a response from the teacher. Clearly the PSTs felt sorry for the learner, and they realised how such responses shut down a conducive learning environment. They gave some inputs on how best to address such a situation in the future and how best they can create and maintain a conducive environment for learners to freely share their opinions. The PSTs went on to evaluate how much in-service teachers are insensitive of learner thinking and environments they create through their actions:

> *PST MP*: I think they are so used to their teachers being hard, "yaa we have just explained this", and here you come calmly so trying to find out what the problem is, and they think you also gonna react the same way their teacher reacts.
>
> *PST MP*: Mrs W … is a colour cookie … she is like … I have just explained, it go read. That's how she is.
>
> *PST S*: Yaah, that's the problem that they have. Even when I was in school when you ask a questions one or two, three times … they will be like, "I have just explained it. Just read, I have just explained it". They don't think, ok but why is this person is not understanding for the third time that's happening. So …
>
> *PST MP*: Teachers are focusing on direct instruction and then not all of us are understanding

The above is a clear testimony of what the PSTs encountered during their practice sessions (WIL). The PSTs reflected on the kinds of environments that they were subjected to when they were still school learners and what they observed as PSTs in the field of teaching, and vowed to bring some changes to the schooling system. The noticeable change observed in this context is the creation of a supportive learning environment that encourages learners to freely express their opinions and thoughts. They suggested that direct instructions given by in-service teachers are one-way teaching from teacher to learners. The PSTs showed concern about the way in-service teachers do not consider learner thinking.

PST S showed concern about the way language became a barrier when she was presenting. She realised that learners did not understand the difference between greater than and less than. When she was presenting, she overlooked the matter because she thought every learner understood what she was referring to. She considered rephrasing complicated terms to words that are easier for learners to understand: “Because maybe, their second, like English is their second or third language, so maybe the greater and less they don’t understand it better. But maybe if we said bigger, smaller, equal”. In her reflection, she realised that it is important for effective communication during teaching and imperative that new concepts are clarified and communicated effectively for every learner to understand.

4.5.2 Consideration of mathematical connections during the post-lesson reflection

Connections come in different forms, such as connections between informal and formal mathematics, between concepts, between topics, between mathematics and other subjects, between mathematics and everyday life. Establishing connections is crucial for learners to rapidly grasp new ideas.

The presenter felt it was necessary to make connections between mathematics and the outside world and other subjects. During their reflection, the PSTs discussed the challenge some learners faced distinguishing between the greater-than and smaller-than signs. They explored the potential benefits of incorporating songs in the classroom to help learners establish connections between the outside world and the concepts being taught with the aim of facilitating understanding. PTS S said the following:

> In Afrikaans we learnt that this one is *kleiner*, means smaller, and then this one, which means bigger or greater, but even this one if it looks like this you can just switch it so, less than you see, and greater than, yaa. There was this rhyme for the denominator and numerator that I saw. If we had like more time or something, I could have played the video for that one. It’s like the rhyme you sing, then you sing the numerator, and then you take the numerator home, and the denominator you need to leave it that wherever. Things that you said about kinaesthetic is very important for learners. Yesterday I was actually in a class with one of the UP [University of Pretoria] students. She was doing her practical, and then she was using for figures of speech, she used a video with Lady Gaga’s song but then they switched the song into a figures of speech, and then, it was a very

> catchy song, and they knew exactly what's happening cause she made it fun for them to enjoy

PST S recognised that learners had difficulty grasping the concepts of greater than and less than, and she believed that incorporating music into the lesson could have been beneficial as it would have allowed learners to see the connection between mathematical concepts and other subjects. Music can make the learning experience enjoyable and fun, and therefore, has the potential to enhance understanding and engagement.

4.5.3 Considering problem-solving during the post-lesson reflection

Problem-solving is the process of finding solutions to complex or challenging situations or questions. It involves using critical thinking, logical reasoning, and creative strategies to identify, analyse, and resolve problems. Instead of training children to be merely arithmetic and algebraic human calculators, the emphasis in the mathematics curriculum should be on nurturing learners' creativity, adaptability, and self-reliance, enabling them to confidently and flexibly tackle novel problems (Foster, 2023).

The teachers realised the need to probe learners to reflect and justify their answers. PST S engaged in reflection on a particular incident when she asked a learner to provide a rationale for their answer, even though the answer given by the learner was correct. She reflected on it as follows:

> Yaah, Yaah, now yaah, with her she she's like when I was doing the activity, I was like, ok why do you say like that. She had it perfect why, but I think when I asked why she thought she is wrong. She is like, I said ok but why do you like it like that … yoh, did I do it wrong now and then she didn't want to explain cause she was afraid to explaining, then I told her, no, the answer is correct, I just want to understand how you got there, and then even then she couldn't tell me. I think she thought she is on a what do we call it *vasvrae* [quiz]. Oh, it's like a trick question. (PST MP: Wanted to catch her out). But yaa, most them in terms of that her confidence was a bit … But in mathematics, neh mathematics you need to have confidence in mathematics. It's one of the things that really like it's out of my past at all. I don't trust my thinking

Even in cases where a learner got the answer correct, the presenter wanted the learner to explain her thought process for the benefit of the learner and others in class. The ability to explain oneself in mathematics is a higher level of understanding that the PSTs wanted to cultivate within learners. Such a scenario is evidence that the PSTs considered learner thinking in their reflections. PST L also approved the practice to probe learners who had the right answers to explain themselves for the benefit of those who did not understand and could not get the right answers. She explained it as follows:

> So, uhm, ok, few learners, ok, I have just written that misconception with the fractions so umm she saw those who had a problem with the fractions, also she would ask them, I think she would hear the wrong answers and ask them why are you saying that instead of asking those who had the right answers. So she would ask them why are you saying that then they would give their reason, and then she corrected them, so after that they would grasp the whole idea so that's what I liked about it.

She believed that those who got the answer wrong should also explain their reasoning. She advocated for the presenter to probe those who got the answers right more.

The teachers appreciated the extension of the learners' ideas through authentic concrete examples and involving them in the process as this challenged them in a fun way. PST M explained it as follows:

> Me, I like the fact that you know during a lesson, if you involve learners, if they participate, they enjoy. It's like playing to them but they are learning at the same time because they were enjoying, like making that whole thing [fraction wall chart]. So, the fun part that one [engaging learners].

Bringing some fun practical work to the class engaged learners and they grasped concepts since they were involved in the process of making the fraction wall. The fraction wall was not imposed on learners but rather they constructed it as a class.

As the interviewer, I asked the PSTs about their frequent use of the phrase 'why do you say that?' The purpose of this question was to gain insight into the underlying rationale behind their chosen strategy. The following an excerpt from the conversation:

Interviewer: Constantly you were using the phrase 'why do you say that?' Do you have the rationale why you were constantly pestering learners to give you reasons behind what they were saying?

PST S: I wanted to probe them to get to the right answer. So, if I saw that they were giving a wrong answer based on a misconception, then I was like why, why do you think that. So that they can get back to the lesson that we did and understand why they were having the misconception. So uhm instead of telling them no its wrong or yes its right, I wanted to know why they were thinking cognitively in that way.

It is evident that the PSTs were aware of the benefits of probing learners to provide the rationale for their answers. When learners are asked to explain the reasoning behind their answers, it prompts them to engage in deeper thinking and analysis. They need to consider the logic and evidence that support their responses, fostering critical thinking skills. It provides an opportunity for them to express their understanding and showcase their knowledge.

4.5.4 Considering mathematical reasoning and proof during the post-lesson reflection

Reasoning and proofing involve two major activities, namely generalising, such as forming educated guesses (conjectures) about potential mathematical relationships, and constructing arguments to determine the validity or falsehood of those generalisations, some of which may meet the criteria for proofs (Stylianides et al., 2013).

Creating and upholding an environment that allows learners to make conjectures is vital for reasoning and proof. The presenter realised one of the learners made a conjecture and she acknowledged it during the reflection session. When the learners were given an activity to order fractions, there was a sum that was already ordered from smallest to biggest, and PST S came across a learner who made a conjecture of how to quickly order fractions. They reflected on it as follows:

PST MP: So they want to rearrange it, thinking that its wrong. And most of them started with the denominator there is bigger … like 1 over 10 … yes.

PST S: That's what one of them said, and I was like wow that's a good way to remember it. In fractions if the denominator is bigger, it's the smallest fraction.

Learners need environments that encourage them to be confident and to make conjectures about what is being taught. The teacher in such an incident could have elaborated further to assist other learners in class. The PSTs did not delve much on how they identified reasoning and proof.

4.5.5 Considering mathematical representations during the post-lesson reflection

Representation involves representing mathematical information in a way that is accessible and meaningful to learners, enabling them to better understand and communicate mathematical concepts. During the reflection process, the main duty of the PSTs was evaluating the effectiveness of their chosen representations to enhance learning within the classroom environment.

One unanimous agreement among the four PSTs was their recognition of the value of using learners' work and representations as a means to deduce their thought processes. They placed a significant emphasis on studying and interpreting the work produced by the learners. They discussed it as follows:

PST S: I do not know if you saw but the line there I thought we can, they weren't quite sure it's a fraction line. They were not sure that they need to write it on top and below. So, the line was that big. So maybe in terms of that practicality, it could have been improved umm yaah.

PST MP: During the first activity, learners were writing boma [such as] 5 over 9 meaning that 5 that was not shaded over 9, so when I asked one of the learners, he said no because 5 is not shaded, and then the parts are 9. Yes, he got the denominator right but the numerator was not correct.

PST S: Coz the question said it needed the shaded part. So, maybe he didn't understand the question.

PST MP: The other thing with the last activity, learners were like ok now they have already arranged. I liked the way you challenged them, like they were like yaa we are confused. So it was challenging for them already the fractions were arranged.

PST MP: The confusion was why would they want us to write the correct answer while the correct answer is already there. So they thought if I write this as is, it might be wrong. So I always refer them back to from smallest to biggest (PST L: The instructions.) Yes, the instructions, but it took them a while to understand.

The student teachers demonstrated keen observation skills as they assessed the ongoing progress while learners were working on their assignments. During this period, each PST was actively engaged in assisting the learners, indicating their careful examination of the learners' thought processes through their work. The PSTs went a step further by providing suggestions on how to enhance the activities to promote a better understanding among the learners. Ultimately, the learners' representations served as the primary source for the student teachers to deduce their thinking and understanding.

From the statement by PST MP in the conversation above, it is evident that the PSTs possess an inaccurate conception regarding the appropriate use of mathematical language. The PSTs consistently used inappropriate terms like "5 over 9" instead of 5 ninths in naming fractions.

The initial remarks made during the reflection process by PST S indicate that the PSTs heavily relied on deducing learners' thinking through representations. In PST S's opening statements during her reflection, she specifically referred to the learners' work, which helped her gain insights into their thought processes. This was further revealed through the unstructured questions I posed to PST S, prompting her to provide specific details rather than generalising her reflections. The following is an excerpt of the conversation:

Interviewer: What was the misconception there? What was the confusion?

PST S: Umm, the confusion. Ok. One misconception one girl had is that she wrote 2 on 2. So there was two shaded parts, and then 2 parts left that weren't shaded, so she wrote 2 on 2, and I am thinking ok, its 2 out of 2 but it was actually 2 out of 4. So that was one of the misconceptions she had in term of numerator and denominator.

Commencing the reflection session by discussing the final part of the lesson presentation highlighted the significant role that representations play in providing valuable insights into learners' thinking for the teacher. To establish the framework for a thorough reflection, I prompted the PSTs with unstructured questions, encouraging them to delve into the specifics and provide detailed insights.

Analysing learners' work provided the PSTs with a clearer understanding of the learners' thinking processes. The following is an account of how the PSTs swiftly identified the underlying cause of a misunderstanding by examining the learners' work:

> *PST S*: But Victor you will see in those activities neh, but especially with Activity 2 a lot of them got equal. Especially with smaller, equal to, greater than, equal to. A lot of them got those wrong.
>
> *Interviewer*: Why?
>
> *PST MP*: They don't understand the signs.

It is often easier to discern learners' thinking through their written work because it provides a tangible representation of their thought processes. When the learners expressed their ideas, solutions, and reasoning in writing, it allowed the PSTs to examine the steps taken, the strategies employed, and any potential misconceptions. Written work offers a snapshot of learners' thinking at a particular moment, enabling teachers to gain insights into their understanding, problem-solving approaches, and areas where further support may be needed.

4.6 Conclusion

In this chapter, I presented the findings of this research that came from an intensive data analysis of the lesson plan, transcribed research lesson observation, and transcribed post-lesson reflection session. The findings of this data analysis chapter revealed that the documents analysed, such as the lesson plan and learner's work samples, provided important context for understanding the observed lesson. Lesson observation allowed the collection of rich and detailed data on classroom interactions and learner thinking and engagements. The post-lesson reflection provided insights into the thought processes and decision-making of the PSTs, as well as potential areas for improvement.

Overall, this chapter highlighted the importance of triangulating the data sources to gain a holistic understanding of teaching and learner thinking. By using a variety of teaching methods, teachers can gain a more complete picture of what transpires in the thinking of their learners, and they can use such information to make informed decisions about instructional practices and learning environments.

CHAPTER 5: DISCUSSION OF FINDINGS, RECOMMENDATIONS AND CONCLUSIONS

5.1 Introduction

This chapter looks at the research outcomes by highlighting the patterns and insights discovered to ultimately arrive at a deeper understanding of how mathematics PSTs grapple with the infusion of KoST within the crucial stages of Lesson Study. I sought to address the following primary research: How do mathematics PSTs infuse KoST into teaching and learning within the Lesson Study cycle? To guide my exploration and delve into the primary question, I formulated the following secondary research questions:

- Which features/components of KoST are reflected in the PSTs' lesson plans?
- How do PSTs apply the features/components of KoST during the teaching of a mathematics lesson?
- To what extent do PSTs consider the features/components of KoST when reflecting on the lesson?

In this chapter I not only scrutinise the areas where KoST was prominently used but also confront the spaces where opportunities for growth and refinement emerged. I contemplate the implications of my findings, and consider the potential for enriching teacher training programs and enhancing the quality of mathematics education in the dynamic context of South Africa.

The actual discussions of the findingssummary of the findings related to the LS stages is presented in Table 4.2.

Table 5.1 Summary of findings

COMPONENTS OF KOST	LS STAGES		
	Lesson Planning	*Lesson presentation*	*Reflection*
COMMUNICATION	• The planning for probing questions • Some appropriate resources were listed in the lesson plan to enhance communication • Planned to cater for divergent thinking and addressing possible misconceptions	The lesson presenter • Encouraged classroom conversations: teacher-learners and learners-learners • used probing questions to engage learners • reinforced the use of correct mathematical language	PSTs: • reflected on the correct use of mathematical language • deliberated on how they could improve the learning environment through communication for it to be conducive for learners • pointed out language barrier as a hindrance to effective learning and how they could improve in their approach to teaching given similar circumstances occur in the future.
CONNECTION	• Reference to prior knowledge in the lesson plan • PSTs stated that they will show learners instead of allowing them to discover on their own. E.g. "I will show them how if you have more people in your group your piece of bread is smaller".	The presenter: • connected the topic of the day to prior knowledge • connected everyday knowledge to the new concept through the use of manipulatives	PSTs reflected on the use of familiar contexts to foster connections e.g. the use of songs
PROBLEM-SOLVING	• PSTs value and recognised learners as problem solvers, especially in the context of fractions. • Incorporating time allocation on every activity giving learners ample time to solve mathematics problems	The continuous use of probing questions for learners to reflect and explain their thinking	PSTs: • reflected on the usefulness of probing questions in enhancing problem-solving. • recognised the importance of authentic concrete examples in enhancing problem-solving in learners.
REASONING AND PROOF	Prevalence of purposeful activities that required reasoning and proof	The lesson presenter used the question 'why' to evoke reasoning and proofing and encouraged conjecturing	PSTs reflected on the creation of an environment that upholds conjecturing and the use of the why question
REPRESENTATION	PSTs planned to use a variety of forms of representation to teach fractions e.g. balls, fraction symbols and loaves of bread which were familiar to learners	• There was evidence of learners' written work as a form of representation of learner thinking • The lesson presenter used a variety of representations which included bean bags, rectangular strips, 2-D shapes etc.	There was a reflection on the usefulness of learners' written work in revealing learner thinking

Although all components of KoST featured in each stage of LS, some were more prominent than others. For instance, representation as an element of KoST during post-lesson reflection features only one aspect which is learners' written work and yet features more aspects under lesson planning. The findings also indicate that certain elements of KoST were less prevalent in different stages of LS.

This chapter is organised according to the structure depicted in Figure 5.1.

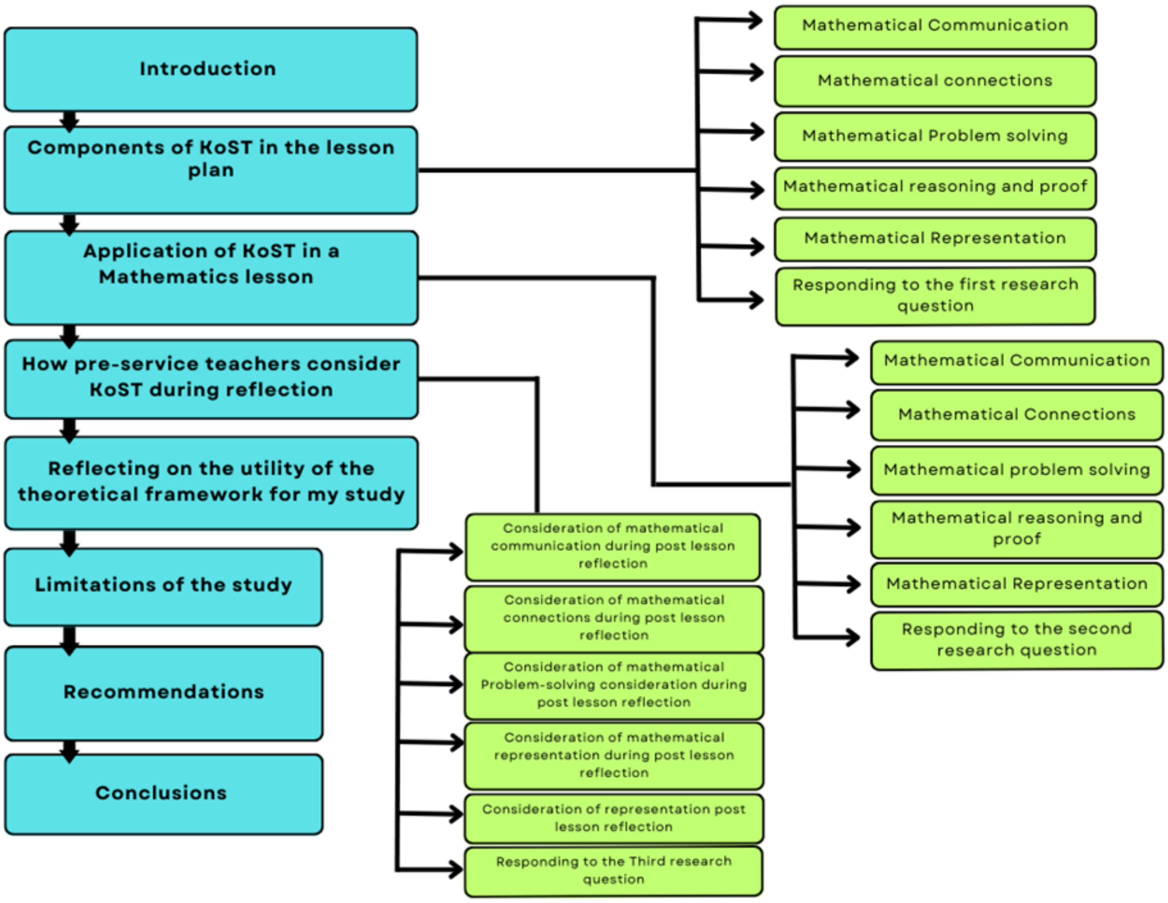

Figure 5.1 Outline for Chapter 5

5.2 Components of KoST in the Lesson Plan

The PSTs planned to teach comparing fractions to Grade 4 learners, and hence, I discuss the findings from their lesson plan. I mainly focus on the features and components of KoST evident in the preservice lesson plans, thereby responding to research question 1 which states that "Which features/components of KoST are reflected in the PSTs' lesson plans?".

5.2.1 Mathematical communication

Communication can be described as the exchange of thoughts or ideas between individuals (Fatimayin, 2018). Communication plays a crucial role in shaping learners' mathematical thinking, making it a key strength that teachers must demonstrate during mathematics instruction and lesson planning. The findings shed light on the strategies employed by the PSTs in planning their research lessons to create an environment that promotes the sharing of ideas and encourages divergent thinking among learners. The incorporation of probing questions emerged as a prominent approach used by the PSTs to facilitate meaningful interactions and engagement in the classroom. Additionally, the emphasis on catering for divergent thoughts and misconceptions indicates the PSTs' awareness of the importance of addressing individual learners' learning needs.

One of the resounding threads that emerged from the analysis of the PSTs' lesson plan was their intent to foster an environment that promotes the sharing of ideas through probing questions. This practice resonates deeply with knowledge of content and teaching within the MKT framework, and the PSTs tapped into foundational mathematical concepts to engender an atmosphere of inquiry and discourse. Their intention to create a platform for idea-sharing signifies the essence of KCT and KCS, which seeks to ignite the curiosity and intellectual engagement of learners. This resonates with Lamb (2015b) findings that the planning process can give PSTs valuable insights into their teaching practice and the needs of their learners, which can help them develop their skills and knowledge as teachers. Lamb (2015b) found that peer feedback and reflection during planning helped PSTs scaffold the exploration of improving their own and their peers' performance in front of a class. When using probing questions, the PSTs engage learners in meaningful dialogues and discussions.

Some elements of the PSTs' lesson planning aimed to enhance communication in the classroom. However, one notable shortcoming was the intent to use puzzles to clarify misconceptions regarding numerators and denominators. While visual representations can assist with fraction understanding, the fundamental concept of fractions involves equal partitioning of a whole. A jigsaw puzzle does not adequately portray this idea of equal shares since its pieces are not uniformly shaped or sized. That part of the lesson plan reveals limitations in the PSTs SCK and KCT. Specifically, they did not recognise that the selected puzzle activity may promote further confusion instead of conceptual clarity. While striving to engage learners visually, overreliance on an imperfect model like a puzzle reveals insufficient knowledge in formulating developmentally appropriate teaching aids understandable by others.

5.2.2 Mathematical connections

The findings revealed significant insights into how the PSTs integrated connections within their lesson planning process. One noteworthy observation was that the PSTs demonstrated a conscious effort to include prior knowledge in their planning for the research lesson. This practice is deeply rooted in the heart of KCS and KCT within the MKT framework. By weaving prior knowledge into their lesson plans, the PSTs created a bridge between the known and the new, fostering a continuum of mathematical understanding. This deliberate inclusion embodies the essence of KCS and KCT, which underscores the importance of building on established mathematical concepts. By tapping into learners' existing knowledge, the PSTs acknowledged the importance of building upon learners' foundations to promote deeper understanding and engagement.

Furthermore, the intent to connect formal mathematics with intuitive informal mathematics underscores an alignment with KCT and KCS within MKT. This dimension emphasises the need for pedagogical strategies that bridge the gap between abstract mathematical concepts and the real-world experiences of learners. The planned connection between formal and informal mathematics resonates with KCT and KCS, which promotes teaching methods that work well for different ways of learning. This is in line with the findings of Selvianiresa and Prabawanto (2017) that by connecting mathematics to real-life contexts and integrating it with other content areas, learners can also develop a deeper understanding of the subject and see its relevance to their lives. Informal mathematics often involves problem-solving and exploration in real-world contexts. SLT suggests that these informal problem-solving experiences can complement formal mathematics education by providing practical applications and reinforcing abstract concepts. This bridge between formal and informal mathematics can enhance learners' understanding and motivation to engage with mathematical concepts.

Conversely, the findings exposed some missed opportunities in PSTs' lesson planning, particularly in relation to excluding investigations. Investigations connect learners' mathematical ideas to real-life experiences in the classroom. The exclusion of investigations from the classroom environment can have significant consequences for both learners and teachers. For learners, this omission may lead to reduced engagement, limited depth of understanding, and a missed opportunity to connect mathematical concepts to real-world applications. This is in line with Da Ponte (2007) findings that investigations can be a valuable tool for promoting learning and engagement in mathematics. Investigations can help learners develop a deeper understanding of mathematical concepts and processes and can make mathematics more engaging and enjoyable for both learners and teachers (Da Ponte, 2007). Therefore, PSTs may find themselves with a narrowed pedagogical repertoire, relying solely on teaching methods that might not cater for diverse learner needs. Moreover, this exclusion indicates gaps in the PSTs' KCT and a limited grasp of how learners learn best. These implications underscore the importance of equipping PSTs with a comprehensive understanding of SCK, KCT and KCS enabling them to provide a rich and varied educational experience that aligns with the diverse needs and aspirations of their learners.

Furthermore, the findings suggested that the PSTs did not extensively link fractions to other subjects in their planning, which is contrary to Selvianiresa and Prabawanto (2017) encouragement to use real-life contexts and integrate it with other content areas. This omission further aligns with the findings by Lee (2019) that, during the early stages of the Lesson Study cycles, the PSTs were mainly focused on group organisation rather than learner thinking or lesson and instructional goals. The implications of not linking fractions with other subjects when planning to teach can have both pedagogical and interdisciplinary consequences. When fractions are not linked to other subjects in the lesson plan, learners may struggle to grasp the real-life applications and significance of fractions, leading to a limited contextual understanding of the concept. Integrating fractions into other subjects creates opportunities for cross-curricular learning, enhancing learners' ability to see the interconnectedness of various disciplines. When fractions are taught in isolation, learners may struggle to transfer their understanding of fractions to other mathematical concepts or problem-solving situations.

5.2.3 Mathematical problem-solving

The findings shed light on the PSTs' perceptions of and approaches to fostering problem-solving skills among learners. One notable observation was that the PSTs believed in learners' capabilities to solve problems. This positive attitude indicates the

PSTs' confidence in learners' problem-solving abilities, which can positively influence learners' self-efficacy and motivation to tackle challenging tasks. Furthermore, the PSTs prioritised giving learners sufficient time to solve problems. Allowing ample time for problem-solving activities signifies the PSTs' understanding of the cognitive processes involved in problem-solving and the need for learners to engage in thoughtful exploration and analysis.

Although the PSTs believed in their learners' problem-solving abilities and allocated sufficient time for problem-solving, they neglected to include non-routine problems in their lesson planning. The essence of problem-solving lies in thoughtfully chosen tasks that teachers present in the classroom. Consequently, omitting such questions can have detrimental effects on the learners. Bailey (2022) cautioned that the absence of problem-solving in the classroom may restrict learners' opportunities to cultivate advanced thinking, reasoning, and problem-solving skills, which are crucial for their future success in mathematics and other academic disciplines. This finding is in line with that of Sahin-Taskin (2017) that PSTs find it difficult to find activities that are appropriate for learners' learning levels, interests, and needs, to determine whether their planned activities meet the learners' levels, and to prepare activities appropriate for their learners' learning levels.

The PSTs' choice of routine problems implies a lack of knowledge of content and learners (KCS), KCT, and SCK. It is evident that the PSTs cannot discern the distinction between routine and non-routine problems. Routine problems involve well-rehearsed procedures with predictable solutions. Opting for such problems instead of non-routine ones neglects the developmental impact that non-routine problems can have on learners and the subject itself. If teachers proceed to teach the planned routine problems in the classroom, learners may become disengaged and lack confidence when encountering non-routine problems outside the classroom. Their exposure to routine problems presented as routine may hinder their ability to effectively approach and solve non-routine problems.

In the planning of activities for the learners, the PSTs utilised a triangle to teach the concept of fractions through equal sharing. However, dividing a triangle into equal parts can be challenging and intricate, and in this case, the parts of the triangle were not equal. This indicates that the PSTs lacked SCK and KCT. SCK is essential for making the features of specific content visible and accessible for learners, while KCT aids teachers in evaluating the instructional advantages and drawbacks of various representations. This highlights the importance of PSTs developing a strong understanding of the subject matter and the most effective ways to represent it for their learners.

5.2.4 Mathematical reasoning and proof

One striking observation from the findings was the resolute intention of the PSTs to integrate concrete manipulatives into their lesson plan. The significance of this observation extends beyond the mere selection of teaching aids; it illuminates the PSTs' profound understanding of the paramount importance of providing learners with tangible tools to underpin their reasoning and nurture their grasp of mathematical concepts. This is consistent with the results obtained by Ndlovu and Chiromo (2019) that the PSTs in their study demonstrated an enhanced proficiency in integrating concrete material into their teaching as a result of increased collaborative learning experiences and enriched mathematical discourse.

The use of concrete objects within lesson planning resonates deeply with the core principles of SCK within MKT. SCK emphasises how important it is for a teacher to really know and understand the subject they are teaching. This includes knowing how the subject is organised, how different parts are connected, and where students might have trouble. By integrating concrete manipulatives, the PSTs demonstrate their nuanced comprehension of the subject's intricacies, and crucially, their ability to translate this understanding into accessible, hands-on experiences for learners.

Furthermore, this deliberate choice of incorporating concrete manipulatives reflects the KCT dimension within the MKT framework. KCT emphasises the synergy between content knowledge and pedagogical expertise. It is a union that enables teachers to create transformative learning experiences.

On the contrary, the results indicated that the lesson plans did not include group work, projects, or investigations. These activities are essential for learners to think and prove their ideas. When learners work together and explore math concepts in different ways, it helps them understand better. Including more group activities and investigations in the lesson plans can make learning more exciting and improve learners' ability to reason and prove their solutions. Mata-Pereira and da Ponte (2017) presented design principles for an intervention aimed at enhancing learners' mathematical reasoning in the classroom, and suggested teacher actions such as inviting learners to participate, challenging or guiding them to participate, encouraging learners to share their ideas, and asking them to go beyond the task can lead learners to generalise and justify their responses.

Teacher actions play a crucial role in enhancing learners' mathematical reasoning, particularly in promoting generalisation and justification in mathematical reasoning processes. This omission hinders learners' opportunities to engage in reasoning and proof activities and to develop their ability to justify and communicate their mathematical thinking effectively. Encouraging PSTs to create such a classroom environment can help learners join more actively in math discussions. This, in turn, will boost their ability to think critically and feel more confident when sharing and supporting their ideas. The omission of group work, projects, or investigations relates to lack of KCS and KCT within MKT, which emphasises the need for diversity in teaching methods which incorporates varied instructional strategies.

5.2.5 Mathematical representation

The findings offered valuable insights into how the PSTs incorporated different forms of representation in their lesson planning process. One significant observation was that the PSTs sought to use learners' written work as a representation. This practice acknowledges the importance of allowing learners to articulate their mathematical thinking in written form, providing an opportunity for the PSTs to assess and support their understanding. The integration of learners' work into the lesson plan aligns with the pedagogical philosophy championed by MKT, particularly within the dimensions of KCT and SCK. This aspect highlights how important it is for teachers to be really good at their subject and also know how to teach it in a way that learners can easily understand and learn from. In this context, the inclusion of learners' work exemplifies KCT and SCK as they reflect the PSTs' deliberate choice to employ a tangible representation which reveal mathematical thinking, fostering a dynamic and participatory classroom environment.

Conversely, the findings indicated the use puzzles which constitute inappropriate representations that led to misconceptions among learners. This dovetails with findings by Sahin-Taskin (2017) that the PSTs in their study found it difficult to find activities that are appropriate for learners' learning levels, interests, and needs. Mainali

(2021) reiterated that the correct choice of representations and how to use them should be based on the nature of the mathematics content being taught. The PSTs in the current study planned to use puzzles to teach fractions. Rather than conveying equal sharing, a puzzle risks implying that fractions denote shapes rather than proportional quantities.

Using wrong representations, such as puzzles, to teach fractions in a mathematics class can have several implications for learners and the teacher. Puzzles lack the mathematical rigour required to build a solid foundation. Puzzles do not accurately represent the concepts of fractions and that may lead to misconceptions and confusion about the true meaning and operations of fractions. When learners are taught fractions through inappropriate representations like puzzles, they will struggle to apply their knowledge to real-life situations or more complex mathematical problems since they lack the fundamentals. On the other hand, teachers may encounter challenges to explain fractions using puzzles, leading to frustration for both teachers and learners.

5.2.6 Responding to the first research question

The discussion within this section was primarily directed towards addressing the first research question: Which features/components of KoST are reflected in the PSTs' lesson plans? In response to this research question, the analytical findings showed that although certain components did not feature prominently, the PSTs included a notable array of elements pertaining to learner thinking within the lesson plan. For instance, problem-solving and reasoning and proof did not feature prominently. Therefore, in answering this research question, I conclude that some of the KoST components featured more than others. PSTs found it challenging to unpack fractions in a manner that is clear, and comprehensible to learners.

5.3 Application of KoST in a mathematics lesson presentation

In these subsections, I explore the findings on how the PSTs applied the features and components of KoST during the teaching of a mathematics lesson thereby responding to research question 2 which states that "How do PSTs apply the features/components of KoST during the teaching of a mathematics lesson?"

5.3.1 Mathematical communication

This section looks closely at how the PSTs apply KoST in teaching comparing fractions and how they embraced the MKT framework as a guiding beacon. Within this framework, communication as a component of KoST revealed a complex pattern of strategies through which the PSTs cultivated a dynamic learning environment that resonates with the dimensions encapsulated within MKT.

One of the salient threads woven into the fabric of the PSTs' instructional practices was the use of interactive methods to help learners learn better by asking probing questions during the mathematics research lesson to stimulate cognitive engagement among learners. This way of teaching is connected to KCS and KCT. The use of probing questions as a teaching approach calls for knowing which questions to ask to determine learners' understanding and understanding how posing certain questions can support or challenge learners at a certain level. The teaching approach by PSTs aligns with Morrissey et al. (2020) findings that the PSTs in their study demonstrated an ability to pose probing questions to emphasise contextual aspects, semantic interpretations, or representations.

However, an intriguing shade of complexity emerges when considering the PSTs' ability to elaborate the fractions concepts effectively. The inability to elaborate on concepts shows how PSTs lack CCK, SCK, KCS and KCT domains and sheds light on the knowledge gap that would be created in the academic trajectory of the learners. This finding aligns with Ramaligela et al. (2019) finding that the classroom practice of most PSTs does not demonstrate a comprehensive grasp of subject matter or a deep understanding of their learners. The PSTs were unable to coherently introduce new concepts and to connect concepts in a procedural manner.

5.3.2 Mathematical connections

One notable aspect of the PSTs' pedagogical practices was the incorporation of prior knowledge into their lesson. This practice closely aligns with the tenets of SCK, KCS and KCT as defined by MKT. By tapping into learners' existing mental frameworks, the PSTs fostered an environment where new mathematical concepts were scaffolded upon a familiar foundation, thereby promoting a smoother integration of new ideas. Manipulatives were prominent too during the lesson to bridge the gap between abstract mathematical concepts and real-world experiences. The integration of manipulatives aligns with KCS's emphasis on tapping into learners' diverse learning modalities and sensory perceptions. This dynamic strategy harnesses learners' intuitive understanding and concretely connects mathematical abstractions to their everyday life.

However, I did witness the exploration of activities that veered off-course from the intended mathematical concepts. An alignment between activities and curriculum objectives is critical for effective instruction because it ensures that every pedagogical endeavour contributes cohesively to learners' academic journey. This finding was not unique to this study, and Ramaligela et al. (2019) witnessed the same occurrence. They found out that the PSTs in their study were unable to coherently introduce new concepts and faced difficulties connecting concepts in a procedural manner. Engaging in activities that stray from the intended concepts can lead to the formation of misconceptions. Learners might develop incorrect ideas about the mathematical concepts being taught. This clearly demonstrates lack of CCK, SCK, and KCT since it involves knowing the content and how to teach mathematical concepts to learners.

5.3.3 Mathematical problem-solving

A discordant note emerged from the study, revealing that the PSTs predominantly employed routine problems and omitted non-routine problems. This raises questions about the PSTs' KCT within MKT, which looks at how effectively they adapt their pedagogical strategies to challenge learners beyond rote problem-solving. The finding is contrary to Leavy and Hourigan (2020) findings that the PSTs in their study showed significant improvements in their ability to pose good problems, including a greater awareness of the openness of problem-solving situations, the use of various strategies, and the possibility of multiple solutions.

The use of routine problems by PSTs signifies a deficiency in their profound comprehension of the subject matter, consequently pointing to absence of SCK. This leads to a constrained KCT as PSTs lack the exposure and expertise required for instructing non-routine problems.

5.3.4 Mathematical reasoning and proof

The practice of sustained self-reflection, manifested through the PSTs' consistent inquiry of 'why' in response to learner contributions throughout the lesson showcased a strong alignment with KCS within MKT. This practice encourages learners to explore the underlying reasons and rationale behind mathematical concepts, fostering a deep understanding that goes beyond surface-level memorisation. The emphasis on 'why' resonates with KCS, which encourages PSTs to delve into learners' cognitive processes to guide and refine instructional practices. Through this lens, PSTs harness their understanding of mathematical content to probe the underlying thought processes of learners, thereby cultivating a culture of critical thinking and self-assessment. While the presenter effectively sought to stimulate deeper thought through "why" inquiries, the technique occasionally failed to align with the intended objective, thus limiting its effectiveness. In some instances, the lines of questioning employed did not fully elicit or explore the critical thinking targeted, failing to yield the hoped-for insight into learners' reasoning.

This discovery is in line with the findings of Morrissey et al. (2020) that indicated that most PSTs believed that asking questions related to information and procedures is equivalent to asking probing thinking questions. This is because questions related to information and procedures do not consider the underlying conceptual understanding of the learner, and do not probe into learner thinking as required.

5.3.5 Mathematical representation

A significant element that ran through the fabric of the PSTs' pedagogical practices was their use of written work as a lens to decode learner thinking. The practice of using written work to decode learner reasoning serves as a testament to the significance of KCS in illuminating the pathways through which learners engage with mathematical ideas. The PSTs' use of written work as a representation form showcases an alignment with the KCS dimensions, as it summarises the cognitive exploration behind learners' mathematical expression.

Decoding learner thinking from their written work is not unique to this study as this was effectively used by Selmer et al. (2021), and they discovered that PSTs engaged in noticing aspects of learner thinking in written work and focusing on both disciplinary and non-disciplinary aspects. In their study, the PSTs developed the capacity to articulate, assess, interpret, and provide responses regarding learner thinking without necessitating a shift towards discerning more generalised aspects through explicit inquiry.

PSTs incorporated three different forms of representing fractions, which is in line with the Grade 4 curriculum which encourages the use of multiple models to enhance understanding of fractions. This is a demonstration of knowledge of the curriculum and SCK.

5.3.6 Responding to the second research question

This section's discourse primarily addressed the second research question: How do PSTs apply the features/components of KoST during the teaching of a mathematics lesson? In response to this question, the analytical findings indicated that individual components of KoST appeared sporadically throughout the lesson, except for communication. The element of communication as a component of KoST demonstrated a more significant presence during the research lesson than all other components. This result suggests that the PSTs effectively utilised communication during the lesson, but they could have improved their application of the other KoST components to better support learners' understanding of the subject matter.

5.4 How preservice teachers consider KoST during reflection

In the following subsections, I discuss the findings on how the PSTs infused features of KoST during the post-lesson reflection session thereby responding to research question 3 which states "To what extent do PSTs consider the features/components of KoST when reflecting on the lesson?"

5.4.1 Consideration of mathematical communication during post-lesson reflection

The PSTs' reflection session was jam-packed with contemplation of the accurate and appropriate use of mathematical language. The PSTs' considerations of the precision and suitability of mathematical language underscore their commitment to ensuring clarity and effectiveness in mathematical discourse. The discovery here is in complete congruence with the findings of Guner and Akyuz (2020) that PSTs predominantly concentrate on classroom lesson execution and pedagogical strategies when engaging in reflective activities during the initial phases of their Lesson Study cycles. In their study, the PSTs directed their attention towards evaluating their instructional methodologies, with a particular emphasis on addressing the challenges they encountered.

In this research, the PSTs displayed a greater degree of concern about the precise and suitable application of mathematical language by the volunteer PST. While it is undeniably crucial to maintain vigilance over the accurate use of mathematical terminology, this particular focus underscores how this heightened awareness diverted the PSTs' attention from concentrating on learner thinking. Consequently, the PSTs failed to engage in a reflective examination of the varied forms of communication, such as the potential incorporation of group work, which could have been employed during the research lesson to enhance their capacity to notice learner thinking.

A reflection on the PSTs' endeavours to create a comfortable environment for learners to share ideas and opinions demonstrated how they valued learner thinking. This practice aligns harmoniously with both KCS and KCT within MKT. By nurturing a classroom atmosphere that encourages open dialogue and evidence-based discourse, the PSTs exemplified KCS by considering learners' perspectives and KCT by enacting pedagogical strategies that facilitate active engagement.

5.4.2 Consideration of mathematical connections during post-lesson reflection

A significant facet that emerged from the analysis of the PSTs' reflections is their consideration of linking the fraction concept with a song. This reflective practice reveals an intention to draw connections between abstract mathematical concepts and engaging real-world experiences, aligning with the essence of SCK within the MKT framework. By employing creative and memorable strategies like a song, the PSTs exemplified SCK, which emphasises the importance of pedagogical techniques that make mathematical concepts accessible and engaging.

It was clear that the PSTs engaged in a profound level of introspection with a specific focus on considering learner thinking. They reflected on alternative pedagogical strategies they could have used to capture learner thinking. This is contrary to findings of Lee (2019), whose research indicated that PSTs demonstrated professional growth during the later stages of Lesson Study reflections, during which they increasingly concentrated on learner thinking and responses, in contrast to the earlier Lesson Study reflections where their discussions revolved around learners' dispositions. It is worth noting that the use of unstructured interview questions in this study may have significantly contributed to guiding the PSTs towards reflective considerations centred on learner thinking. It is worth highlighting that there was no reference to the incorporation of prior knowledge in their reflections until a specific inquiry was made regarding the significance of the prior knowledge they integrated into the introduction of the research lesson.

There was no mention during the entire reflection session of the significance of using manipulatives or real-life examples. This omission raises questions about the exploration of pedagogical strategies that tap into the cognitive processes of KCS within MKT. Evidently, the PSTs primarily focused their reflections on elements they intended to incorporate into the lesson and areas that required improvement.

5.4.3 Consideration of mathematical problem-solving consideration during post-lesson reflection

There was no special mention of the types of problems used during the lesson, namely routine problems, and no mention of the problem-solving strategies employed. This is a clear evidence of lack of analytical lenses for scanning the material, aids, and tasks given to learners to evaluate their effectiveness to enhance learning. The PSTs were worried about whether the learners were able to do calculations from the given routine sums and complete them. This particular observation is not an isolated occurrence and finds resonance in Amador and Weiland (2015) finding that PSTs similarly reflected on the learners' tendency to prioritise task completion over engaging in mathematical reasoning. In instances where the PSTs did offer interpretive comments on classroom occurrences, these insights were occasionally tempered by references to classroom management or specific lesson-related actions.

The inability to differentiate between routine and non-routine problems has an obvious impact on the developmental trajectory of the PSTs' capacity to formulate meaningful questions that stimulate learning. Furthermore, it may lead the PSTs to draw inaccurate conclusions regarding learners' comprehension of concepts, potentially overlooking the necessity for further instructional support.

5.4.4 Consideration of mathematical reasoning and proof during post-lesson reflection

An intriguing observation from the reflection session was the PSTs' consideration of the significance of probing learners to reflect and justify their answers. The PSTs' reflections on the constant use of the question 'why' throughout the lesson reflected a profound commitment to pedagogical practices that resonate with KCT and KCS within MKT. By consistently prompting learners to question and justify their reasoning, the PSTs embodied KCT and KCS, emphasising the importance of helping learners construct and articulate their mathematical understanding. This practice mirrors the principle of KCT, which encourages PSTs to integrate content knowledge with effective instructional strategies.

The reflection on the tendency by the PSTs in this study to frequently prompt learners to substantiate their reasoning stands in contrast to the discoveries made by Guner and Akyuz (2020) in the first cycles of Lesson Study because in their study the PST neither attended nor respond to learner ideas in ways that could reveal learner thinking. The PST ignored wrong answers from learners as she proceeded with solving the problems, and she did not ask for reasons behind learner strategies. The PST directly communicated the correct answers whenever she was correcting learner mistakes instead of prompting them to reason it out on their own.

5.4.5 Consideration of mathematical representation during post-lesson reflection

A noteworthy and conspicuous observation picked up from the analysis of the PSTs reflections was their reliance on learners' work as a pivotal focal point of their discussions. However, it is vital to acknowledge the potential pitfalls of an excessive dependence on learners' work as the primary means of deciphering learner thinking. This overreliance can potentially mislead teachers, contingent upon the nature of questions posed to learners. Teachers may erroneously infer a comprehensive understanding of a concept by learners when, in reality, the questions presented were routine and primarily assessed procedural knowledge rather than conceptual comprehension.

5.4.6 Responding to the third research question

This section of the discussion chiefly focuses on addressing the third research question: To what extent do PSTs consider the features/components of KoST when reflecting on the lesson? In response to the research question, the findings revealed that most components of KoST were not utilised during the reflection session. However, there was a noticeable focus on two elements of KoST namely problem-solving and communication. These elements were given greater consideration by the PSTs. While all the other components were featured only sparingly, there was a significant absence of reasoning and proof, which aligns with other research findings that PSTs often struggle with reflecting and grasping reasoning and proof.

5.5 Reflecting on the utility of the theoretical framework for my study

In this study, I employed two theoretical frameworks, namely SLT and MKT. The two frameworks blended seamlessly and created a synergy that greatly influenced my research. They played pivotal roles in shaping the depth and relevance of the literature review. SLT underscored the importance of learning within authentic contexts, which triggered the need to examine research that investigated how mathematical concepts and skills were best acquired and applied in a Lesson Study context. Therefore, the SLT framework guided my literature review towards studies that explored the effectiveness of pedagogical practices rooted in SLT principles. In parallel, MKT offered a specific lens through which to analyse and interpret the literature. It prompted me to focus on the specialised mathematical knowledge that teachers require to effectively convey mathematical concepts, and therefore, directed the review towards research that delved into the development and impact of such knowledge. Combining these two theoretical frameworks provided a robust foundation for conducting a literature review that not only comprehensively captured the current state of mathematics education research but also how learners learn in real classrooms and what PSTs need to become competitive teachers.

The two frameworks significantly enhanced the framing of the methodology for this study. SLT, which emphasise learning within real-world contexts, synergised with the interpretivist approach, enabling me to delve deeply into the intricate aspects of learner thinking within the context of Lesson Study. SLT informed the choice of data gathering techniques, particularly participant observation, as it enabled me to immerse myself in an authentic learning and teaching environment. MKT complemented this perspective as it informed the interview process by ensuring that the study explored the unique mathematical knowledge infused by PSTs.

SLT and MKT played a pivotal role in shaping and enriching the research findings of this study from which five key themes emerged, namely communication, problem-solving, reasoning and proof, representation, and connections. The use of SLT ensured the research findings remained closely tied to the fundamental contexts of Lesson Study. This allowed for the emergence of rich insights into how the PSTs infused KoST throughout the various stages of Lesson Study, from planning to observation and reflection. Furthermore, MKT served as a tool to measure the specific mathematical knowledge essential for effective teaching, thereby enhancing the depth and refinement of the research findings within the identified themes.

Finally, the two frameworks played a pivotal role in shaping the discussion chapter. SLT with its emphasis on learning within authentic contexts allowed for a deep exploration of the implications of the research findings within the real-world settings of Lesson Study. This framework provided a solid foundation for discussing the practical relevance of the research findings, highlighting their significance for PSTs and learners in authentic classroom environments. By integrating MKT into the discussion, this chapter gained depth as it delved into the implications of the research findings for teacher preparation and development. The synergy of these two frameworks enriched my discussion chapter as they assisted me to thoroughly look into how the research findings connect with and add to the bigger picture of mathematics education, ultimately offering practical insights and implications for teachers and researchers.

5.6 Limitations of the Study

As researcher, I faced several challenges during the data collection phase. One significant constraint was time. In South Africa, the Lesson Study model involves five stages to complete one full cycle, and therefore, completing multiple cycles is ideal to collect accurate data and identify trends effectively. However, the participants in this study were PSTs still in college, and due to their tight WIL schedules, we could only manage to complete one full Lesson Study cycle. Gathering the PSTs for lesson planning posed an additional challenge since it had to be scheduled for weekends. Collaborative teacher learning approaches like Lesson Study require a substantial amount of planning time, which can be difficult to integrate into already busy college programs.

The COVID-19 pandemic further complicated data collection. Restrictions limited the number of people allowed in a room, making it challenging to recruit the originally planned six PSTs for the study. In the end, only four could participate. Furthermore, I could not include an outside expert during the research lesson presentation, which deviated from the recommended Lesson Study practice.

5.7 Recommendations

Considering the time constraints and the compressed schedules faced particularly by PSTs, there is a pressing need to adapt and optimise the Lesson Study cycle to accommodate their unique circumstances. Collaborative lesson planning can be particularly challenging for PSTs due to the logistical hurdles to coordinate group meetings. Lesson Study, traditionally designed for in-service teachers to collaboratively plan, teach, and reflect on lessons, requires a tailored approach to cater for PSTs. The adaptation may involve aligning the Lesson Study cycle with their educational training. PSTs may be awarded the opportunity to prepare and refine their lesson plans before embarking on their WIL. Tailoring the Lesson Study process to PSTs will better support them in their learning, development, and preparedness for their future roles as mathematics teachers.

To facilitate effective learning, learners should engage with materials, questions, and tasks that stimulate their thinking. It is the responsibility of teachers to possess creative skills that enable them to craft non-routine mathematics problems. These non-routine problems are instrumental in encouraging learners to engage in critical and creative thinking, fostering a deeper mathematical understanding. I strongly advocate for the inclusion of training programs for PSTs focused on designing such problems. Equipping PSTs with this skill will empower them to create meaningful and engaging learning experiences to enhance their ability to engage learners in meaningful learning experiences.

A classroom serves as an authentic learning environment where an avalanche of information flies continuously throughout lessons. For novice PSTs, these moments can be overwhelming and incomprehensible. To empower PSTs to effectively harness this information for the benefit of both learners and themselves, it is strongly recommended that teacher educators place greater emphasis on helping PSTs refine their ability to observe and understand learner thinking and its components. The skill of recognising learners' thought processes is fundamental for effective teaching because it enables teachers to customise their instruction to address the distinct needs of individual learners. Additionally, teacher educators should encourage PSTs to incorporate the question 'why' into their teaching approaches. This style of inquiry has the potential to enrich learners' understanding by prompting them to provide explanations for their thought processes. Furthermore, PSTs are advised to attentively address all learners' inquiries and responses, fostering active participation and nurturing a supportive and conducive learning atmosphere.

5.8 Conclusions

This study aimed to investigate KoST among mathematics PSTs during their WIL experiences facilitated through the Lesson Study framework. This research endeavour was motivated by the pressing issue of the overall quality of initial teacher education in South Africa, which raises questions about its effectiveness. The central research question guiding this study was How do mathematics PSTs infuse KoST into teaching and learning within the Lesson Study cycle?'

This study was conducted as a case study and followed a qualitative research approach. Through document analysis, observation, and interviews, data were collected and carefully analysed. The analysis unveiled five prominent themes related to learner thinking, namely problem-solving, reasoning and proof, representation, communication, and connections. The findings of this research indicated that while certain components of KoST were not fully considered during the Lesson Study stages of lesson planning, lesson presentation and observation, and the post-lesson reflection session, there was a noticeable presence of crucial elements of learner thinking that were indeed considered. These findings underscore the complexity of PSTs infusing KoST into the Lesson Study cycle and highlight areas of improvement. For PSTs to effectively teach fractions to learners, they must first have a strong conceptual understanding of fractions themselves and yet this has proven to be a deficit in this study. Having a shallow procedural understanding of fractions rather than a deep conceptual grasp makes it hard for PSTs to see beyond surface-level responses to deeper levels of thinking. The CCK deficit prevents reflection on different approaches and learners' misconceptions since they have limited experience with the content themselves. PSTs fail to anticipate common errors or have a plan to address them, leading to misconceptions not getting resolved. Formative assessment is weakened due to CCK deficit as witnessed in the nature of assessments administered in this study.

In the wider context of teacher education, this study contributes valuable insights into the challenges and possibilities of preparing mathematics teachers who can effectively incorporate learner thinking into their teaching practices. Further research can be done to explore the impact of incorporating learner thinking in authentic classrooms on the overall learner performance. It is my hope that these findings will inform future teacher training programs, enhance the quality of mathematics education, and ultimately, contribute to the growth and success of future generations of learners in South Africa and beyond.

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Annexures

Annexure A: Document Analysis Instrument

Criteria	Observed episodes
Communication	
Enriched use of Mathematical language	
Environment where learners share ideas	
Problem Solving	
Extension of learner's ideas through authentic examples	
Probing learners to justify their answer	
Reasoning and proof	
Use concrete material to discover mathematics	
Probe learners beyond their observations	
Nurture the ability to explain one's self	
Representation	
Study and interpreting learners' work	
The use of well-crafted activities	
Analysing learner's representations	
Connections	
Incorporate topics previously taught	
Connections between Mathematics and other subjects	
Link classroom Mathematics with everyday learner experiences	

Annexure B: Observation Tool to Reveal the Components of KoST (adapted from studies by An et al., 2004; Çelik and Güzel (2017))

Observation criteria	Notes
Building on learner's prior Mathematics ideas	
Predicting learners' possible ideas and approaches	
Considering learners' mathematics challenges	
Considering learner misconceptions and errors	
Considering and invoking divergent thoughts	
Promoting learners' thinking of mathematics	

Annexure C: Ethics Approval

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

Amendment

Ethics Committee
21 May 2021

Mr V Chakawanei

Dear Mr V Chakawanei

REFERENCE: UP 19 03 01 SEKAO 21-03

We received the proposed amendments to your existing project. Your amendment is thus **approved**. The decision covers the entire research process, until completion of the study report, and not only the days that data will be collected. The approval is valid for two years for a Masters and three for Doctorate.

The approval by the Ethics Committee is subject to the following conditions being met:

1. The research will be conducted as stipulated on the application form submitted to the Ethics Committee with the supporting documents.
2. Proof of how you adhered to the Department of Basic Education (DBE) policy for research must be submitted where relevant.
3. In the event that the research protocol changed for whatever reason the Ethics Committee must be notified thereof by submitting an amendment to the application (Section E), together with all the supporting documentation that will be used for data collection namely; questionnaires, interview schedules and observation schedules, for further approval before data can be collected. Non-compliance implies that the Committee's approval is null and void. The changes may include the following but are not limited to:
   - Change of investigator,
   - Research methods any other aspect therefore and,
   - Participants.

The Ethics Committee of the Faculty of Education does not accept any liability for research misconduct, of whatsoever nature, committed by the researcher(s) in the implementation of the approved protocol.

Upon completion of your research you will need to submit the following documentations to the Ethics Committee for your

Clearance Certificate:

- Integrated Declaration Form (Form D08),
- Initial Ethics Approval letter and,
- Approval of Title.

Please quote the reference number UP 19 03 01 SEKAO 21-03 in any communication with the Ethics Committee.

Best wishes

Prof Funke Omidire
Chair: Ethics Committee
Faculty of Education

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Faculty of Education
Fakulteit Opvoedkunde
Lefapha la Thuto

Annexure D: College MD Approval

Mr. Victor Chakawanei
406 James Dewrance Street
Eersterust, 0022
Email: vachakawanei@gmail.com

Dear Mr. V. Chakawanei

LETTER OF CONSENT TO CONDUCT THE RESEARCH STUDY

I, Jaco Bernardo, the MD of SANTS voluntarily and willingly permit Mr. V. Chakawanei to conduct a research study titled: ***Mathematics pre-service teachers' knowledge of learner thinking using lesson study.*** I understand that the participation of the Intermediate Phase pre-service teachers in the afore-mentioned study for which I am granting permission, will involve pre-service teachers:

a) being observed and audio/video recorded during three of the five stages of the Lesson Study cycle i.e. collaborative lesson planning, lesson presentation & observation and post-lesson reflection.
b) being part of the interview that will be audio/video recorded.
c) availing their collaboratively prepared lesson plan for analysis.

I declare that I understand the purpose of the study and that you subscribe to the ethical research principles, including the informed consent, safety, privacy (confidentiality and anonymity) and trust.

In addition, I grant the University of Pretoria permission to use data provided for this study, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and using the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

The findings published in the MEd thesis will be the University of Pretoria student's personal interpretation of the data and not that of SANTS as an institution.

Given the above information, I grant permission for our pre-service teachers' participation in the study.

Jaco Bernardo | [signature] | 12/8/21

(Name and surname) Signature Date

Annexure E: Consent Letter for Preservice Teachers

Faculty of Education

Mr. Victor Chakawanei
406 James Dewrance Street
Eersterust, 0022
Email: vachakawanei@gmail.com

Dear Mr. V. Chakawanei

LETTER OF CONSENT TO CONDUCT THE RESEARCH STUDY

I,.., a pre-service teacher at SANTS, voluntarily and willingly permit Mr. V. Chakawanei to conduct a research study titled: ***Mathematics pre-service teachers' knowledge of learner thinking using lesson study***. I understand that my participation as an Intermediate Phase pre-service teacher in the afore-mentioned study to which I am consenting, will involve:

a) being observed and audio/video recorded during three of the five stages of the Lesson Study cycle i.e. collaborative lesson planning, lesson presentation & observation and post-lesson reflection.
b) being part of the interview that will be audio/video recorded.
c) availing the collaboratively prepared lesson plan for analysis.

I declare that I understand the purpose of the study and that you subscribe to the ethical research principles, including informed consent, safety, privacy (confidentiality and anonymity) and trust.

In addition, I grant the University of Pretoria permission to use data provided for this study, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and using the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Given the above information, I consent to participate in the study.

______________________	______________	____________
(Name and surname)	Signature	Date

Annexure F: Assent for Learners

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA
Faculty of Education

Mr. Victor Chakawanei
406 James Dewrance Street
Eersterust, 0022
Email: vachakawanei@gmail.com

Dear Mr. Chakawanei

LETTER OF ASSENT TO PARTICIPATE IN THE RESEARCH STUDY

I,................................ Intermediate Phase learner at.................................., voluntarily and willingly agree to participate in the study titled: ***Mathematics pre-service teachers' knowledge of learner thinking using lesson study.*** I understand that my participation in the afore-mentioned study to which I am assenting, will involve being observed and video/audio recorded during the lesson presented by the SANTS pre-service teacher.

I declare that I understand the purpose of the study and that you subscribe to the ethical research principles, including the informed consent, safety, privacy (confidentiality and anonymity) and trust.

In addition, I grant the University of Pretoria permission to use the data provided for this study, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and using the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Given the above information, I give assent to voluntarily participate in the study.

______________________	______________	____________
(**Name** and **surname**)	Signature	Date

Annexure G: Parental Consent

Mr. Victor Chakawanei
406 James Dewrance Street
Eersterust, 0022
Email: vachakawanei@gmail.com

Dear Mr Chakawanei

LETTER OF CONSENT TO PARTICIPATE IN THE RESEARCH STUDY

I, ……………………………………………………, parent of………………………………………………………, voluntarily and willingly permit my child to participate in Mr. V. Chakawanei's research study titled: ***Mathematics pre-service teachers' knowledge of learner thinking using lesson study***. I understand that the participation of my child in the afore-mentioned study to which I am consenting, will involve being observed during the lesson taught by a student teacher from SANTS Private Higher Education Institution. I declare that I understand the purpose of the study and that you subscribe to the ethical research principles, including *informed consent*, *safety*, *privacy* and *trust*.

In addition, I grant the University of Pretoria permission to use the data provided for this study, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and using the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Given the above information, I give consent for my child's participation in the study.

______________________	______________	____________
(Name and surname)	Signature	Date

Annexure H: Lesson Plan

Lesson Plan: Fractions Is Fun. Grade 4.

LESSON OUTCOMES:

A. Develop learners' ability to write fractions.

- Learners must be able to differentiate between a numerator and denominator.

B. Learners must be able to compare fractions of different sizes and recognize which fractions are smaller or larger.

ANTICIPATION OF MISCONCEPTIONS IN FRACTIONS:

Nr	**Misconception**	**Strategy to address misconceptions.**
1	The denominator represents some parts in a whole.	Address it with the visual representation of the numerator and denominator and where they need to be written.
2	Fractions cannot be represented by more than one shape but only represent by circles.	The use of different shapes and objects to represent fractions.
3	Big numbered fractions represents large fractions. For example 1/8 is larger than ½.	The fraction wall chart will assist them in comparing fractions that is larger or smaller than another.

LSTM:

1. Playpen balls. Red (1) and white (3)
2. Black board.
3. Worksheets 1- 3 (attached)
4. A3 paper to make "loafs of bread"
5. Prestic

LESSON PHASES -

THEME:

FRACTIONS IS FUN!

INTRODUCTION:

- Introduce yourself as a student teacher. (5 minutes)
- I will ask the learners what they think fractions are and if they can remember it from grade 3.

- I will continue to show them a visual representation of the numerator and denominator and where they are written in the fraction. I will use a puzzle. The puzzle pieces will be complete below the line (denominator) and only 2 on top of the line (numerator). (10 minutes). This is where the information sheet will be available. (See addendum A)
- I will ask 4 learners to stand in front and pick a ball from the bag. Three learners will pick white balls and one learner will pick a red ball.
- I will ask the class what fraction of the balls are white and write it with them according to their understanding of the numerator and denominator. (REMEMBER- Do not tell them, they must conclude the answer themselves.)
  = ¾
- I will ask the class what fraction of the balls are red and write it with them on the board.
  = ¼
- I will hand out activity one and allow 10 minutes for the learners to do the activity. (See Addendum B)
- This part all the student teachers will walk through the class and see weather the students grasp the concepts and or whether they are forming new misconceptions that was not considered yet.

DEVELOPMENT:

- In this part I will start with the loafs of bread.
- I will pick different students in different groups to share a loaf of bread. One person will get a loaf of bread for themselves.
- Each group will cut out their share of the wall chart (loaf of bread) and paste it on the board.
- Here I will show them how if you have more people in your group your piece of bread is smaller. So even though the fraction looks bigger ¼ you still have less than someone who is ½.
- I will give out the activity where students will be able to compare their fractions to see what is smaller and larger.
- In this I will allow for divergent thought from learners during the lesson and address them accordingly.
- During the concept development phase I will probe learner thinking through questions and address these accordingly.

CONSOLIDATION:

At the end of the lesson we will collect the worksheets and reflect on learner thinking and how the lesson accommodated for possible misconceptions.

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NOTES AND OBSERVATIONS: