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Multiscale Analysis of Prandtl-Ishlinskii Operators

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Abstract. Homogenization is a cost reducting mathematical method used to model composite materials. It replaces rapidly varying coefficients with constant ones, resulting in an idealized homogeneous material that exhibits similar macroscopic behavior, both qualitatively and quantitatively, to the actual material. The current paper focuses on the deterministic homogenization of the heat equation with hysteresis, which involves the Prandtl-Ishlinskii operator of play type. This equation serves as a model for heat conduction with phase transitions, accounting for undercooling and superheating effects. We consider a sequence of problems with spatially varying coefficients and utilize the concept of sigma-convergence to demonstrate the convergence of the corresponding solutions to the solution of the homogenized problem.

1. Setting of the Problem

Our work furthers the work of [16], which was concerned with the well-posedness of hysteresis phenomena driven by diffusion problems. We dealt there with a Cauchy problem for a nonlinear parabolic equation containing a continuous hysteresis operator and a nonlinear monotone operator in the diffusion term. The existence, uniqueness, and long-time behavior of solutions to the following parabolic partial differential equation (P.D.E.) were studied in a smooth bounded spatial domain $\Omega \subset \mathbb{R}^d$ and a time interval $T > 0$, with the intention of letting T tend to ∞.

$$\frac{\partial}{\partial t}\left(c(x)u + w\right) - \operatorname{div} \boldsymbol{a}(x, \nabla u) = f(x,t).$$

The operator $\nabla \equiv \nabla_x$ denotes the usual gradient, i.e., $\nabla = \left(\frac{\partial}{\partial x_i}\right)_{1\leq i\leq d}$, and div the divergence with respect to the variable x. The mapping $\operatorname{div} \boldsymbol{a}$ is assumed to be

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the gradient of a convex potential with growth bounded below and above with $|\nabla u|^p$, $p \geq 2$. The term w under the time derivative is considered in the form

$$w = \mathcal{P}[u(x, \cdot); x](t),$$

where $\mathcal{P}$ stands for a hysteresis operator. The P.D.E is complemented with a standard initial condition and homogeneous Dirichlet boundary conditions:

$$u = 0 \text{ on } \partial\Omega \times (0, T) \text{ and } u(x, 0) = u^0(x) \text{ in } \Omega.$$

The problem was studied in [16] using semigroup theory, which helped justify its well-posedness. Diffusion processes with hysteresis effects play a key role in science and engineering. They help explain materials and systems with memory effects. In [16], we proved the well-posedness of a nonlinear parabolic equation involving a continuous hysteresis operator and a nonlinear monotone operator in the diffusion term. However, many real materials have properties that change across space, a factor not fully explored in previous studies. Our goal is to analyze the homogenization of diffusion processes with hysteresis effects to better understand materials with memory-dependent properties. This approach also simplifies numerical computations since the homogenized model has stable coefficients.

The present work investigates the asymptotic behavior of a sequence of problems modeling diffusion with spatially inhomogeneous Prandtl-Ishlinskii hysteresis. Here, the diffusion or heat conduction equation replaces the usual linear relation with a spatially dependent Prandtl-Ishlinskii hysteresis operator. The equation is as follows:

$$\frac{\partial}{\partial t}\left(c^{\varepsilon} u_{\varepsilon} + \mathcal{P}[u_{\varepsilon}(x, \cdot); x](t)\right) - \operatorname{div} \boldsymbol{a}^{\varepsilon}(\cdot, \nabla u_{\varepsilon}) = g \text{ in } Q = \Omega \times (0, T), \tag{1.1}$$

where $\varepsilon > 0$ approaching zero is a small parameter representing the scale of the inhomogeneities which are small compared to the global size of the material Ω, a bounded open set in $\mathbb{R}^d$ (integer $d \geq 1$). The function $\boldsymbol{a}^{\varepsilon}(\cdot, \nabla u_{\varepsilon})$ is defined on Q by $\boldsymbol{a}^{\varepsilon}(\cdot, \nabla u_{\varepsilon})(x, t) = \boldsymbol{a}(x/\varepsilon, \nabla u_{\varepsilon}(x, t))$. We suppose that the coefficients in (1.1) satisfy the following hypotheses:

- (**A1**) $\boldsymbol{a} = (a_i)_{1\leq i\leq d}$ is a function defined by $a_i(y, \lambda) = \frac{\partial J}{\partial \lambda_i}(y, \lambda)$ where the function $J : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ verifies the conditions below:
  - (i) $J(\cdot, \lambda)$ is measurable $\forall \lambda \in \mathbb{R}^d$,
  - (ii) $J(y, \cdot)$ is strictly convex for $a.a \quad y \in \mathbb{R}^d$,
  - (iii) There exist three constants $p \geq 2$, $\alpha_1 > 0$ and $\alpha_2 > 0$ verifying

$$\alpha_1 |\lambda|^p \leq J(y, \lambda) \leq \alpha_2 (1 + |\lambda|^p) \tag{1.2}$$

  for all $\lambda \in \mathbb{R}^d$ and for $a.a \quad y \in \mathbb{R}^d$.
- (**A2**) $c^{\varepsilon}(x) = c(\frac{x}{\varepsilon})$ is a function, with $0 < \alpha \leq c \in L^{\infty}(\mathbb{R}^d)$, where α is a constant not dependent on $y \in \mathbb{R}^d$.
- (**A3**) $g \in L^2(Q)$ and $u^0 \in W_0^{1,p}(\Omega)$.
- (**A4**) For any $x \in \Omega$, the Prandtl-Ishlinskii operator $\mathcal{P}[\cdot; x]$ is continuous on $\mathcal{C}([0, T])$ and piecewise increasing. Furthermore $\mathcal{P}$ is affine bounded and there exist a

function $\kappa_0 \in L^2(\Omega)$ and a constant γ_0 which is positive and such that, for all $\ell \in \mathbb{N}$, the parameterized final value mapping

$$(s,x) \mapsto \mathcal{P}_f(s;x), \;\; s = (v_0, ..., v_\ell) \in S$$

is measurable and verifies

$$|\mathcal{P}_f(s;x)| \leq \kappa_0(x) + \gamma_0 \left\|s\right\|_\infty . \tag{1.3}$$

Here, $\mathcal{P}_f$ denotes the generating functional of the Prandtl-Ishlinskii operator $\mathcal{P}$ and S stands for the set of all finite strings of real numbers, a string being as usual a vector having either finitely or countably infinitely many real components. In the sequel we will set

$$w_\varepsilon(x,t) = \mathcal{P}[u_\varepsilon(x,\cdot);x](t) \text{ in } Q.$$

Some information concerning the operator $\mathcal{P}$, useful for our work, is given in Section 2 below. Apart from piecewise monotonicity and continuity, we need further hypotheses on the Prandtl-Ishlinskii operator $\mathcal{P}[\cdot;x]$, which will guarantee the uniqueness of the solution of (1.1) (see [16, Theorem 5.1 and Corollary 5.1]):

(**A5**) For every $x \in \Omega$, the Prandtl-Ishlinskii operator $\mathcal{P}[\cdot;x]$ maps $W^{1,1}(0,T)$ into itself, and there exist $\gamma_1 > 0$ and $\kappa_1 \in L^2(\Omega)$ such that the condition

$$|(\mathcal{P}[v;x])'(t)| \leq \kappa_1(x) + \gamma_1 \left|v'(t)\right| \quad \forall x \in \Omega, \text{ for a.e. } t \in (0,T)$$

holds for every $v \in W^{1,1}(0,T)$.

In (**A5**), v' stands for the time derivative (in the classical sense of distributions) of a function $v \in W^{1,1}(0,T)$.

The homogenization of operators with hysteresis has already been studied in the literature. Some key works include [5, 4, 6, 8, 9, 14]. In [5], the author examined a Prandtl-Ishlinskii hysteresis operator of play type, defined by a distribution function, with a linear diffusion term. The homogenized equation was derived using the two-scale convergence method. In [6], researchers focused on spatially inhomogeneous Prandtl-Ishlinskii operators. These operators were homogenized by analyzing a sequence of equations with spatially periodic data. The study in [9] explored Preisach operators and applied two-scale convergence to obtain the effective operator. For parabolic equations involving monotone operators but without hysteresis effects, see [10, 12, 13].

The current work extends previous results by considering a nonlinear monotone operator in the diffusion term, derived from a convex energy functional. Instead of restricting the analysis to a periodic setting, a more general deterministic framework is used. This includes periodic, almost periodic, and other cases. The presence of a spatially inhomogeneous Prandtl-Ishlinskii hysteresis operator adds complexity. Traditional methods like Γ-convergence may not fully capture the system's behavior as $\varepsilon \to 0$. Although useful for variational problems, Γ-convergence struggles with correctors in nonlinear monotone problems, which are essential for computing homogenized coefficients. In contrast, the Σ-convergence method is better suited for this type of problem. A major advantage of this approach is the ability to compute effective parameters numerically. The corrector equation is studied in the classical sense of distributions in $\mathbb{R}^d$, following [23].

The paper is divided up as follows. Section 2 is devoted to a survey of relevant results for general Prandtl-Ishlinskii hysteresis operators. In Section 3, we state a well-posedness result for (1.1) (for each freely fixed $\varepsilon > 0$). Details of the existence and uniqueness result can be found in [16] and thus are omitted here. We also establish some useful uniform estimates. Fundamental of the Σ-convergence concept are gathered in Section 4 while Section 5 is concerned with the deterministic homogenization process, and we prove therein the main homogenization result for (1.1).

2. General Prandtl-Ishlinskii Hysteresis Operators

Hysteresis operators can be seen as nonlinear causal functional operators ([2, 19]). One of the main characteristics of these operators is the fact that they have memory character, *i.e.* the value at some time t does not only depend on the value of t at this precise moment, but it also depends on the previous evolutions and on inputs up to the time t.

The role of corner stones in the construction of Prandtl-Ishlinskii hysteresis operator is played by the notion of *rate independent functionals*. Let us denote by $M_{pm}[0,T]$ the set of all piecewise monotone functions and by $\mathcal{C}_{pm}[0,T]$ the subspace of $M_{pm}[0,T]$ for functions which are continuous. The so-called rate independence can be seen as the classical property of Prandtl-Ishlinskii hysteresis operator. It is fundamental for our approach and can be expressed in the form

$$\mathcal{P}\left[v \circ \varphi\right] = \mathcal{P}\left[v\right] \circ \varphi, \tag{2.1}$$

for all admissible time transformations φ.

Equality (2.1) reflects the fact that the amount of information that the input function needs is rather discrete than continuous. As a consequence, we are able to analyze the memory effect of the Prandtl-Ishlinskii hysteresis operators without taking into account their properties of regularity and continuity. We first present the notion of the rate independent functionals define on the space $\mathcal{C}_{pm}[0,T]$ of continuous and piecewise monotone functions. This notion will play the role of corner stones in our construction.

Definition 2.1. A functional $\mathcal{H} : \mathcal{C}_{pm}[0,T] \to \mathbb{R}$ is said to be rate independent if the following equality holds:

$$\mathcal{H}\left[v \circ \varphi\right] = \mathcal{H}\left[v\right],$$

for all $v \in \mathcal{C}_{pm}[0,T]$ and all continuous increasing functions $\varphi : [0,T] \to [0,T]$ verifying $\varphi(0) = 0$ and $\varphi(T) = T$. Such a function φ is called an admissible time transformation.

As a consequence of the previous definition, one can notice that only the local extremal values of the input function v should have a real influence on $\mathcal{H}\left[v\right]$. To explain it, the finite string $(v(t_0), ..., v(t_\ell))$ should be used to determine $\mathcal{H}\left[v\right]$, where $0 = t_0 < t_1 < \cdots < t_\ell = T$ denote the local extrema of v in $[0,T]$. This observation allows to obtain a normal form of rate independent functionals. We have to introduce some usual notations, both for functions and for strings.

Definition 2.2. Let us consider a set X. We denote by $Map(X)$ the set of all real-valued functions defined on X. Let $I \subset \mathbb{R}$ be a non empty interval. A function $v \in Map(I)$ is said to be

- increasing on I, if $v(\tau) \leq v(t)$ whenever $\tau, t \in I$ and $\tau < t$,
- strictly increasing on I, if $v(\tau) < v(t)$ whenever $\tau, t \in I$ and $\tau < t$,
- monotone on I, if either v or $-v$ is increasing on I,
- strictly monotone on I, if either v or $-v$ is strictly increasing on I.

Moreover, let us say that $\Delta = \{t_i\}_{1\leq i\leq \ell}$, $0 = t_0 < t_1 < \cdots < t_\ell = T$ *is a monotonicity partition for* $v : [0,T] \to \mathbb{R}$, *if* v *is monotone on all subintervals* $[t_i, t_{i+1}]$. *For all* $v \in M_{pm}[0,T]$, *the usual monotonicity partition of* v *can be defined by setting* $t_0 = 0$ *and*

$$t_{i+1} = \max\{t \in [t_i, T] \text{ such that } v \text{ is monotone on } [t_i, t]\}, \text{ if } 0 \leq t_i \leq T.$$

The number of monotonicity intervals in the usual monotonicity partition of v is denoted by $\ell_{mon}(v)$. As soon as $v \in Map(I)$ is not piecewise monotone, $\ell_{mon}(v) = +\infty$. We also use the standard pointwise ordering for functions $v_1, v_2 \in Map(I)$: we note $v_1 \leq v_2$ whenever $v_1(t) \leq v_2(t)$ for any $t \in I$.

Definition 2.3. Let S be the set of all finite strings of real numbers,

$$S = \{(v_0, v_1, ..., v_\ell) \text{ with } \ell \in \mathbb{N}^*, v_i \in \mathbb{R}, 1 \leq i \leq \ell\},$$

and let S_A be the subset of alternating strings

$$S_A = \{(v_0, v_1, ..., v_\ell) \text{ with } \ell \in \mathbb{N}^*, (v_{i+1} - v_i)(v_i - v_{i-1}) < 0,\ 0 < i < \ell\}.$$

For all $s = (v_0, ..., v_\ell)$, we define the number $\ell_{mon}(s)$ and the usual monotonicity partition $0 = i_0 < ... < i_{\ell_{mon}(v)} = \ell$ of the index set obviously. The length of s is $\ell + 1$.

Let us define the restriction operator $\rho_A : M_{pm}[0,T] \to S_A$, by

$$\rho_A(v) = (v(t_0), ..., v(t_\ell)), \tag{2.2}$$

where $\{t_i\}_{1\leq i\leq \ell}$ is the standard monotonicity partition of v. (2.2) will help us to pass from input functions to input strings or vice versa.

Conversely speaking, if we link with every $s = (v_0, ..., v_\ell) \in S$ the linear interpolate $v : [0,T] \to \mathbb{N}$ for the values $v(\frac{i}{\ell}T) = v_i$, we get the prolongation operator $\pi_A : S \to \mathcal{C}_{pm}[0,T]$.

Moreover, let us consider on $\mathcal{C}_{pm}[0,T]$ the equivalence relation induced by the property of rate independence, defined as

$$v \sim \bar{v} \Leftrightarrow \exists \varphi, \bar{\varphi} \text{ with } v \circ \varphi = \bar{v} \circ \bar{\varphi},$$

where φ and $\bar{\varphi}$ are admissible time transformations. Precisely, $\mathcal{H}[v] = \mathcal{H}[\bar{v}]$ if $v \sim \bar{v}$. The following result holds.

Lemma 2.1. *Let $v, \bar{v} \in \mathcal{C}_{pm}[0,T]$. Then $\rho_A[v] = \rho_A[\bar{v}]$ if and only if $v \sim \bar{v}$.*

Proof. On the one hand, suppose that $v \sim \bar{v}$, so for any admissible time transformation φ, we have $\rho_A[v \circ \varphi] = \rho_A[v]$ and then $\rho_A[v] = \rho_A[\bar{v}]$. Conversely, suppose on the other hand that $\rho_A[v] = \rho_A[\bar{v}]$ and suppose without loss of generality that v and $\bar{v}$ have the same usual monotonicity partition $\{t_i\}_{1 \leq i \leq \ell}$. If v is piecewise strictly monotone on $I_i = [t_i, t_{i+1}]$, then we define

$$\bar{\varphi} \mid_{I_i} = id \mid_{I_i}, \quad \varphi \mid_{I_i} = (v \mid_{I_i})^{-1} \circ \bar{v} \mid_{I_i} .$$

If neither v nor $\bar{v}$ are strictly monotone on I_i, then we can choose an increasing surjective function $f_i : I_i \to J_i = [v(t_i), v(t_{i+1})]$ such that, for any $x \in J_i$, $f_i^{-1}(X)$ is a singleton only if both $v^{-1}(x)$ and $\bar{v}^{-1}(x)$ are singletons. Then we are able to construct $\varphi \mid_{I_i}$ and $\bar{\varphi} \mid_{I_i}$ such that $v \circ \varphi \mid_{I_i} = f_i = \bar{v} \circ \bar{\varphi} \mid_{I_i}$. ☐

Owing to Lemma 2.1, we are able to interpret any rate independent functional $\mathcal{H}$ as a real-valued mapping $\tilde{\mathcal{H}}$ acting on S_A, and vice versa. In fact, it is clear that $\rho_A[v] = \rho_A[(\pi_A \circ \rho_A)[v]]$ for any $v \in \mathcal{C}_{pm}[0,T]$, and thus $\mathcal{H} = \mathcal{H} \circ \pi_A \circ \rho_A$. Whence the following result holds.

Proposition 2.1. *Let us consider the decomposition below:*

$$\mathcal{H} = \tilde{\mathcal{H}} \circ \rho_A, \quad \text{with } \tilde{\mathcal{H}} = \mathcal{H} \circ \pi_A. \tag{2.3}$$

The relation (2.3) defines a bijective correspondence between the elements of $Map(S_A)$ and the rate independent functionals $\mathcal{H}$ on $\mathcal{C}_{pm}[0,T]$.

Proof. This result is an immediate consequence of the above considerations. ☐

Remark 2.1 (Canonical Extension to S and $M_{pm}[0,T]$)**.** Using the formula

$$\mathcal{H}(v) = \mathcal{H}[(\pi_A \circ \rho_A)[v]], \quad \tilde{\mathcal{H}}(s) = \tilde{\mathcal{H}}\left((\rho_A \circ \pi_A)(s)\right),$$

we can respectively extend any rate independent functional $\mathcal{H}$ from $\mathcal{C}_{pm}[0,T]$ to $\mathcal{C}_{pm}[0,T]$ and any real-valued mapping $\tilde{\mathcal{H}}$ from S_A to S.

Example 2.1 (Maximum Norm and Total Variation)**.** The maximum norm and the total variation are given by:

$$\|v\|_\infty = \max_{t \in [0,T]} |v(t)|, \quad var[v] = \sup_{\Delta} \sum_{i=0}^{\ell-1} |v(t_{i+1}) - v(t_i)|,$$

where the supremum is taken with respect to any finite partitions $\Delta = \{t_i\}_{1 \leq t \leq \ell}$ of $[0,T]$. Obviously, both define a rate independent functional. Consequently, their string versions can be written as

$$\|s\|_\infty = \max_{t \in [0,T]} |v_i|, \quad var[s] = \sum_{0 \leq i \leq \ell} |v_{i+1} - v_i|, \quad s = (v_0, ..., v_\ell).$$

Since

$$var(s) = \sum_{j=0}^{\ell_{mon}-1} \left| v_{i_{j+1}} - v_{i_j} \right|$$

for the usual monotonicity partition, we have for any two strings $s_1, s_2 \in S$ of equal length

$$|var(s_1) - var(s_2)| \leq 2 \left\| s_1 - s_2 \right\|_\infty \ell(s_1, s_2)$$

where

$$\ell(s_1, s_2) = \max\left\{\ell_{mon}(s_1), \ell_{mon}(s_2)\right\} + 1.$$

Considering two functions $v_1, v_2 : [0,T] \to \mathbb{R}$, the above inequality becomes

$$|var(v_1) - var(v_2)| \leq 2 \left\| v_1 - v_2 \right\|_\infty \ell(v_1, v_2).$$

Now is the time to give the definition of the so-called Prandtl-Ishlinskii hysteresis operator. With this mind, let $\mathcal{H} : M_{pm}[0,T] \to \mathbb{R}$ be a rate independent functional. We are able to define an operator

$$\mathcal{P} : M_{pm}[0,T] \to Map([0,T])$$

by

$$\mathcal{P}[v](t) = \mathcal{H}[v_t], \ \ t \in [0,T] \ \ v \in M_{pm}[0,T], \tag{2.4}$$

where v_t stands for the truncation of v at t, defined by

$$v_t(\tau) = \begin{cases} v(\tau) & \text{for } 0 \leq \tau \leq t \\ v(t) & \text{for } t \leq \tau \leq T. \end{cases}$$

Moreover, the rate independent functional $\mathcal{H}$ induces an operator $\tilde{\mathcal{P}} : S \to S$ by

$$\tilde{\mathcal{P}}(s) = \left(\tilde{\mathcal{H}}(v_0), \tilde{\mathcal{H}}(v_0, v_1), ..., \tilde{\mathcal{H}}(s)\right) \tag{2.5}$$

where $\tilde{\mathcal{H}} : S \to \mathbb{R}$ is the uniquely determined mapping that satisfy (2.3). According to Proposition 2.1, the operators $\mathcal{P}$ and $\tilde{\mathcal{P}}$ must be closely connected. Indeed, they provide equivalent descriptions of the same hysteretic behaviour. In order to introduce our notion of Prandtl-Ishlinskii hysteresis operators, let us now use the formulas (2.4) and (2.5).

Definition 2.4. An operator $\mathcal{P} : M_{pm}[0,T] \to Map([0,T])$, is called a Prandtl-Ishlinskii hysteresis operator on $M_{pm}[0,T]$ if and only if (2.4) holds for some rate independent functional $\mathcal{H}$ on $M_{pm}[0,T]$. Also, an operator $\tilde{\mathcal{P}} : S \to S$ is called Prandtl-Ishlinskii hysteresis operator on S if and only if (2.5) holds with $\mathcal{H} = \tilde{\mathcal{H}} \circ \rho_A$ for some rate independent functional $\mathcal{H}$ on $M_{pm}[0,T]$.

At first sight, this definition seems very abstract and seems to have little to do with what is usually understood by hysteresis. It will soon turn out that this is not the case and that our approach is actually quite natural. To support this, we derive some consequences from the definition.

Firstly, in the defining equation (2.4), we replace t by T to observe that

$$\mathcal{P}[v](t) = \mathcal{H}[v_T] = \mathcal{H}[v], \ \ \forall v \in M_{pm}[0,T]. \tag{2.6}$$

Therefore, to any Prandtl-Ishlinskii hysteresis operator $\mathcal{P}$ on $M_{pm}[0,T]$ corresponds one and only one rate independent functional $\mathcal{H}$ satisfying equation (2.4). This rate independent functional is called the generating functional of $\mathcal{P}$, denoted by $\mathcal{P}_f$. In the same way, for any Prandtl-Ishlinskii hysteresis operator $\tilde{\mathcal{P}}$ on S the mapping $\tilde{\mathcal{P}_f} = \tilde{\mathcal{H}}$ satisfying (2.5) is unique. In view of Proposition 2.1, it is easy to see that relation (2.3) induces a correspondence between the Prandtl-Ishlinskii hysteresis operators $\mathcal{P}$ on $M_{pm}[0,T]$ and the Prandtl-Ishlinskii hysteresis operators $\tilde{\mathcal{P}}$ on S. So we are able to identify $\tilde{\mathcal{P}}$ with $\mathcal{P}$ and, also, $\tilde{\mathcal{P}_f}$ with $\mathcal{P}_f$. In the following, we will no longer distinguish between these operators and functionals, respectively, and we will denote them both by $\mathcal{P}$ and $\tilde{\mathcal{P}_f}$, respectively. In this sense, they will be applied to both input functions and input strings as arguments. Depending on the circumstances the real meaning will always be clear, so that no confusion can occur. Let us draw other consequences from the definition. It is simple to see that

- $\mathcal{P}[v]$ remains constant over any subinterval of $[0,T]$ where v is kept constant,
- $\mathcal{P}$ is entirely determined by the output values at the end of the evolution, and
- $\mathcal{P}$ verifies the Volterra property, that is, whenever $v, \bar{v} \in M_{pm}[0,T]$ and $t \in [0,T]$, then $v_t = \bar{v}_t$ implies that $(\mathcal{P}[v])_t = (\mathcal{P}[\bar{v}])_t$.

  Also,
- any Prandtl-Ishlinskii hysteresis operator is rate independent.

  Indeed, we have for all admissible time transformation φ,

  $$\mathcal{P}[v\circ\varphi](t) = \mathcal{P}_f[(v\circ\varphi)_t] = \mathcal{P}_f[(v_{\varphi(t)}\circ\varphi)] = \mathcal{P}_f[v_{\varphi(t)}] = \mathcal{P}[v]\circ\varphi(t);$$

  whence $\mathcal{P}[v\circ\varphi] = \mathcal{P}[v]\circ\varphi$.

The considerations we have just enumerated indicate that our notion of Prandtl-Ishlinskii hysteresis operator has exactly the properties of assumptions (**A4**)-(**A5**). The most convincing fact, however, is given by the following result which indicates that our class of Prandtl-Ishlinskii hysteresis operators coincides with the class of operators having both properties of Volterra and rate independence properties.

Proposition 2.2 (Characterization of a Prandtl-Ishlinskii Hysteresis Operator)**.** *Let* $\mathcal{P} : \mathcal{C}_{pm}[0,T] \to Map([0,T])$ *have both the Volterra property and the rate independence property. Then* $\mathcal{P}$ *is a Prandtl-Ishlinskii hysteresis operator.*

Proof. Let $\mathcal{H} = \mathcal{P}_f$ defined by (2.6). It is clearly checked that $\mathcal{H}$ is a rate independent functional. Equation (2.4) follows from

$$\mathcal{P}[v](t) = \mathcal{P}[v_t](t) = \mathcal{P}[v_t](T) = \mathcal{H}[v_t],$$

where the second equality holds true since $\mathcal{P}[v_t](\tau) = \mathcal{P}[v_t](\rho(\tau))$ for all $\tau \in [t,T]$ and for all admissible time transformations $\rho = id$ on $[0,T]$. □

Remark 2.2. Assume that $\mathcal{H}$ is a rate independent functional, and let $\mathcal{P}$ denote a Prandtl-Ishlinskii hysteresis operator that maps $\mathcal{C}_{pm}[0,T]$ into $M_{pm}[0,T]$. Then, since

$$(\mathcal{H}\circ\mathcal{P})[v\circ\varphi] = \mathcal{H}[\mathcal{P}[v]]\circ\varphi = \mathcal{H}[\mathcal{P}[v]],$$

the composition $\mathcal{H} \circ \mathcal{P}$ is also a rate independent functional. In particular, the mappings

$$v \longmapsto \|\mathcal{P}[v]\|_\infty \text{ and } v \longmapsto var[\mathcal{P}[v]],$$

are rate independent functionals.

Example 2.2 (Superposition Operator)**.** Let $g : \mathbb{R} \to \mathbb{R}$ a function. Then g induces a superposition operator by

$$w(t) = \mathcal{P}[v](t) = g(v(t)), \ \ t \in [0, T].$$

$\mathcal{P}$ is a Prandtl-Ishlinskii hysteresis operator with the generating functional

$$\mathcal{P}_f(v_0, ..., v_\ell) = g(v_\ell).$$

3. Preliminaries

3.1. Existence and uniqueness result

Let $\varepsilon > 0$ be fixed. Owing to assumption (**A1**), we can observe that the function

$$\boldsymbol{a}^\varepsilon : (x, \lambda) \mapsto \boldsymbol{a}^\varepsilon(x, \lambda) := \boldsymbol{a}(x/\varepsilon, \lambda) \tag{3.1}$$

from $\Omega \times \mathbb{R}^d$ to $\mathbb{R}^d$ verifies the following well-known hypotheses:

(H)$_1$ For each $\lambda \in \mathbb{R}^d$, the function $x \mapsto \boldsymbol{a}^\varepsilon(x, \lambda)$ is measurable from Ω into $\mathbb{R}^d$.
(H)$_2$ There exists a positive constant α_3 such that $\boldsymbol{a}^\varepsilon(x, \lambda) \cdot \lambda \geq \alpha_1 |\lambda|^p - \alpha_3$.
(H)$_3$ There is a constant $C_2 > 0$, such that, a.e. in $x \in \Omega$, for $\lambda_1, \lambda_2 \in \mathbb{R}^d$,

$$(\boldsymbol{a}^\varepsilon(x, \lambda_1) - \boldsymbol{a}^\varepsilon(x, \lambda_2)) \cdot (\lambda_1 - \lambda_2) \geq 0,$$

$$|\boldsymbol{a}^\varepsilon(x, \lambda_1) - \boldsymbol{a}^\varepsilon(x, \lambda_2)| \leq C_2(1 + |\lambda_1| + |\lambda_2|)^{p-2} |\lambda_1 - \lambda_2|,$$

where the dot stands for the usual Euclidean inner product in $\mathbb{R}^d$, and $|\cdot|$ the associated norm.

Thanks to [12, Proposition 2.1], the diffusion term of the differential operator in (1.1) is well defined. More precisely, let $u \in L^p(0, T; W^{1,p}(\Omega))$; then $\boldsymbol{a}^\varepsilon(\cdot, \nabla u) \in L^{p'}(Q)^d$, as pointed out above. But we may as well see $\boldsymbol{a}^\varepsilon(\cdot, \nabla u)$ as a function in $L^{p'}(0, T; L^{p'}(\Omega)^d)$ (where $p' = \frac{p-1}{p}$). Hence, $\operatorname{div} \boldsymbol{a}^\varepsilon(\cdot, \nabla u)$ turns out to rigorously represent the function $t \mapsto \operatorname{div} \boldsymbol{a}^\varepsilon(\cdot, \nabla u(\cdot, t))$ of $(0, T)$ into $W^{-1,p'}(\Omega)$, which lies in $L^{p'}(0, T; W^{-1,p'}(\Omega))$.

We are now able to define the notion of weak solution we will deal with in the sequel.

Definition 3.1. Let us assume that Assumptions (**A1**)-(**A4**) hold. Then, a function $u_\varepsilon : Q \to \mathbb{R}$ is said to be a weak solution of (1.1) if

$$\begin{cases} u_\varepsilon \in L^\infty(0, T; W_0^{1,p}(\Omega)) \text{ with } u_\varepsilon' \in L^2(0, T; L^2(\Omega)), \\ w_\varepsilon \in L^2(Q) \cap L^2(\Omega; \mathcal{C}([0, T])) \text{ with } w_\varepsilon' \in L^{p'}(0, T; W^{-1,p'}(\Omega)) \end{cases}$$

and u_ε verifies Eq. (3.2) below

$$\left\{\begin{array}{l} \int_Q c^\varepsilon u_\varepsilon'(x,t)\varphi(x,t)dxdt + \int_0^T \langle w_\varepsilon'(\cdot,t), \varphi(\cdot,t)\rangle\, dt \\ \quad + \int_Q \boldsymbol{a}^\varepsilon(x, \nabla u_\varepsilon(x,t)) \cdot \nabla\varphi(x,t)dxdt = \int_Q g(x,t)\varphi(x,t)dxdt \\ \text{for all } \varphi \in L^p(0,T; W_0^{1,p}(\Omega)). \end{array}\right. \tag{3.2}$$

The next result will be of interest in the work.

Lemma 3.1. *Consider the function* $\boldsymbol{a}$ *defined by* (**A1**), *i.e.* $\boldsymbol{a}(y,\lambda) = \nabla_\lambda J(y,\lambda)$ *where* ∇_λ *stands for the gradient with respect to* λ *of the function* J *defined in* (**A1**) *and verifying* (i)-(iii) *therein. For a freely fixed* $\varepsilon > 0$, *let* $\boldsymbol{a}^\varepsilon$ *be defined by* (3.1) *and verifying* $(\mathrm{H})_1$-$(\mathrm{H})_3$ *above. Suppose that* $u_\varepsilon \in L^\infty(0,T; W_0^{1,p}(\Omega))$, $u_\varepsilon' \in L^2(0,T; H_0^1(\Omega))$ *and* $\operatorname{div} \boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon) \in L^{p'}(0,T; W^{-1,p'}(\Omega))$. *Then the function*

$$t \mapsto \sigma(u_\varepsilon(t)) := \int_\Omega J(\cdot, \nabla u_\varepsilon(t))dx \tag{3.3}$$

is absolutely continuous on $(0,T)$ *and*

$$\frac{d}{dt}\sigma(u_\varepsilon(t)) = -\left\langle \operatorname{div} \boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon(t)), u_\varepsilon'(t)\right\rangle \text{ for a.e. } t \in [0,T] \tag{3.4}$$

where $u_\varepsilon(t) = u_\varepsilon(\cdot, t)$ *and* $\langle\cdot,\cdot\rangle$ *denotes the duality pairings between* $L^{p'}(0,T; W^{-1,p'}(\Omega))$ *and* $L^p(0,T; W_0^{1,p}(\Omega))$.

The detailed proof of Lemma 3.1 can be found in [15, 16].

Remark 3.1. It is worth noticing that Lemma 3.1 remains true if the assumptions

$$u_\varepsilon' \in L^2(0,T; H_0^1(\Omega)) \text{ and } \operatorname{div} \boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon) \in L^{p'}(0,T; W^{-1,p'}(\Omega))$$

are replaced by the following ones therein:

$$u_\varepsilon' \in L^2(0,T; L^2(\Omega)) \text{ and } \operatorname{div} \boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon) \in L^2(0,T; L^2(\Omega)), \tag{3.5}$$

the other ones remaining unchanged. In that case, the duality pairings $\langle\cdot,\cdot\rangle$ can be replaced by the inner product in $L^2(0,T; L^2(\Omega))$ and we may proceed by approximation like in [1, Proposition 2.11] to get (3.4) for the approximating sequence and then conclude like in [1, Lemma 3.3] for the passage to limit.

Theorem 3.1. *Let the assumptions* (**A1**)-(**A4**) *be in force. Then for each* $\varepsilon > 0$ *there exists at least one solution* u_ε *in the sense of Definition* 3.1. *Moreover if assumption* (**A5**) *holds, then* u_ε *is unique and the following estimate is satified:*

$$\alpha \int_{t_1}^{t_2}\int_\Omega \left|u_\varepsilon'(x,t)\right|^2 dxdt + 2\sigma(u_\varepsilon(t_2)) - 2\sigma(u_\varepsilon(t_1)) \leq \frac{1}{\alpha}\int_{t_1}^{t_2}\int_\Omega |g(x,t)|^2\, dxdt \tag{3.6}$$

for all $0 \leq t_1 \leq t_2 \leq T$. *Here* $\alpha > 0$ *is the same as in assumption* (**A2**) *and* $\sigma(\cdot)$ *is given by* (3.3).

This theorem has been established in [16] in which an existence and uniqueness result is stated and proved by using an implicit time discretization scheme together with a fundamental inequality due to M. Hilpert [7].

Remark 3.2. Specifically, u_ε lies in

$$V^p = \{v \in L^p(0,T;W_0^{1,p}(\Omega)); v' = \frac{\partial v}{\partial t} \in L^2(0,T;L^2(\Omega))\}.$$

Endowed with the norm

$$\|v\|_{V^p} = \|v\|_{L^p(0,T;W^{1,p}(\Omega))} + \|v'\|_{L^2(0,T;L^2(\Omega))} ,$$

V^p is a Banach space. For further needs it is worth remarking that, since $p \geq 2$, the space $W_0^{1,p}(\Omega)$ is densely and continuously embedded in $L^2(\Omega)$. Consequently, identifying $L^2(\Omega)$ with his dual, it readily follows

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega)$$

with continuous embeddings. This has two important consequences:

(1) We will use the same symbol, denoting both the inner product in $L^2(\Omega)$ and the duality pairing between the space $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

(2) The space V^p is continuously embedded in $\mathcal{C}([0,T];L^2(\Omega))$ (this is a well-known result). Hence, we can define $v(t)$ for $v \in V^p$ and $0 \leq t \leq T$, and further the mapping $v \to v(t)$ sends continuously V^p into $L^2(\Omega)$. Thus, we can consider the space $V_0^p = \left\{v \in V^p : v(0) = u^0\right\}$, a closed convex hull, which turns out to contain the solution u_ε of (1.1).

3.2. A priori estimates

Lemma 3.2. *Let assumptions* (**A1**)-(**A4**) *be in force. Then the solution* u_ε *of the problem* (1.1) *verifies the following estimates:*

(3.7)
$$\int_0^T \int_\Omega |u_\varepsilon'|^2 dxdt + \sup_{0\leq t\leq T} \left(\|\nabla u_\varepsilon(t)\|^p_{L^p(\Omega)} + \|\boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon))\|^p_{L^{p'}(0,T,W^{-1,p'}(\Omega)}\right) \leq C,$$

$$\|u_\varepsilon\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leq C, \quad \|\boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon)\|_{L^{p'}(Q)} \leq C, \quad \|w_\varepsilon\|_{L^2(Q)} \leq C, \tag{3.8}$$

and

$$\left\|\frac{\partial}{\partial t}\left(c^\varepsilon u_\varepsilon + w_\varepsilon\right)\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C,$$

where the constant C *depends on the domain* Ω*, the norm of* g *in* $L^{p'}(0,T,W^{-1,p'}(\Omega))$*,* $u^0 \in W_0^{1,p}(\Omega)$ *and the constant* α*.*

Proof. Let us test (1.1) by $u_\varepsilon' = \frac{\partial u_\varepsilon}{\partial t}$ and integrate over Ω to get, for $t > 0$,

(3.9)
$$\int_\Omega c^\varepsilon \left|u_\varepsilon{}'(t)\right|^2 dx + \int_\Omega u_\varepsilon{}'(t)\frac{\partial w_\varepsilon}{\partial t}(t)dx - \langle \operatorname{div} \boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon(t)), u_\varepsilon'(t)\rangle = \int_\Omega u_\varepsilon{}'(t)g(t)dx,$$

where we have considered the abbreviation $u_\varepsilon(t) = u_\varepsilon(\cdot,t)$. From Assumption (**A4**), $\mathcal{P}[\cdot;x]$ is piecewise monotone for every $x \in \Omega$ and thus $u_\varepsilon' \frac{\partial w_\varepsilon}{\partial t} \geq 0$, so that the second term of the left-hand side of (3.9) becomes non-negative, *i.e.* $\int_\Omega {u_\varepsilon}'(t)\frac{\partial w_\varepsilon}{\partial t}(t)dt \geq 0$. Since (see (3.4))

$$-\left\langle \operatorname{div} \boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon(t)), u'_\varepsilon(t)\right\rangle = \frac{d}{dt}\sigma(u_\varepsilon(t)),$$

where

$$\sigma(u_\varepsilon(t)) = \int_\Omega J(x, \nabla u_\varepsilon(x,t))dx, \tag{3.10}$$

we integrate (3.9) with respect to t and apply suitable Young's inequality to its right-hand side to get

$$\alpha \int_0^t \int_\Omega \left|u'_\varepsilon(\tau)\right|^2 dxd\tau + \sigma(u_\varepsilon(t)) - \sigma(u^0) \leq \frac{1}{2\alpha}\int_0^t \int_\Omega |g|^2 dxd\tau + \frac{\alpha}{2}\int_0^t \int_\Omega \left|u'_\varepsilon(\tau)\right|^2 dxd\tau,$$

where we have also considered the inequality $c^\varepsilon \geq \alpha$ from assumption (**A2**). We utilize the left-hand side of inequality (1.2), we infer

$$\int_0^t \int_\Omega \left|u'_\varepsilon(\tau)\right|^2 dxd\tau + \left\|\nabla u_\varepsilon(t)\right\|^p_{L^p(\Omega)} \leq C \int_0^T \int_\Omega |g|^2 dxdt + C\left\|\nabla u^0\right\|^p_{L^p(\Omega)},$$

where C depends only on α, α_1 and p. Hence we find the a priori estimate

$$\int_0^t \int_\Omega \left|u'_\varepsilon(\tau)\right|^2 dxd\tau + \sup_{0\leq t\leq T}\left\|\nabla u_\varepsilon(t)\right\|^p_{L^p(\Omega)} \leq C \int_0^T \int_\Omega |g|^2 dxdt + C\left\|\nabla u^0\right\|^p_{L^p(\Omega)} \tag{3.11}$$

where the constant C in (3.11) depends only on α, α_1 and p. We infer from (3.11) that

$$\int_Q |u'_\varepsilon|^2 dxdt + \sup_{0\leq t\leq T}\left\|\nabla u_\varepsilon(t)\right\|^p_{L^p(\Omega)} \leq C, \tag{3.12}$$

where the positive constant C in (3.12) depends only on α, α_1, p, u^0 and g. Now, from the second inequality in $(\mathrm{H})_3$ we get

$$\begin{aligned}|\boldsymbol{a}^\varepsilon(x,\lambda)| &\leq C_2(1+|\lambda|)^{p-2}|\lambda| + \sup_{y\in\mathbb{R}^d}|\boldsymbol{a}(y,0)| \\ &\leq C(1+|\lambda|)^{p-1},\end{aligned}$$

where $C = \max(C_2, \sup_{y\in\mathbb{R}^d}|\boldsymbol{a}(y,0)|)$, and where we recall that $\sup_{y\in\mathbb{R}^d}|\boldsymbol{a}(y,0)| < \infty$ since J satisfies (1.2). Hence

$$|\boldsymbol{a}^\varepsilon(x,\lambda)|^{p'} \leq C(1+|\lambda|)^p \leq C(1+|\lambda|^p),$$

the constant in the last inequality above being depending on C_2, $\boldsymbol{a}$, and p. It follows readily from (3.12) that

$$\left\|\boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon)\right\|^{p'}_{L^{p'}(Q)} \leq C(1 + \left\|\nabla u_\varepsilon\right\|^p_{L^p(Q)}) \leq C, \tag{3.13}$$

where C is a positive constant depending on the measure of Ω, C_2, $\boldsymbol{a}$, p and T.

Also

$$\|u_\varepsilon\|_{L^p(0,T;W_0^{1,p}(\Omega))\cap H^1(0,T;L^2(\Omega))} \leq C. \tag{3.14}$$

The properties of the monotone operator $\boldsymbol{a}^\varepsilon$ together with (3.14) yield

$$\|\boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon)\|_{L^{p'}(Q)} \leq C.$$

Finally we find from (3.2) and (3.7) that for any $\varphi \in L^p(0,T;W_0^{1,p}(\Omega))$,

$$\left|\int_Q \frac{\partial w_\varepsilon}{\partial t}\varphi dxdt\right| \leq C\,\|\varphi\|_{L^p(0,T;W_0^{1,p}(\Omega))}, \tag{3.15}$$

so that

$$\left\|\frac{\partial w_\varepsilon}{\partial t}\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C. \tag{3.16}$$

According to assumption (**A4**), $\mathcal{P}$ is affine bounded, i.e. there exist $L > 0$ and $\upsilon \in L^2(\Omega)$ such that for every measurable function $u : \Omega \to \mathcal{C}([0,T])$ we have

$$\|\mathcal{P}(u_\varepsilon)(x,\cdot)\|_{\mathcal{C}([0,T])} \leq L\,\|u_\varepsilon(x,\cdot)\|_{\mathcal{C}([0,T])} + \upsilon(x) \text{ a.e. in } \Omega, \tag{3.17}$$

and using (3.14) and (3.17), we get

$$\|w_\varepsilon\|_{L^2(Q)} \leq \sqrt{T}\,\|w_\varepsilon\|_{L^2(\Omega;\mathcal{C}([0,T]))} \leq \sqrt{T}L\,\|u_\varepsilon\|_{L^2(\Omega;\mathcal{C}([0,T]))} + \sqrt{T}\,\|\upsilon\|_{L^2(\Omega)} \leq C.$$

So we obtain

$$\|w_\varepsilon\|_{L^2(Q)} \leq C.$$

The same reasoning as in (3.15) yields

$$\left\|\frac{\partial}{\partial t}\left(c^\varepsilon u_\varepsilon + w_\varepsilon\right)\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq C.$$

The last point is to check that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in V^p. To this end, observe that

$$\int_0^T \left(c^\varepsilon {u_\varepsilon}'(t), v(t)\right) dt + \int_\Omega \left(\frac{\partial w_\varepsilon}{\partial t}(t), v(t)\right) dx +$$

$$\int_Q \boldsymbol{a}^\varepsilon(x,\nabla u_\varepsilon(x,t))\cdot\nabla v(x,t)dxdt = \int_0^T (g(t),v(t))dt \tag{3.18}$$

for all $v \in V^p$, where $\varepsilon > 0$ is arbitrarily fixed. Taking in particular $v = u_\varepsilon$ and using the series of inequalities

$$0 \leq \frac{1}{2}\alpha\,\|u_\varepsilon(T)\|_{L^2(\Omega)}^2 = \alpha\int_0^T \left({u_\varepsilon}'(t), u_\varepsilon(t)\right) dt \leq \int_0^T \left(c^\varepsilon {u_\varepsilon}'(t), u_\varepsilon(t)\right) dt \tag{3.19}$$

and the properties of $\boldsymbol{a}$, we obtain by mere routine

$$\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^p(0,T,W_0^{1,p}(\Omega))} < \infty. \tag{3.20}$$

Using the hypothesis $(\mathrm{H})_1$-$(\mathrm{H})_3$, it follows

$$\sup_{\varepsilon>0} \|\boldsymbol{a}^{\varepsilon}(\cdot,\nabla u_{\varepsilon})\|_{L^{p'}(Q)^d} < \infty, \tag{3.21}$$

hence $\sup_{\varepsilon>0} \|\boldsymbol{a}^{\varepsilon}(\cdot,\nabla u_{\varepsilon})\|_{L^{p'}(0,T,W^{-1,p'}(\Omega))} < \infty$. We deduce by (1.1) that

$$\sup_{\varepsilon>0} \left\| \frac{\partial u_{\varepsilon}}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} < \infty, \tag{3.22}$$

which combines with (3.20) to show that the sequence $(u_{\varepsilon})_{\varepsilon\in E}$ is bounded in V^p. ☐

4. Sigma-Convergence

We recall in this section the main properties and some basic facts about the concept of sigma-convergence. We refer the reader to [18, 21, 22] for the details regarding most of the results of this section.

4.1. Algebra with mean value

Let A be an algebra with mean value on $\mathbb{R}^d$, that is, a closed subalgebra of the Banach algebra $\mathrm{BUC}(\mathbb{R}^d)$ (of bounded uniformly continuous real-valued functions on $\mathbb{R}^d$) that contains the constants, is translation invariant ($\tau_a u = u(\cdot + a) \in A$ for any $u \in A$ and $a \in \mathbb{R}^d$) and is such that any of its elements possesses a mean value in the following sense: for every $u \in A$,

$$M(u) = \lim_{R\to\infty} \fint_{B_R} u(y)dy \tag{4.1}$$

where B_R stands for the open ball in $\mathbb{R}^d$ of radius R centered at the origin and $\fint_{B_R} = \frac{1}{|B_R|}\int_{B_R}$.

Let $u \in \mathrm{BUC}(\mathbb{R}^d)$ and assume that $M(u)$ exists. Then defining the sequence $(u^{\varepsilon})_{\varepsilon>0} \subset \mathrm{BUC}(\mathbb{R}^d)$ by $u^{\varepsilon}(x) = u(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}^d$, we have

$$u^{\varepsilon} \to M(u) \text{ in } L^{\infty}(\mathbb{R}^d)\text{-weak} * \text{ as } \varepsilon \to 0.$$

This is an easy consequence of the fact that the set of finite linear combinations of the characteristic functions of open balls in $\mathbb{R}^d$ is dense in $L^1(\mathbb{R}^d)$.

Let A be an algebra with mean value. Define the space A^{∞} by

$$A^{\infty} = \left\{u \in A : D_y^{\alpha} u \in A \text{ for every } \alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d\right\}.$$

Then endowed with the family of norms $|||\cdot|||_m$ defined by $|||u|||_m = \sup_{|\alpha|\leq m} \sup_{y\in\mathbb{R}^d} |D_y^{\alpha} u|$ where $D_y^{\alpha} = \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} ... \partial y_d^{\alpha_d}}$, A^{∞} is a Fréchet space.

In order to define the generalized Besicovitch space, we first need to define the Marcinkiewicz space $\mathfrak{M}^p(\mathbb{R}^d)$ $(1 \leq p < \infty)$, which is the space of functions $u \in L^p_{loc}(\mathbb{R}^d)$ satisfying $\limsup_{R\to\infty} \fint_{B_R} |u(y)|^p \, dy < \infty$. Endowed with the seminorm

$$\|u\|_p = \limsup_{R\to\infty} \left(\fint_{B_R} |u(y)|^p \, dy \right)^{\frac{1}{p}},$$

$\mathfrak{M}^p(\mathbb{R}^d)$ is a complete seminormed space. Next we define the generalized Besicovitch space $B_A^p(\mathbb{R}^d)$ $(1 \leq p < \infty)$ associated to the algebra with mean value A as the closure in $\mathfrak{M}^p(\mathbb{R}^d)$ of A with respect to $\|\cdot\|_p$. It is easy to see that for $f \in A$ and $0 < p < \infty$, $|f|^p \in A$, so that

$$\|f\|_p = \left(\lim_{R\to\infty} \fint_{B_R} |f(y)|^p\right)^{\frac{1}{p}} \equiv \left(M(|f|^p)\right)^{\frac{1}{p}}. \tag{4.2}$$

The equality (4.2) extends by continuity to any $f \in B_A^p(\mathbb{R}^d)$. Equipped with the seminorm (4.2), $B_A^p(\mathbb{R}^d)$ is a complete seminormed space. We refer the reader to [11, 21, 22] for further details about these spaces. Namely, the following holds true:

(1) The space $\mathcal{B}_A^p(\mathbb{R}^d) = B_A^p(\mathbb{R}^d)/\mathcal{N}$, (where $\mathcal{N} = \{u \in B_A^p(\mathbb{R}^d) : \|u\|_p = 0\}$) is a Banach space under the norm $\|u + \mathcal{N}\|_p = \|u\|_p$ for $u \in B_A^p(\mathbb{R}^d)$.

(2) The mean value $M : A \to \mathbb{R}$ extends by continuity to a continuous linear mapping (still denoted by M) on $B_A^p(\mathbb{R}^d)$. Furthermore, considered as defined on $B_A^p(\mathbb{R}^d)$, M extends in a natural way to $\mathcal{B}_A^p(\mathbb{R}^d)$ as follows: for $u = v+\mathcal{N} \in \mathcal{B}_A^p(\mathbb{R}^d)$, we set $M(u) := M(v)$; this is well defined since $M(v) = 0$ for any $v \in \mathcal{N}$.

To the space $B_A^p(\mathbb{R}^d)$ we attach the corrector space defined as follows:

$$B_{\#A}^{1,p}(\mathbb{R}^d) = \{u \in W_{loc}^{1,p}(\mathbb{R}^d) : \nabla u \in \left(B_A^p(\mathbb{R}^d)\right)^d \text{ and } M(\nabla u) = 0\}.$$

In $B_{\#A}^{1,p}(\mathbb{R}^d)$ we identify two elements by their gradients: $u = v$ in $B_{\#A}^{1,p}(\mathbb{R}^d)$ if and only if $\nabla(u - v) = 0$, i.e. $\|\nabla(u - v)\|_p = 0$. We may therefore equip $B_{\#A}^{1,p}(\mathbb{R}^d)$ with the gradient norm $\|u\|_{\#,p} = \|\nabla u\|_p$, which makes it a Banach space [3, Theorem 3.12].

In the current work, we will deal with the concept of *ergodic* algebras with mean value. A function $u \in \mathcal{B}_A^1(\mathbb{R}^d)$ is said to be *invariant* if for any $y \in \mathbb{R}^d$, $\|u(\cdot + y) - u\|_1 = 0$. This being so, an algebra with mean value A is ergodic if every invariant function u is constant in $\mathcal{B}_A^1(\mathbb{R}^d)$, i.e. if $\|u(\cdot + y) - u\|_1 = 0$ for any $y \in \mathbb{R}^d$, then $\|u - c\|_1 = 0$ where c is a constant. We assume that all the algebras with mean value used in the sequel are ergodic.

4.2. Sigma-convergence

We begin with one underlying notion. By a *fundamental sequence* is meant any ordinary sequence of real numbers $0 < \varepsilon_n \leq 1$ such that $\varepsilon_n \to 0$ as $n \to \infty$. From now on, the letter E will stand for any subset of positive real numbers admitting 0 as accumulation point. We will always write $\varepsilon \to 0$ instead of $\varepsilon_n \to 0$ as $n \to \infty$. Let Ω be an open bounded set in $\mathbb{R}^d$, $1 \leq p < \infty$ and p' defined by $\frac{1}{p} + \frac{1}{p'} = 1$. We set $Q = \Omega \times (0, T)$, where $T > 0$ is fixed. Let A be an algebra with mean value on $\mathbb{R}^d$. In what follows, we keep using the same notations as in the preceding subsection.

Definition 4.1. A sequence $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(Q)$ $(1 \leq p < \infty)$ is said to *weakly* Σ-*converge* in $L^p(Q)$ to some $u_0 \in L^p(Q; \mathcal{B}_A^p(\mathbb{R}^d))$ if as $E \ni \varepsilon \to 0$, we have

$$\int_Q u_\varepsilon(x,t)v\left(x,t,\frac{x}{\varepsilon}\right)dxdt \to \int_Q M\left(u_0(x,t,\cdot)v(x,t,\cdot)\right)dxdt \tag{4.3}$$

for any $v \in L^{p'}(Q;A)$, for a.e. $(x,t) \in Q$.

We express this by writing "$u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ."

Remark 4.1. The convergence (4.3) still holds true for $v \in \mathcal{C}(\overline{Q}; B_A^{p',\infty}(\mathbb{R}^d))$, where $B_A^{p',\infty}(\mathbb{R}^d) = B_A^{p'}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$; see [10] for the justification.

Moreover the uniqueness of the limit u_0 is ensured, and it is also a fact that the weak Σ-convergence in L^p implies the weak convergence in L^p; see e.g. [10, 12, 21].

Definition 4.2. A sequence $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(Q)$ $(1 \leq p < \infty)$ is said to *strongly* Σ-*converge* in $L^p(Q)$ to $u_0 \in L^p\left(Q;\mathcal{B}_A^p(\mathbb{R}^d)\right)$ if Definition 4.1 holds true and further $\|u_\varepsilon\|_{L^p(Q)} \to \|u_0\|_{L^p(Q;\mathcal{B}_A^p(\mathbb{R}^d))}$ as $E \ni \varepsilon \to 0$.

We denote it by "$u_\varepsilon \to u_0$ in $L^p(Q)$-strong Σ".

The following are the main properties of the concept of Σ-convergence; they are of utmost importance in the forthcoming homogenization process. We refer the reader to [21, 22] for their proofs.

Theorem 4.1. *Let $(u_\varepsilon)_{\varepsilon\in E}$ (where E is a fundamental sequence) be a bounded sequence in $L^p(Q)$, $(1 < p < \infty)$. Then there exists a subsequence E' from E and a function $u \in L^p\left(Q,\mathcal{B}_A^p(\mathbb{R}^d)\right)$ such that the sequence $(u_\varepsilon)_{\varepsilon\in E'}$ weakly Σ-converges in $L^p(Q)$ to u.*

Theorem 4.2. *Let $(u_\varepsilon)_{\varepsilon\in E}$ (E a fundamental sequence) be a bounded ordinary sequence in $V^p = \{v \in L^p(0,T;W_0^{1,p}(\Omega)) : v' = \frac{\partial v}{\partial t} \in L^2(0,T;L^2(\Omega))\}$. Then there exist a subsequence E' of E and a couple $(u_0,u_1) \in V^p \times L^p(Q, B_{\#A}^{1,p}(\mathbb{R}^d))$ such that as $E' \ni \varepsilon \to 0$,*

- *$u_\varepsilon \to u_0$ in V^p-weak*
- *$u_\varepsilon \to u_0$ in $L^2(Q)$-strong*
- *$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}$ in $L^p(Q)$-weak Σ $(1 \leq j \leq d)$.*

Theorem 4.3. *Let $1 \leq p,q,r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ and $v_\varepsilon \to v_0$ in $L^q(Q)$-strong Σ, then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^r(Q)$-weak Σ.*

5. Homogenization Result

In what follows we suppose that A is an ergodic algebra with mean value on $\mathbb{R}^d$. We also assume that the distribution of the microstructures can be distributed anyhow in the medium in the following way:

(A6) $\boldsymbol{a}(\cdot,\lambda) \in (B_A^{p'}(\mathbb{R}^d))^d$ for all $\lambda \in \mathbb{R}^d$.

This means that the microstructures in the medium can be displayed anyhow in the deterministic fashion, provided that they are located in the way that there exists a mean value for their distribution function.

5.1. Passage to the limit

Let us set the space $Y = L^\infty(0,T; W_0^{1,p}(\Omega)) \cap H^1\left(0,T; L^2(\Omega)\right)$. Endowed with the norm $\|u\|_Y = \|u\|_{H^1(0,T;L^2(\Omega))} + \|u\|_{L^\infty(0,T;W_0^{1,p}(\Omega))}$, Y is a Banach Space. Owing to Lemma 3.2, the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in Y, so that the following preliminary result holds.

Proposition 5.1. *Let us consider $(u_\varepsilon)_{\varepsilon\in E}$ (where E is a fundamental sequence) be a bounded sequence in Y. Then there exist a subsequence E' of E and a couple $(u_0, u_1) \in Y \times L^p(Q, B^{1,p}_{\#A}(\mathbb{R}^d))$ such that as $E' \ni \varepsilon \to 0$,*

- *$u_\varepsilon \to u_0$ in $L^\infty(0,T; W_0^{1,p}(\Omega))$-weak $*$*
- *$u'_\varepsilon \to u'_0$ in $L^2(Q)$-weak*
- *$u_\varepsilon \to u_0$ in $L^2(Q)$-strong*
- *$\nabla u_\varepsilon \to \nabla u_0 + \nabla_y u_1$ in $L^p(Q)^d$-weak Σ.*

Proof. The proof is straightforward and is an easy consequence of both Theorem 4.2 and Lemma 3.2. ☐

Coming from the estimates in Lemma 3.2, and given the subsequence E' of Proposition 5.1, there exist a subsequence of E' not relabeled and functions $\mathbf{v} \in L^{p'}(Q; \mathcal{B}^{p'}_A(\mathbb{R}^d))^d$ and $w \in L^2(Q)$ such that

$$\boldsymbol{a}^\varepsilon(\cdot, \nabla u_\varepsilon) \to \mathbf{v} \text{ in } L^{p'}(Q)^d\text{-weak } \Sigma \tag{5.1}$$

and

$$w_\varepsilon \to w \text{ in } L^2(Q)\text{-weak.} \tag{5.2}$$

We infer from Proposition 5.1 that $\mathbf{u} = (u_0, u_1)$ belongs to $\mathbb{F}_0^p$ where $\mathbb{F}_0^p = V^p \times L^p(Q; B^{1,p}_{\#A}(\mathbb{R}^d))$, which is a Banach space with an obvious norm. It is an easy task showing that the space $\mathcal{F}_0^\infty = \mathcal{C}_0^\infty(Q) \times (\mathcal{C}_0^\infty(Q) \otimes A^\infty)$ is dense in $\mathbb{F}_0^p$. Here, the symbol $\otimes$ represents the algebraic tensor product and the space $\mathcal{C}_0^\infty(Q) \otimes A^\infty$ stands for the space of smooth functions with compact support on Q that take values in the algebra A^∞.

For $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^p$, we set $\mathbb{D}_i \mathbf{v} = \frac{\partial v_0}{\partial x_i} + \frac{\partial v_1}{\partial y_i}$ and $\mathbb{D}\mathbf{v} = (\mathbb{D}_i \mathbf{v})_{1\le i\le d} \equiv \nabla v_0 + \nabla_y v_1$. For $\Phi = (\psi_0, \psi_1) \in \mathcal{F}_0^\infty$, we define $\mathbb{D}\Phi$ accordingly.

Proposition 5.2. *The couple $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^p$ and the function w determined above solve the following variational problem*

$$\begin{cases} -\int_Q (M(c)u_0 + w)\frac{\partial \psi_0}{\partial t} dxdt + \int_Q M\left(\mathbf{a}(\cdot, \mathbb{D}\mathbf{u}) \cdot \mathbb{D}\Phi\right) dxdt \\ = \int_Q g\psi_0 dxdt \text{ for all } \Phi = (\psi_0, \psi_1) \in \mathcal{F}_0^\infty. \end{cases} \tag{5.3}$$

Furthermore the function w has the representation $w = \mathcal{P}(u_0; \cdot)$ a.e. in Q.

Proof. To view this, we shall pass to the limit in the variational formulation of (1.1) provided that the assumptions (**A1**)-(**A6**) are fulfilled. For the sake of simplicity, we may omit throughout this section to precise that $E' \ni \varepsilon \to 0$ when dealing with a convergence result, although this will be kept in mind once for good. Bearing this in mind, we proceed as follows. Let $\Phi_\varepsilon = \psi_0 + \varepsilon\psi_1^\varepsilon$ be defined by

$$\Phi_\varepsilon(x,t) = \psi_0(x,t) + \varepsilon\psi_1\left(x,t,\frac{x}{\varepsilon}\right) \ \ ((x,t) \in Q)$$

where $\psi_0 \in \mathcal{C}_0^\infty(Q)$ and $\psi_1 \in \mathcal{C}_0^\infty(Q) \otimes A^\infty$. Then $\Phi_\varepsilon \in \mathcal{C}_0^\infty(Q)$ and

$$\begin{aligned} &\Phi_\varepsilon \to \psi_0 \text{ in } L^p(0,T;W_0^{1,p}(\Omega))\text{-weak} \\ &\frac{\partial\Phi_\varepsilon}{\partial t} = \frac{\partial\psi_0}{\partial t} + \varepsilon\left(\frac{\partial\psi_1}{\partial t}\right)^\varepsilon \to \frac{\partial\psi_0}{\partial t} \text{ in } L^p(0,T;W^{1,p}(\Omega))\text{-weak} \\ &\nabla\Phi_\varepsilon = \nabla\psi_0 + \varepsilon(\nabla\psi_1)^\varepsilon + (\nabla_y\psi_1)^\varepsilon \to \nabla\psi_0 + \nabla_y\psi_1 \text{ in } L^p(Q)^d\text{-strong } \Sigma. \end{aligned} \tag{5.4}$$

Considering Φ_ε as a test function in the weak formulation of (1.1), we get

$$-\int_Q (c^\varepsilon u_\varepsilon + w_\varepsilon)\frac{\partial\Phi_\varepsilon}{\partial t}dxdt + \int_Q \boldsymbol{a}^\varepsilon(\cdot,\nabla u_\varepsilon)\cdot\nabla\Phi_\varepsilon dxdt = \int_Q g\Phi_\varepsilon dxdt. \tag{5.5}$$

Therefore utilizing Proposition 5.1, convergence results (5.1), (5.2), (5.4) and the monotonicity of $\boldsymbol{a}$, the passage to the limit in (5.5) is an easy exercise; see e.g., [12, 20]. We are led to (5.3) at once.

In order to conclude the proof of the proposition, we have to characterize the function w in terms of u_0.

We already noticed that the a priori estimates we found yield

$$u_\varepsilon \to u_0 \text{ in } L^p(0,T;W_0^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))\text{-weak.} \tag{5.6}$$

On the other hand, we can deduce that, it is possible to extract another subsequence from E',

$$u_\varepsilon \to u_0 \text{ uniformly in } [0,T] \text{ and a.e. in } \Omega.$$

Making use of the strong continuity of the operator $\mathcal{P}$, we obtain that

$$\mathcal{P}(u_\varepsilon;\cdot) \to \mathcal{P}(u_0;\cdot) \text{ uniformly in } [0,T] \text{ and a.e. in } \Omega.$$

Now, we define the functions

$$z_\varepsilon(x,t) = \mathcal{P}[u_\varepsilon(x,\cdot);x](t) \text{ and } z_0(x,t) = \mathcal{P}[u_0(x,\cdot);x](t).$$

It is an easy matter to see that $u_\varepsilon \to u_0$ in $L^2(\Omega;\mathcal{C}([0,T]))$-strong; in particular, $u_\varepsilon(x,\cdot) \to u_0(x,\cdot)$ in $\mathcal{C}([0,T])$, for a.e $x \in \Omega$. The fact that $w = \mathcal{P}(u_0;\cdot)$ can be showed arguing as in [19, Section IV.1], in particular we have to utilize some interpolation results and use the continuity of the Prandtl-Ishlinskii hysteresis operator $\mathcal{P}$ uniformly in time, a.e. in space, which can be deduced from the local Lipschitz continuity property of $\mathcal{P}$. Hence, using the continuity of $\mathcal{P}$ supposed in (**A4**), we obtain $z_\varepsilon(x,\cdot) \to z(x,\cdot)$ in $\mathcal{C}([0,T])$, for a.e. $x \in \Omega$. In the following, in view of (1.3), we get

$$\sup_{0\leq t\leq T}|z_\varepsilon(x,t)| \leq \kappa_0(x) + \gamma_0 \sup_{0\leq t\leq T}|u_\varepsilon(x,t)| \text{ for a.e. } x \in \Omega,$$

where the right-hand side converges strongly in $L^2(\Omega)$. Thus $z_\varepsilon \to z$ in $L^2(\Omega;\mathcal{C}([0,T]))$-strong. Knowing that w_ε is the linear interpolate of z_ε, an analogous argument shows that $w_\varepsilon - z_\varepsilon \to 0$ in $L^2(\Omega;\mathcal{C}([0,T]))$-strong.

In summary, as $w_\varepsilon(x,\cdot)$ is the time interpolate, we get $w_\varepsilon \to \mathcal{P}(u_0;\cdot)$ uniformly in $[0,T]$ and a.e. in Ω. Therefore, we obtain $w = \mathcal{P}(u_0;\cdot)$ a.e. in Q. The sequence $(\|w_\varepsilon(\cdot,t)\|_{\mathcal{C}([0,T])})_\varepsilon$ is uniformly integrable in Ω as the same holds for u_ε. Hence we have shown that $w_\varepsilon \to w = z$ in $L^2(\Omega;\mathcal{C}([0,T]))$-strong. □

5.2. Homogenized problem

In order to obtain the homogenized result, we have to deal with an equivalent expression of problem (5.3). As we can see, this problem is equivalent to the system (5.7)-(5.8) stated below

$$\left\{\begin{array}{l} \int_Q (M(c)u_0 + w)\frac{\partial\psi_0}{dt}dxdt + \int_Q M\left(\boldsymbol{a}\left(\cdot,\nabla u_0 + \nabla_y u_1\right)\cdot\nabla_x\psi_0\right)dxdt \\ = \int_Q g\psi_0 dxdt \text{ for all } \psi_0 \in \mathcal{C}_0^\infty(Q) \\ w = \mathcal{P}(u_0;\cdot) \text{ a.e. in } Q \end{array}\right. \tag{5.7}$$

and

$$\int_Q M\left(\boldsymbol{a}\left(\cdot,\nabla u_0 + \nabla_y u_1\right)\cdot\nabla_y\psi_1\right)dxdt = 0 \quad \forall\psi_1 \in \mathcal{C}_0^\infty(Q)\otimes A^\infty. \tag{5.8}$$

Let us first deal with (5.8). We choose $\psi_1(x,t,y) = \varphi(x,t)\phi(y)$ with $\phi \in A^\infty$ and $\varphi \in \mathcal{C}_0^\infty(Q)$, (5.8) becomes

$$M\left(\boldsymbol{a}(\cdot,\mathbb{D}\mathbf{u})\cdot\nabla_y\phi\right) = 0 \text{ for all } \phi \in A^\infty, \tag{5.9}$$

which is precisely the weak form in the duality arising from the mean value, of the following equation (in the usual sense of distributions in $\mathbb{R}^d$)

$$-\operatorname{div}_y \boldsymbol{a}(\cdot,\nabla u_0 + \nabla_y u_1) = 0 \text{ in } \mathbb{R}^d. \tag{5.10}$$

Then we fix $r \in \mathbb{R}^d$ and we hold the following corrector problem

$$\text{Find } \chi_r \in B^{1,p}_{\#A}(\mathbb{R}^d) \text{ such that } -\operatorname{div}_y \boldsymbol{a}(\cdot, r + \nabla_y\chi_r) = 0 \text{ in } \mathbb{R}^d. \tag{5.11}$$

Then thanks to [17, 23], we derive the existence of a function $\chi_r \in B^{1,p}_{\#A}(\mathbb{R}^d)$ solution of (5.11) such that its gradient $\nabla\chi_r$ is uniquely defined. Now if we take $r = \nabla u_0$ in (5.11) and compare the resulting equation with (5.10) and next using the uniqueness of the gradient of the corresponding solution, we end up with $u_1 = \chi_{\nabla u_0}$ a.e. in Q, i.e. $u_1(x,t,y) = \chi_{\nabla u_0(x,t)}(y)$.

That said, let

$$\boldsymbol{a}^*(r) = M(\boldsymbol{a}(\cdot, r + \nabla_y\chi_r)) \text{ for } r \in \mathbb{R}^d. \tag{5.12}$$

The function $\boldsymbol{a}^*$ is well defined and is the so-called *homogenized coefficient*. We can verify that $\boldsymbol{a}^*$ satisfies properties similar to those of $\boldsymbol{a}$.

The next result provides us with the upscaled model of (1.1), which is *homogenized problem*, for which u_0 is the solution.

Proposition 5.3. *The function u_0 is the solution of the boundary value problem*

$$\begin{cases} \dfrac{\partial}{\partial t}(cu_0 + w) - \operatorname{div} \boldsymbol{a}^*(\nabla u_0) = g \text{ in } Q \\ w(x,t) = \mathcal{P}[u_0(x,\cdot);x](t) \text{ in } Q \\ u_0 = 0 \text{ on } \partial\Omega \times (0,T) \text{ and } u_0(x,0) = u^0(x) \text{ in } \Omega. \end{cases} \tag{5.13}$$

Proof. We just need to replace u_1 by $\chi_{\nabla u_0}$ in (5.7) and choose there $\psi_0(x,t) = \varphi(x,t)$, with $\varphi \in \mathcal{C}_0^\infty(Q)$. Then we readily get (5.13). ☐

The uniqueness of u_0 in (5.13) is ensured by the following result.

Proposition 5.4. *Consider u_0 and u_0^* be two solutions of (5.13) with the same initial condition u^0, then $u_0 = u_0^*$.*

Proof. Set $v_0 = u_0 - u_0^*$. Then appealing to [16, Theorem 5.1 and Corollary 5.1], it emerges that $v_0 = 0$. ☐

We are now able to state the main result of the work.

Theorem 5.1. *Let us suppose that assumptions (**A1**)-(**A6**) hold. For each $\varepsilon > 0$ let u_ε be the unique solution of (1.1). Then the sequence $(u_\varepsilon)_{\varepsilon>0}$ strongly converges in $L^2(Q)$ and weakly star in $L^\infty(0,T;W_0^{1,p}(\Omega))$ to the unique solution of the problem (5.13).*

Proof. Since the solution of (5.13) is unique, the conclusion of Theorem 5.1 follows from Propositions 5.1 and 5.3. ☐

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

Data Availability

The author declares that data are available.

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