Compactness property associated with the quasi-normed integral operator ideals

by

Brian Chihinga Ndumba

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Declaration

I, Brian Chihinga Ndumba, the undersigned declare that the thesis, which I hereby submit for the degree **Philosophiae Doctor in Mathematical Sciences** at the University of Pretoria is my own independent work and has not previously been submitted by me or any other person for any degree at this or any other tertiary institution.

Signature:

Name: Brian Chihinga Ndumba

Date: November 14, 2024

Dedication

I dedicate this thesis to my late supervisor, Dr. S. M. Maepa and to my family for the love, care, and support they rendered to me during my Ph.D. studies, especially to my amazing wife, Milimo Kaula, who laboured and sacrificed for me and our three children, Brian Chihinga Jnr, Flora Buumba and Zangi Breanna to make it possible for me to complete my studies. Not forgetting my late mother Flora Chikanku Ndumba and father Ronald .K. Ndumba, the golden vessels through wish I came to this world.

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Abstract

In this thesis, we conduct a study on the (p,r)-compactness and mid (p,r)-compactness of subsets in Banach spaces for $1 \leq p \leq \infty$, and $1 \leq r \leq p^*$, where p^* is the conjugate index of p. We begin by introducing and studying a compactness property which a Banach space may or may not have. This compactness property will be denoted by $\mathcal{C}_p^r$ and it is the class of all Banach spaces X such that X belongs to $\mathcal{C}_p^r$ if for every bounded subset A of X, A is relatively (p,r)-compact if, and only if, U_A^* belongs to the injective hull of the $(p,r^*,1)$-integral operators where U_A^* is the adjoint of the operator $U_A : \ell_1(A) \longrightarrow X$. Our main interest is to investigate the relationship between the (p,r)-compactness of sets and the $\mathcal{C}_p^r$ Property of Banach spaces. Moreover, we will also prove a characterization that a Banach space Y has the $\mathcal{C}_p^r$ Property precisely when the (p,r)-compact operators from X into Y equals the surjective hull of the dual of the $(p,r^*,1)$-integral operators from X into Y for every Banach space X (that is, $\mathcal{K}_{(p,r)}(X,Y) = (\mathcal{I}_{(p,1,r^*)}^{dual})^{sur}(X,Y)$). Other results with regard to the $\mathcal{C}_p^r$ Property of Banach spaces will also be proved.

We also introduce and study mid (p,r)-compact sets and operators. We begin by introducing and defining the mid (p,r)-compact subsets of a Banach space X and the mid (p,r)-compact operators between Banach spaces X and Y. The set of mid (p,r)-compact operators between Banach spaces X and Y is denoted by $\mathcal{K}_{(p,r)}^{\text{mid}}(X,Y)$. We prove that the ideal $(\mathcal{K}_{(p,r)}^{\text{mid}}(X,Y), \kappa_{(p,r)}^{\text{mid}}(\cdot))$ is a quasi-Banach operator ideal.

Finally, we introduce and study the (p,r)-limited subsets in Banach spaces. We prove that every mid (p,r)-compact subset of X is (p,r)-limited and that the set $\mathcal{K}_{(p,r)}^{\text{mid}}(X,Y)$ consists of (p,r)-limited sets. Other results with regard to this ideal $(\mathcal{K}_{(p,r)}^{\text{mid}}(X,Y), \kappa_{(p,r)}^{\text{mid}}(\cdot))$ and the (p,r)-limited sets will also be proved.

Contents

Chapter 1

Introduction

In this chapter, we give an overview of the thesis and introduce the concepts and the notations that will be used throughout this thesis. We begin by giving a background and summary of the thesis and then introduce the concepts and notations to be used in this thesis.

1.1 Overview

In his paper [24], Grothendieck in 1955 discussed the theory of relatively compact subsets of Banach spaces. He characterized the relatively compact subsets as subsets that sit inside the closed convex hull of norm null sequences. This result by Grothendieck aroused a lot of interest in studying relatively compact subsets of Banach spaces. Several authors studied the subsets of Banach spaces that sit in the closed convex hull of other classes of null sequences.

In the early 1980s, O. Reinov and J. Bourgain described a stronger form of compactness: A property enjoyed by those subsets of a Banach space that sit in the closed convex hulls of norm p-summable sequences for $1 \leq p < \infty$. Sinha and Karn in the year 2002 investigated this class of sets for Banach spaces. In the manner of Sinha and Karn in [43], a subset of a Banach space is relatively p-compact if the subset sits inside the p-convex hull of a norm p-summable sequence for $1 \leq p \leq \infty$.

It is well known that a linear operator from a Banach space into another Banach space is *compact* if it maps bounded sets to relatively compact sets. For $1 \leq p \leq \infty$, a *p-compact operator* is defined as a linear operator which maps bounded sets to relatively p-compact ones.

The p-compact operators and their properties described in the manner of Sinha and Karn in [43] and [44] aroused great interest in the study of p-compact operators. Some of the important properties of these p-compact operators in the sense of Sinha and Karn were studied in the following papers [4], [7], [8], [10], [11], [12], [13], [19], [21], [25], [26], [27], [28], [30], [31], [34], [38], and [39]. One such notable property is a result that was proved in [13] by Delgado, Piñeiro, and Serrano where they proved that an operator is p-compact if, and only if, its adjoint is quasi p-nuclear. Unfortunately, the p-compact operators in Reinov and Bourgain in [6] and [41] where not adequately given attention to.

In 2012, Ain, Lillemets, and Oja in their paper [2] bridged the gap and eliminated the short-coming in the study of p-compact operators in the sense of Sinha and Karn and Reinov and Bourgain when they extended the notion of relatively p-compact sets and p-compact operators to relatively (p, r)-compact sets and (p, r)-compact operators, respectively for $1 \leq p \leq \infty$, and $1 \leq r \leq p^*$, where p^* is the conjugate index of p. They described the relatively (p, r)-compact subsets of a Banach space as those subsets which sit inside the (p, r)-convex hulls of norm p-summable sequences (or norm null sequences for $p = \infty$). They further defined a (p, r)-compact operator as a linear operator which maps bounded sets to relatively (p, r)-compact sets. They showed that p-compactness in the sense of Reinov and Bourgain corresponds to $(p, 1)$-compactness. They further proved that the (p, p^*)-compactness coincide with the p-compactness in the sense of Sinha and Karn.

One of our interests in our investigation is to look at the p-compact operators studied in 2010 by Delgado, Piñeiro, and Serrano in their paper [13]. Of particular interest in [13] is the result where they proved that a bounded subset A of an arbitrary Banach space X is relatively p-compact if, and only if, the corresponding evaluation map

$$\begin{aligned} U_A^* : X^* &\longrightarrow \ell_\infty(A) \\ x^* &\mapsto (\langle x^*, a\rangle)_{a\in A}. \end{aligned}$$

is p-summing where U_A^* is the adjoint of the operator $U_A : \ell_1(A) \longrightarrow X$ and the operator U_A is of the form

$$\begin{aligned} U_A : \ell_1(A) &\longrightarrow X \\ (\xi_a)_{a\in A} &\mapsto \sum_{a\in A} \xi_a a \end{aligned}$$

where we write U_X instead of U_{B_X}.

In 2014, Delgado, while working with Piñeiro strengthened this result and introduced a new class of Banach spaces X in [11] satisfying the condition that X belongs to this class of Banach spaces if for every bounded subset A of X, A is relatively p-compact if, and only if, U_A^* is p-summing. They denoted this class by $\mathcal{C}_p$. Our interest is to extend the notion of this class $\mathcal{C}_p$ introduced by Delgado and Piñeiro in [11] to multiple indexes and denote this new class by $\mathcal{C}_p^r$, where $\mathcal{C}_p^{p^*} = \mathcal{C}_p$. We say a Banach space X is a member of the class $\mathcal{C}_p^r$ if for every bounded subset A of X, A is relatively (p, r)-compact if, and only if, U_A^* belongs to the injective hull of the $(p, r^*, 1)$-integral operators.

In extending the results from p-compactness, to (p, r)-compactness for $1 \le p \le \infty$, and $1 \le r \le p^*$, where p^* is the conjugate index of p, Ain, Lillemets, and Oja in 2012 considered a discrete set $A = \{x_n : n \in \mathbb{N}\}$ and proved that the operator Φ_{x_n} from a Banach space ℓ_r into a Banach space X is $(p, 1, r^*)$-nuclear. The first main interest of this thesis is to investigate the relationship between the (p, r)-compactness of sets and the $\mathcal{C}_p^r$ Property of Banach spaces. To achieve this, we shall extend some of the results in [11] and [13] concerning p-compactness to (p, r)-compactness by considering any bounded subset of a Banach space instead of the discrete subset considered by Ain, Lillemets, and Oja in 2012. One of the results of this thesis is that the operator U_A from $\ell_1(A)$ into an arbitrary Banach space X is $(p, 1, r^*)$-nuclear whenever a bounded subset A of X is relatively (p, r)-compact. We shall also provide a necessary and sufficient condition under which a Banach space may enjoy this $\mathcal{C}_p^r$ Property. We will only provide an application of this $\mathcal{C}_p^r$ Property where we proved in section 4.3 that $X \in \mathcal{C}_p^r$ whenever $X^{**} \in \mathcal{C}_p^r$. Other examples of Banach spaces having this $\mathcal{C}_p^r$ Property will not be looked into in this thesis.

It is also well known that the theory of limited sets originated from the error of Gelfand found in [22]. Recall that a nonempty subset A of a Banach space X is called *limited* in X if for every weak*-null sequence (f_n) in X^* (that is, $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in X$), it holds that $f_n \to 0$ uniformly on A. Alternatively ([45]), for every weak*-null sequence (f_n) in X^*, there exists $(\alpha_n) \in c_0$ such that $|f_n(x)| \le \alpha_n$ for all $x \in A$ and all $n \in \mathbb{N}$. It is known that every compact set is limited. The converse is, however, false and this was initially mistakenly thought to be true by Gelfand ([22]) but was later refuted by Phillips ([40]) who produced an example of non-compact limited set (see[45], §2, second paragraph).

In 2015, Sinha and Karn in [45] extended the concept of limited sets to the p-level case for $1 \le p < \infty$: a subset A of X is said to be *p-limited* in X $(1 \le p < \infty)$ if for every weak*-p-summable sequence (f_n) in X^* (that is, $\sum_{n=1}^{\infty} |f_n(x)|^p < \infty$ for all $x \in X$), there exists an $(\alpha_n) \in \ell_p$ such that $|f_n(x)| \le \alpha_n$ for all $x \in A$ and $n \in \mathbb{N}$. They also introduced a vector space which the authors of [5] called $\ell_p^{mid}(X)$ of X-valued sequences such that

$$\ell_p(X) \subseteq \ell_p^{mid}(X) \subseteq \ell_p^w(X)$$

where

$$\ell_p^{mid}(X) := \{(x_j)_{j=1}^{\infty} \in \ell_p^w(X) | ((x_n^*(x_j))_{j=1}^{\infty})_{n=1}^{\infty} \in \ell_p(\ell_p) \text{ whenever } (x_n^*)_{n=1}^{\infty} \in \ell_p^w(X^*)\}$$

while $\ell_p(X)$ is the vector space of p-summable X-valued sequences and $\ell_p^w(X)$ is the vector space of weakly p-summable X-valued sequences, respectively.

Botelho, Campos and Santos in [5] equipped this vector space $\ell_p^{mid}(X)$ introduced in [45] with a norm

$$\|(x_j)_{j=1}^{\infty}\|_{mid,p} := \sup_{(x_n^*)_{n=1}^{\infty} \in B_{\ell_p^w(X^*)}} \left\{ \left(\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |x_n^*(x_j)|^p \right)^{1/p} \right\}$$

under pointwise operations under which it is a Banach space.

The other main interest of this thesis is to expound the theory of mid (p,r)-compact sets and operators. We do this by introducing the relatively mid (p,r)-compact sets in X by defining the (p,r)-convex hull of a sequence $(x_n) \in \ell_p^{\text{mid}}(X)$ and the (p,r)-limited subset A of X in X for $1 \le p < \infty$ and $1 \le r \le p^*$ where p^* is the conjugate index of p. One key result under the mid (p,r)-compact sets and operators is that the class $\mathcal{K}_{(p,r)}^{\text{mid}}$ of mid (p,r)-compact operators is an operator ideal as well as that the ideal $\mathcal{K}_{(p,r)}^{\text{mid}}$ is quasi-Banach operator ideal (see Proposition 5.1.8 and Proposition 5.1.10, respectively). The proof that the class $\mathcal{K}_{(p,r)}^{\text{mid}}$ of mid (p,r)-compact operators is an operator ideal follows a similar style to that of the class $\mathcal{K}_{(p,r)}$ of (p,r)-compact operators in ([1], Proposition 3.8) and [2] and uses the definition of the $\ell_p^{\text{mid}}(X)$. This then divides the thesis as follows:

Chapter 1 gives an introduction of (p,r)-compactness and its background as well as the notation used in this thesis.

Chapter 2 gives an overview of the concepts and results that will be needed in the next chapters. These concepts and results include the theory and results on operator ideals including an overview of regular, injective, surjective, and maximal hulls of operator ideals, and the (t,u,v)-nuclear and integral operators for $0 < t \leq \infty$, $1 \leq u, v \leq \infty$ and $\frac{1}{u} + \frac{1}{v} \leq 1 + \frac{1}{t}$. We refer to [37] for the theory on operator ideals.

Chapter 3 is devoted to the development and proofs of some of the quasi-Banach ideal theoretic properties of the operator U_A^* and its preadjoint U_A which will be used as the probing tools in the investigation of Banach spaces which enjoy certain isolated properties.

In chapter 4, we introduce a property that a Banach space may or may not enjoy and this property will be known as the $\mathcal{C}_p^r$ Property of Banach spaces. We will also provide a necessary and sufficient condition (with proof) for a Banach space Y to belong to $\mathcal{C}_p^r$, that is, $Y \in \mathcal{C}_p^r$, if, and only if, the (p,r)-compact operators from X into Y equals the surjective hull of the dual of the $(p,r^*,1)$-integral operators from X into Y for every Banach space X. Other results with regard to the $\mathcal{C}_p^r$ Property of Banach spaces are also proved.

Chapter 5 expounds the theory of mid (p,r)-compact sets and operators and the (p,r)-limited sets. Some of the main results under this chapter are:

(i) The class $\mathcal{K}_{(p,r)}^{\text{mid}}$ of mid (p,r)-compact operators is an operator ideal.

(ii) The ideal $\mathcal{K}_{(p,r)}^{\text{mid}}$ of mid (p,r)-compact operators is a quasi-Banach operator ideal.

(iii) Every mid (p,r)-compact subset of a Banach space X is (p,r)-limited, and

(iv) If $T \in \mathcal{K}_{(p,r)}^{\text{mid}}(X,Y)$, then T is (p,r)-limited where X and Y are Banach spaces.

Finally, chapter 6 gives some conclusive remarks and the list of research topics that can be addressed in connection with this research.

1.2 Notation

We shall use standard notation.

We denote Banach spaces by X and Y, and by $\mathbb{K} = \mathbb{R}/\mathbb{C}$ the field of real or complex numbers. For a Banach space X, B_X will denote the closed unit ball of X, while S_X will denote the unit sphere of X, and the space of continuous linear operators from X to Y will be denoted by $\mathcal{L}(X,Y)$. Recall that continuous linear operators between Banach spaces are bounded. We shall refer to the element of $\mathcal{L}(X,Y)$ as *operators*. The dual of X is denoted by $X^* := \mathcal{L}(X,\mathbb{K})$; its typical element will be denoted by x^*. For $x \in X$, we shall write $\langle x^*, x\rangle = x^*(x)$ (or $\langle x, x^*\rangle$) for the action of x^* on x. We shall also denote by κ_X the canonical isometric embedding from X into X^{**} where X^{**} is the dual of X^*. The subspaces of all finite rank and all compact operators will be denoted by $\mathcal{F}(X,Y)$ and $\mathcal{K}(X,Y)$, respectively. For any operator $T \in \mathcal{L}(X,Y)$, $N(T)$ and $R(T)$ denote the null space and range (space) of T respectively. For any $T \in \mathcal{L}(X,Y)$, T^* shall denote the adjoint operator of T from Y^* into X^*. For the characterization of relatively compact sets in Banach spaces, we recall the following:
Let $p \geq 1$. We denote by $\ell_p^w(X)$ the set of all weakly p-summable sequences in X. Then $\ell_p^w(X)$ is a Banach space with the norm

$$\|(x_n)\|_p^w = \sup\{\|(x^*(x_n))\|_p : x^* \in B_{X^*}\}.$$

$\ell_p^w(X)$ is isometrically isomorphic to $\mathcal{L}(\ell_{p^*}, X)$ if $p > 1$ where $p^* = p(p-1)^{-1}$. For $p = 1$, $\ell_1^w(X)$ is isometrically isomorphic to $\mathcal{L}(c_0, X)$. Denote these isometries by $\Phi : \ell_p^w(X) \longrightarrow \mathcal{L}(\ell_{p^*}, X)$ (respectively, $\Phi : \ell_1^w(X) \longrightarrow \mathcal{L}(c_0, X)$): $x = (x_n) \mapsto \Phi_x$, where $\Phi_x((\alpha_n)) = \sum_{n=1}^{\infty} \alpha_n x_n$ and $(\alpha_n) \in \ell_{p^*}$ (respectively, c_0). We write $p\text{-conv}(x_n) := \Phi_x(B_{\ell_{p^*}})$ for $1 < p < \infty$, $1\text{-conv}(x_n) := \Phi_x(B_{c_0})$, and $\infty\text{-conv}(x_n) := \text{abs-conv}(x_n) = \Phi_x(B_{\ell_1})$, the absolute convex hull of (x_n). (Here, $\ell_{p^*} = c_0$ if $p = 1$.)
Next, let $\ell_p(X)$ be the subspace of $\ell_p^w(X)$ of all p-summable sequences in X $(1 \leq p \leq \infty)$, and $c_0(X)$ the space of all norm null sequences in X. Then $\ell_p(X)$ is a Banach space with the norm

$$\|(x_n)\|_p = \Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

Also, $\ell_\infty(X)$ is a Banach space in the norm

$$\|(x_n)\|_\infty = \|(x_n)\|_\infty^w = \sup_{n\in\mathbb{N}} \|x_n\|.$$

Moreover, $c_0(X)$ is a closed subspace of $\ell_\infty(X)$ in the above norm. We write $\Phi(\ell_\infty^p(X)) = \Phi_p(X)$ $(1 \leq p \leq \infty)$ and $\Phi(c_0(X)) = \Phi_0(X)$.

Suppose X is a Banach space and let $p \geq 1$ be a real number. Following [43], the *p-convex hull* of some sequence $(x_k) \in \ell_p(X)$ is defined by

$$p\text{-conv}(x_k) = \Big\{ \sum_{k=1}^{\infty} a_k x_k : (a_k) \in B_{\ell_{p^*}} \Big\}.$$

Suppose $1 \leq p \leq \infty$ and let $1 \leq r \leq p^*$. Following [3], the (p,r)*-convex hull* of some sequence $(x_k) \in \ell_p(X)$ (where $(x_k) \in c_0(X)$ if $p = \infty$) is defined by

$$(p,r)\text{-conv}(x_k) = \Big\{ \sum_{k=1}^{\infty} a_k x_k : (a_k) \in B_{\ell_r} \Big\}.$$

As in [3] and [1], we give the following definitions:

Definition 1.2.1. *Suppose that $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$.*

(1) A subset K of a Banach space X is said to be relatively (p,r)-compact *if $K \subset (p,r)$-conv(x_n) for some sequence $(x_n) \in \ell_p(X)$ for $1 \leq p < \infty$ (and in the case where $p = \infty$, we have $(x_n) \in c_0(X)$).*

(2) A linear operator $T : X \longrightarrow Y$ is said to be (p,r)-compact *if the image set $T(B_X)$ is a relatively (p,r)-compact subset of Y. Denote by $\mathcal{K}_{(p,r)}$ the class of (p,r)-compact operators acting between arbitrary Banach spaces.*

We note the following with regard to compactness:

Proposition 1.2.2. (Generalized Inclusion Theorem ([1], Theorem 3.6)) *Let X be a Banach space. Let $1 \leq p \leq q \leq \infty, 1 \leq r \leq p^*$, and $1 \leq s \leq q^*$. Let*

$$\frac{1}{q} + \frac{1}{s} \leq \frac{1}{p} + \frac{1}{r}.$$

If a subset of X is relatively (p, r)-compact, then it is relatively (q, s)-compact. In particular, if a subset of X is relatively (p, r)-compact, then it is relatively $(\infty, 1)$-compact, and hence weakly compact.

We recall the following concepts as found in [16]:

Definition 1.2.3. *Let X and Y be Banach spaces. An operator $T : X \to Y$ is said to be* p-summing *if there exists a constant $C \geq 0$ such that for $x_1, \cdots, x_n \in X$ and regardless of the natural number n, we have for all $1 \leq p < \infty$ that*

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p \right)^{\frac{1}{p}} \leq C \cdot \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} |x^*(x_i)|^p \right)^{\frac{1}{p}}.$$

The least C for which the above inequality holds will be denoted by $\pi_p(T)$, and we also denote by $\Pi_p(X, Y)$ the set of all p-summing operators from X into Y.

Definition 1.2.4. *Let $1 \leq p \leq \infty$ and suppose X and Y are Banach spaces. An operator $T : X \to Y$ is* p-integral *if there are a probability measure μ and (bounded linear) operators $B : L_p(\mu) \to Y^{**}$ and $A : X \to L_\infty(\mu)$ which gives rise to the following commutative diagram*

$$\begin{array}{ccccc} L_\infty(\mu) & & \xrightarrow{I_{\infty,p}} & & L_p(\mu) \\ {\scriptstyle A}\uparrow & & & & \downarrow{\scriptstyle B} \\ X & \xrightarrow{T} & Y & \xrightarrow{\kappa_Y} & Y^{**} \end{array}$$

where $I_{\infty,p} : L_\infty(\mu) \to L_p(\mu)$ is the formal identity and $\kappa_Y : Y \to Y^{**}$ is the canonical isometric embedding.

The collection of all p-integral operators from X to Y will be denoted by $\mathcal{I}_p(X, Y)$, and for each $T \in \mathcal{I}_p(X, Y)$, the p-integral norm is given by,

$$\iota_p(T) = \inf \|A\| \cdot \|B\|,$$

where the infimum is taken over all measures μ and operators A and B as in the above diagram.

Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. We will have the occasion to appeal to the equivalence of

$r \leq p^*$ and $r^* \geq p$ since

$$\begin{aligned} r &\leq p^* \\ 1/r &\geq 1/p^* \\ -1/r &\leq -1/p^* \\ 1 - 1/r &\leq 1 - 1/p^* \\ 1/r^* &\leq 1/p \\ r^* &\geq p. \end{aligned}$$

Hence $1 \leq p \leq r^*$.

1.3 Published papers and preprints

This thesis constitutes two papers on the compactness property associated with the quasi-normed integral operator ideals as follows:

(1) Salthiel Malesela Maepa and Brian Chihinga Ndumba . "*On mid (p, r)-compact operators*". *Quaestiones Mathematicae*, ***7***, *1 - 22, 2024.*

(2) Brian Chihinga Ndumba and Salthiel Malesela Maepa. "*On $\mathcal{C}_p^r$ Property of Banach spaces*". *Submitted.*

Chapter 2

Preliminaries on operator ideals

In this chapter, we present an overview of the concepts and results which will be needed in the next chapters. We introduce and state some well-known results on the regular hull, injective hull, surjective hull, maximal hull and the dual ideal of operator ideals. The concepts and results on the (t, u, v)-nuclear and (t, u, v)-integral operators for $0 \leq t \leq \infty$, $1 \leq u, v \leq \infty$, and $\frac{1}{u} + \frac{1}{v} \leq 1 + \frac{1}{t}$ will also be introduced and discussed as well as the approximation property with some of their well-known results stated without proofs. The concepts and results are based on Pietsch monographs in [36] and [37], while others are from [42] and [16].

2.1 Operator ideals

We recall some well known concepts as found in [37], [42] and [16].

Suppose X and Y are Banach spaces. We denote by $\mathcal{L}$, the class of continuous linear operators between Banach spaces X and Y. An operator $T \in \mathcal{L}(X, Y)$ is of *finite rank* if $T(X)$ is finite dimensional. In the special case where $T(X)$ is one dimensional, then $T \in \mathcal{L}(X, Y)$ is said to be of *rank one* . We shall denote by $\mathcal{F}(X, Y)$, the subspace of $\mathcal{L}(X, Y)$ of finite rank operators from X into Y. We simply write $\mathcal{F}(X)$ for $\mathcal{F}(X, X)$. It is also known that any $T \in \mathcal{F}(X, Y)$ can be written as

$$T = \sum_{i=1}^{n} x_i^* \otimes y_i$$

with $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ and $y_1, y_2, \ldots, y_n \in Y$. Thus,

$$Tx = \sum_{i=1}^{n} x_i^*(x) y_i$$

for all $x \in X$. If $T \in \mathcal{L}(X, Y)$ is of rank one, then $T = x^* \otimes y$ for some non-zero $x^* \in X^*$ and $y \in Y$.

Definition 2.1.1. *For a given $\mathbb{K}$-vector space X, the map $\|\cdot\|$ from X to non-negative real numbers $\mathbb{R}^+$ is said to be a* quasi-norm *if the conditions N_1 to N_3 below are satisfied:*

(N_1) $\|x\| = 0$ implies $x = 0$ for every $x \in X$

(N_2) $\|x + y\| \leq c[\|x\| + \|y\|]$ where $c \geq 1$ is a constant and is independent of the choice of x and y in X.

(N_3) $\|\beta x\| = |\beta|\|x\|$ for $x \in X$ and $\beta \in \mathbb{K}$.

We note that the quasi-triangle inequality (N_2) generalizes the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$. If the following s-triangle inequality is satisfied,

(N_4) $\|x + y\|^s \leq \|x\|^s + \|y\|^s$

*then, a quasi-norm is known as an s-*norm *where $0 < s \leq 1$.*

Remark 2.1.2. *From Definition 2.1.1, we note the following:*

(1) If $c = 2^{1/s-1}$ in (N_2), then (N_4) implies (N_2). Conversely, every quasi-norm is equivalent to an s-norm where c and s are related by $c = 2^{1/s-1}$. (see [36], 3.2.5.4).

(2) For $0 < t < s \leq 1$ with $c = 2^{1/s-1}$ in (N_2), then the 1-norm is known as a norm, while an s-norm is just a t-norm.

(3) For a given $\mathbb{K}$-vector space X, a quasi-norm on X is said to be a quasi-Banach space *if X is complete with respect to the uniform structure derived from the quasi-norm. In the case where the quasi-norm is an s-norm, then X is called an s-Banach space.*

Definition 2.1.3. *A subclass of $\mathcal{L}$ denoted by $\mathcal{A}$ consisting of continuous linear operators between Banach spaces X and Y is said to be an* operator ideal *if the following three conditions are satisfied:*

(1) $x^* \otimes y \in \mathcal{A}(X, Y)$ for $x^* \in X^*$ and $y \in Y$. That is, $\mathcal{A}$ contains all rank one operators.

(2) $U_1 + U_2 \in \mathcal{A}(X, Y)$ for $U_1, U_2 \in \mathcal{A}(X, Y)$.

(3) If $V \in \mathcal{L}(X_0, X)$, $U \in \mathcal{A}(X, Y)$, and $T \in \mathcal{L}(Y, Y_0)$, then $TUV \in \mathcal{A}(X_0, Y_0)$ where X_0, and Y_0 are Banach spaces.

Examples of operator ideals include the class of approximable operators $\overline{\mathcal{F}}$, compact operators $\mathcal{K}$, weakly compact operators $\mathcal{W}$, and the finite rank operators $\mathcal{F}$. Operator ideals are usually considered over the class of all Banach spaces. The ideal of finite rank operators is the smallest operator ideal while the ideal of continuous linear operators is the largest operator ideal.
We define a *quasi-normed operator ideal* as an operator ideal $\mathcal{A}$ with a quasi-norm such that the components $\mathcal{A}(X, Y) := \mathcal{A} \cap \mathcal{L}(Y, Y)$ are all linear topological Hausdorff spaces. A *quasi-Banach operator ideal* is a quasi-normed operator ideal $\mathcal{A}$ such that all components $\mathcal{A}(X, Y)$ are complete.

Definition 2.1.4. *Suppose $0 < s \leq 1$. Let $\alpha : \mathcal{A} \longrightarrow \mathbb{R}^+$ be an s-norm. We say that an operator ideal $\mathcal{A}$ is an* s-Banach operator ideal *and this is denoted by $[\mathcal{A}, \alpha]$ if the following three conditions are satisfied:*

(1) $[\mathcal{A}(X, Y), \alpha]$ is an s-Banach space.

(2) $\alpha(x^* \otimes y) = \|x^*\|\|y\|$ for all $x^* \in X^*$ and $y \in Y$.

(3) $\alpha(BTA) \leq \|B\|\alpha(T)\|A\|$ for given Banach spaces X_0 and Y_0 where $A \in \mathcal{L}(X_0, X)$, $T \in \mathcal{A}(X, Y)$, and $B \in \mathcal{L}(Y, Y_0)$.

When $s = 1$, the 1-*Banach operator ideal* is referred to as a *Banach operator ideal*. Some examples of operator ideals which are also known as Banach operator ideals with respect to a given operator norm include; approximable operators $\overline{\mathcal{F}}$, compact operators $\mathcal{K}$, and the weakly

compact operators $\mathcal{W}$. However, the finite rank operators $\mathcal{F}$ are not Banach operator ideals (see [36], 6.3.2.4.). For a quasi-normed operator ideal $[\mathcal{A}, \alpha]$, we have $\|T\| \leq \alpha(T)$.
Suppose $[\mathcal{A}, \alpha]$ and $[\mathcal{B}, \beta]$ are two quasi-normed operator ideals. We shall write $[\mathcal{A}, \alpha] \subset [\mathcal{B}, \beta]$ if regardless of the Banach spaces X and Y, we have $\mathcal{A}(X, Y) \subset \mathcal{B}(X, Y)$ with $\beta(T) \leq \alpha(T)$ for all $T \in \mathcal{A}(X, Y)$. $[\mathcal{A}, \alpha] = [\mathcal{B}, \beta]$ means that $[\mathcal{A}, \alpha] \subset [\mathcal{B}, \beta]$ and $[\mathcal{B}, \beta] \subset [\mathcal{A}, \alpha]$ both hold simultaneously, whence, $\beta(T) = \alpha(T)$ whenever $T \in \mathcal{A}(X, Y)$ where X and Y are the given Banach spaces.

2.2 Hull procedures

Let $\mathcal{A}$ be an operator ideal. We recall and state some basic concepts and results on the regular, injective, surjective and maximal hulls of operator ideals as found in [36], [37], and [38].
Suppose that X is a Banach space. Recall from ([37], C.3.3) that

$$X^{inj} := \ell_\infty(B_{X^*}) \qquad \text{and} \qquad X^{sur} := \ell_1(B_X).$$

Then $J_X : X \longrightarrow X^{inj}$ is a metric injection from X into X^{inj} and $Q_X : X^{sur} \longrightarrow X$ is a metric surjection from X^{sur} onto X given by

$$J_X(x) = (\langle x, x^* \rangle)_{x^* \in B_{X^*}} \quad \text{and} \quad Q_X((\beta_x)_{x \in B_X}) = \sum_{x \in B_X} \beta_x x,$$

respectively.

Given Banach spaces X_0, and X, recall that a Banach space Y has the *extension property* (respectively, is *injective*) if for every operator $T \in \mathcal{L}(X_0, Y)$ (respectively, every subspace $X_0 \subset X$), and every injection $J_{X_0} \in \mathcal{L}(X_0, X)$, there is an extension $\tilde{T} \in \mathcal{L}(X, Y)$ with the property that $T = \tilde{T} \circ J_{X_0}$. If for every metric injection $J_{X_0} \in \mathcal{L}(X_0, X)$ and every operator $T \in \mathcal{L}(X_0, Y)$, there is a constant $\lambda \geq 1$ such that $\|\tilde{T}\| \leq \lambda \|T\|$, then Y is said to have the λ-*extension property*. We say that Y has the *metric extension property* if $\lambda = 1$. The space X^{inj} has the metric extension property.
Given Banach spaces Y_0 and Y, a Banach space X has the *lifting property* if for every surjection

$Q \in \mathcal{L}(Y, Y_0)$ and operator $T \in \mathcal{L}(X, Y_0)$, there is a lifting $\tilde{T} \in \mathcal{L}(X, Y)$ such that $T = Q \circ \tilde{T}$. We say that X has the *metric lifting property* if given $\varepsilon > 0$, it holds that for every metric surjection $Q \in \mathcal{L}(Y, Y_0)$ and operator $T \in \mathcal{L}(X, Y_0)$, there exists $\tilde{T} \in \mathcal{L}(X, Y)$ such that $T = Q \circ \tilde{T}$ and $\|\tilde{T}\| \leq (1 + \varepsilon)\|T\|$. The space X^{sur} has the lifting property.

Let $\mathcal{A}$ be a given operator ideal. Then from ([37], Theorem 4.6.9), $\mathcal{A}$ is said to be *injective* if for every metric injection $J \in \mathcal{L}(Y_0, Y)$ and every operator $\tilde{T} \in \mathcal{L}(X, Y_0)$, it follows from $J\tilde{T} \in \mathcal{A}(X, Y)$ that $\tilde{T} \in \mathcal{A}(X, Y_0)$. A quasi-normed operator ideal $[\mathcal{A}, \alpha]$ is said to be *injective* if, furthermore, it holds that $\alpha(T) = \alpha(J\tilde{T})$. We also recall from ([37], Theorem 4.7.9) that $\mathcal{A}$ is said to be *surjective* if for every metric surjection $Q \in \mathcal{L}(X, X_0)$ and every operator $\tilde{T} \in \mathcal{L}(X_0, Y)$, it follows from $\tilde{T}Q \in \mathcal{A}(X, Y)$ that $\tilde{T} \in \mathcal{A}(X_0, Y)$. A quasi-normed operator ideal $[\mathcal{A}, \alpha]$ is said to be *surjective* if, furthermore, it holds that $\alpha(T) = \alpha(\tilde{T}Q)$.

Definition 2.2.1. *(1) A rule new : $[\mathcal{A}, \alpha] \longrightarrow [\mathcal{A}^{new}, \alpha^{new}]$ which defines a new quasi-normed operator ideal $[\mathcal{A}^{new}, \alpha^{new}]$ written as $[\mathcal{A}, \alpha]^{new}$ is called a* procedure.

(2) A procedure is said to be monotone *if $[\mathcal{A}, \alpha] \subset [\mathcal{B}, \beta]$ implies that $[\mathcal{A}, \alpha]^{new} \subset [\mathcal{B}, \beta]^{new}$, while a procedure is said to be* idempotent *if $([\mathcal{A}, \alpha]^{new})^{new} = [\mathcal{A}, \alpha]^{new}$ for all $[\mathcal{A}, \alpha]$. The* dual ideal *of $\mathcal{A}$ denoted by $\mathcal{A}^{dual}$ is given by*

$$\mathcal{A}^{dual} = \{T \in \mathcal{A}(X, Y) : T^* \in \mathcal{A}(Y^*, X^*)\}.$$

We note that every quasi-norm on $\mathcal{A}$ gives a dual quasi-norm

$$\alpha^{dual}(T) = \alpha(T^*) \tag{2.1}$$

on $\mathcal{A}^{dual}$. If $[\mathcal{A}, \alpha]$ is also an s-Banach ideal, then $[\mathcal{A}^{dual}, \alpha^{dual}]$ is also known as an s-Banach operator ideal with (2.1) satisfied.

(3) A monotone and idempotent procedure is called a hull procedure *if $[\mathcal{A}, \alpha] \subset [\mathcal{A}, \alpha]^{new}$, and a* kernel procedure *if $[\mathcal{A}, \alpha] \supset [\mathcal{A}, \alpha]^{new}$.*

(4) For a quasi-normed operator ideal $[\mathcal{A}, \alpha]$, $T \in \mathcal{L}(X, Y)$ belongs to the regular hull *of $[\mathcal{A}, \alpha]$ denoted by $[\mathcal{A}, \alpha]^{reg}$ if $\kappa_Y T \in \mathcal{A}(X, Y^{**})$ where κ_Y is the canonical isometric embedding*

from Y *into* Y^{**} *with* $\alpha^{reg}(T) = \alpha(\kappa_Y T)$. *In the case where* $J_Y T \in \mathcal{A}(X, Y^{inj})$, *then* $T \in \mathcal{L}(X,Y)$ *belongs to the* injective hull *of* $[\mathcal{A}, \alpha]$ *denoted by* $[\mathcal{A}, \alpha]^{inj}$ *where* $J_Y : Y \longrightarrow Y^{inj}$ *is a metric injection considered above with* $\alpha^{inj}(T) = \alpha(J_Y T)$. *Now, if* $T \in \mathcal{L}(X,Y)$ *is such that* $TQ_X \in \mathcal{A}(X^{sur}, Y)$, *then* T *is said to belong to the* surjective hull *of* $[\mathcal{A}, \alpha]$ *denoted by* $[\mathcal{A}, \alpha]^{sur}$ *where* $Q_X : X^{sur} \longrightarrow X$ *is a metric surjection considered above, with* $\alpha^{sur}(T) = \alpha(TQ_X)$. *Note that the injective hull* $[\mathcal{A}, \alpha]^{inj}$ *is the smallest injective operator ideal of* $[\mathcal{A}, \alpha]$ *that contains* $[\mathcal{A}, \alpha]$, *while the surjective hull* $[\mathcal{A}, \alpha]^{sur}$ *is the smallest surjective operator ideal of* $[\mathcal{A}, \alpha]$ *that contains* $[\mathcal{A}, \alpha]$. *Finally,* $T \in \mathcal{L}(X,Y)$ *belongs to the* maximal hull *of* $[\mathcal{A}, \alpha]$ *denoted by* $[\mathcal{A}, \alpha]^{max}$ *if* $STR \in \mathcal{A}(X_0, Y_0)$ *where* $R \in \overline{\mathcal{F}}(X_0, X)$, $S \in \overline{\mathcal{F}}(Y, Y_0)$ *and* X_0 *and* Y_0 *are arbitrary Banach spaces. For a given quasi-normed operator ideal* $[\mathcal{A}, \alpha]$, *the regular hull, the injective hull, the surjective hull and the maximal hull are all hull procedures.*

(5) If $[\mathcal{A}, \alpha] = [\mathcal{A}, \alpha]^{reg}$, *then* $[\mathcal{A}, \alpha]$ *is said to be regular, and if* $[\mathcal{A}, \alpha] = [\mathcal{A}, \alpha]^{inj}$, *then* $[\mathcal{A}, \alpha]$ *is said to be injective. A surjective quasi-normed operator ideal* $[\mathcal{A}, \alpha]$ *is one in which* $[\mathcal{A}, \alpha] = [\mathcal{A}, \alpha]^{sur}$ *while a maximal quasi-normed operator ideal is one in which* $[\mathcal{A}, \alpha] = [\mathcal{A}, \alpha]^{max}$.

(6) For a given quasi-Banach operator ideal $[\mathcal{A}, \alpha]$, $T \in \mathcal{L}(X,Y)$ *belongs to the* maximal hull *denoted by* $[\mathcal{A}, \alpha]^{max}$ *if there is a constant* $\kappa \geq 0$ *such that*

$$\alpha(STR) \leq \kappa \|S\| \|R\|$$

where $R \in \mathcal{F}(X_0, X)$, $S \in \mathcal{F}(Y, Y_0)$ *and* X_0 *and* Y_0 *are arbitrary Banach spaces. We set*

$$\alpha^{max}(T) = \inf \kappa.$$

For a quasi-Banach operator ideal $[\mathcal{A}, \alpha]$, $[\mathcal{A}, \alpha]^{reg}$, $[\mathcal{A}, \alpha]^{inj}$ *and* $[\mathcal{A}, \alpha]^{sur}$ *are also quasi-Banach operator ideals with*

$$\alpha^{reg}(T) = \alpha(K_Y T),$$

$$\alpha^{inj}(T) = \alpha(J_Y T),$$

and

$$\alpha^{sur}(T) = \alpha(TQ_X),$$

respectively for every $T \in \mathcal{L}(X,Y)$ such that $T \in [\mathcal{A},\alpha]^{reg}$, $[\mathcal{A},\alpha]^{inj}$, and $[\mathcal{A},\alpha]^{sur}$, respectively.

The following are some of the properties of an (quasi-Banach) operator ideal. These properties can be found in ([37], Theorem 8.5.9), ([37], Proposition 8.5.12), ([37], Proposition 8.7.13), and ([37], Proposition 8.7.14).

Proposition 2.2.2. *Let $[\mathcal{A},\alpha]$ be a quasi-Banach operator ideal. We have the following:*

(1) $([\mathcal{A},\alpha]^{dual})^{sur} = ([\mathcal{A},\alpha]^{inj})^{dual}$.

(2) $([\mathcal{A},\alpha]^{dual})^{inj} \subseteq ([\mathcal{A},\alpha]^{sur})^{dual}$.

(3) $([\mathcal{A},\alpha]^{inj})^{sur} = ([\mathcal{A},\alpha]^{sur})^{inj}$.

(4) $([\mathcal{A},\alpha]^{inj})^{max} = ([\mathcal{A},\alpha]^{max})^{inj}$.

(5) $([\mathcal{A},\alpha]^{max})^{sur} = ([\mathcal{A},\alpha]^{sur})^{max}$.

2.3 Nuclear and integral operators

In this section, we use the notation found in [3] and [37].

Definition 2.3.1. *([37], 18.1.1.) Let $0 < t \leq \infty$. Suppose $1 \leq u, v \leq \infty$, and $\frac{1}{u} + \frac{1}{v} \leq 1 + \frac{1}{t}$. A bounded linear operator $T \in \mathcal{L}(X,Y)$ is said to be* (t,u,v)-nuclear *if*

$$T = \sum_{n=1}^{\infty} \delta_n x_n^* \otimes y_n, \tag{2.2}$$

where ($\delta_n \in \ell_t$ (or c_0 if $t = \infty$), $(x_n^*) \in \ell_{v^*}^w(X^*)$ and $(y_n) \in \ell_{u^*}^w(Y)$. We denote the (t,u,v)-nuclear norm by

$$\nu_{(t,u,v)}(T) = \inf\{\|(\delta_n)\|_t \|(x_n^*)\|_{v^*}^w \|(y_n)\|_{u^*}^w\}$$

where we take the infimum over all (t,u,v)-nuclear representations (2.2) of T. The set of all (t,u,v)-nuclear operators will be denoted by $(\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}(\cdot))$. On setting $\frac{1}{s} = \frac{1}{t} + \frac{1}{u^*} + \frac{1}{v^*}$, we note that $s \in (0,1)$. It is well known that the set of all (t,u,v)-nuclear operators is an s-Banach operator ideal (see [37], 18.1.2).

Moreover, if $1 \leq p \leq \infty$ recall that for any $T \in \mathcal{L}(X,Y)$, $T \in \mathcal{N}_p(X,Y)$ (respectively $T \in \mathcal{N}^p(X,Y)$) if T has a representation of the form $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$, with $(x_n^*) \in \ell_p(X^*)$ (or $c_0(X^*)$ if $p = \infty$), and $(y_n) \in \ell_{p^*}^w(Y)$ (respectively with $(x_n^*) \in \ell_{p^*}^w(X^*)$, and $(y_n) \in \ell_p(Y)$ (or $c_0(Y)$ if $p = \infty$).
The norm of T is defined by

$$\nu_p(T) = \inf\{\|(x_n^*)\|_p \|(y_n)\|_{p^*}^w\}$$

(respectively $\nu^p(T) = \inf\{\|(x_n^*)\|_{p^*}^w \|(y_n)\|_p\}$) where the infimum is taken over all representations of T of the given form. Then $(\mathcal{N}_p, \nu_p(\cdot))$ (respectively $(\mathcal{N}^p, \nu^p(\cdot))$ is called the Banach operator ideal of *p-nuclear operators* (respectively, *right p-nuclear operators*).
By definition, if $1 \leq p \leq \infty$, then

$$(\mathcal{N}^p, \nu^p(\cdot)) = (\mathcal{N}_{(p,1,p)}, \nu_{(p,1,p)}(\cdot)) \quad \text{(cf.} \quad ([37], 18.1.1) \quad \text{(or} \quad [42], p.140),$$

and

$$(\mathcal{N}_p, \nu_p(\cdot)) = (\mathcal{N}_{(p,p,1)}, \nu_{(p,p,1)}(\cdot)) \quad \text{(cf.} \quad ([37], 18.2.1).$$

Remark 2.3.2. *Concerning the operators defined in equation 2.2 as the (t,u,v)-nuclear operators, the following observation obtains: These operators are defined in [29] as the (v,u)-nuclear operators with the condition $\frac{1}{t} + \frac{1}{u} + \frac{1}{v} = 1$ so that they form a Banach operator ideal, where the $\|(x_n^*)\|_u^w$, $\|(y_n)\|_v^w$ are used instead in the definition of the $\nu_{(t,u,v)}(\cdot)$-norm. Observe that in ([37], 18.1.2) the corresponding condition $\frac{1}{t} + \frac{1}{u^*} + \frac{1}{v^*} = 1$, $\|(x_n^*)\|_{v^*}^w$, and $\|(y_n)\|_{u^*}^w$ are used instead in the definition of the $\nu_{(t,u,v)}(\cdot)$-norm. The corresponding characterizing factorization diagrams in the two cases are therefore also consequently different (cf. ([29], Theorem 1.5) and ([37], 18.1.3) respectively). In particular, whereas the 1-nuclear operators are $(\mathcal{N}_{(1,1,1)}, \nu_{(1,1,1)}(\cdot))$ following ([37],18.2.1), these are $(\mathcal{N}_{(1,\infty,\infty)}, \nu_{(1,\infty,\infty)}(\cdot))$ following [29].*

Theorem 2.3.3. *([37], Theorem 18.1.6) Given a quasi-Banach operator $(\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}(\cdot))$, we have*

$$(\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}(\cdot))^{dual} = (\mathcal{N}_{(t,v,u)}, \nu_{(t,v,u)}(\cdot))^{reg}.$$

Definition 2.3.4. *([37], 19.1.1) Suppose that $0 < t \leq \infty$. Let $1 \leq u, v \leq \infty$ and $1 + \frac{1}{t} \geq \frac{1}{u} + \frac{1}{v}$. An operator $T \in \mathcal{L}(X,Y)$ is said to be* (t,u,v)-integral *if it belongs to the quasi-Banach operator* *ideal*

$$[\mathcal{I}_{(t,u,v)}, \iota_{(t,u,v)}] := [\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]^{max}.$$

In ([37], Theorem 19.1.6), the following result (Theorem 2.3.5) was proved.

Theorem 2.3.5. *([37], Theorem 19.1.6) Suppose* $0 < t < \infty$. *Let* $1 \leq u, v \leq \infty$ *and* $\frac{1}{u} + \frac{1}{v} < 1 + \frac{1}{t}$. *A bounded linear operator* $T \in \mathcal{L}(X, Y)$ *is* (t, u, v)-integral *if and only if there exists a commutative diagram*

$$\begin{array}{ccccc} \ell_{v^*} & & \xrightarrow{D_\lambda} & & \ell_u \\ \uparrow{\scriptstyle A} & & & & \downarrow{\scriptstyle B} \\ X & \xrightarrow{T} & Y & \xrightarrow{K_Y} & Y^{**} \end{array}$$

such that $A \in \mathcal{L}(X, \ell_{v^*})$, $B \in \mathcal{L}(\ell_{v^*}, Y^{**})$, *and* $D_\lambda \in \mathcal{L}(\ell_{v^*}, \ell_u)$ *is the diagonal operator of the form* $D_\lambda(\alpha_n) = (\lambda_n \alpha_n)$ *with* $(\lambda_n) \in \ell_t$.
The collection of all (t, u, v)*-integral operators from* X *to* Y *will be denoted by*

$$\mathcal{I}_{(t,u,v)}(X, Y).$$

With each $T \in \mathcal{I}_{(t,u,v)}(X, Y)$, *its* (t, u, v)*-integral norm is defined by,*

$$\iota_{(t,u,v)}(T) = \inf \|B\| \|(\lambda_n)\|_t \|A\|$$

where the infimum is taken over all possible factorizations as above.

We now recall the notion of an ultrastable quasi-Banach operator ideal (see [37], 8.8.1, 8.8.3, and 8.8.5). Let $(X_i)_{i \in I}$ be a family of Banach spaces and $\mathcal{U}$ be an ultrafilter on an arbitrary index set I. If $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ are families of Banach spaces, and $T_i \in \mathcal{L}(X_i, Y_i)$ is a family of operators, then the operator

$$(T_i)_{\mathcal{U}} : (\prod_{i \in I} X_i)_{\mathcal{U}} \longrightarrow (\prod_{i \in I} Y_i)_{\mathcal{U}}$$

defined by

$$(T_i)_{\mathcal{U}}(x_i)_{\mathcal{U}} = (T_i x_i)_{\mathcal{U}}, \quad (x_i)_{\mathcal{U}} \in (\prod_{i \in I} X_i)_{\mathcal{U}},$$

is called the *ultraproduct of the operators* $T_i \in \mathcal{L}(X_i, Y_i)$ with respect to the ultrafilter $\mathcal{U}$ where $(\prod_{i\in I} X_i)_{\mathcal{U}}$ denotes the *ultraproduct* of the family $(X_i)_{i\in I}$. A quasi-Banach operator ideal $\mathcal{A}$ is said to be *ultrastable* if

$$(T_i)_{\mathcal{U}} \in \mathcal{A}((\prod_{i\in I} X_i)_{\mathcal{U}}, (\prod_{i\in I} Y_i)_{\mathcal{U}})$$

and

$$\|(T_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|T_i\|_{\mathcal{A}}.$$

for every ultrafilter $\mathcal{U}$ on I and any family of operators $T_i \in \mathcal{A}(X_i, Y_i)$ satisfying the condition that $\sup_{i\in I} \|T_i\|_{\mathcal{A}} < \infty$.

To prove the next two theorems (Theorem 2.3.7 and Theorem 2.3.8), we need the following:

Remark 2.3.6. *(a) Let $[\mathcal{A}, \alpha]$ be a quasi-Banach operator. Then by ([37], Proposition 8.7.10), we have that*

$$[\mathcal{A}, \alpha]^{reg} \subset [\mathcal{A}, \alpha]^{max}.$$

In the case where $[\mathcal{A}, \alpha]$ is an ultrastable quasi-Banach operator ideal, then the injective and surjective hulls are ultrastable (see [37], 8.8.8 and 8.8.9) and $[\mathcal{A}, \alpha]^{max} = [\mathcal{A}, \alpha]^{reg}$ (see [37], Theorem 8.8.6.). In the event that $[\mathcal{A}, \alpha]$ is an s-Banach operator ideal, the equality

$$\Big([\mathcal{A}, \alpha]^{dual}\Big)^{max} = \Big([\mathcal{A}, \alpha]^{max}\Big)^{dual}$$

holds by ([37], Proposition 8.7.12).

(b) If $0 < t < \infty$ and $\frac{1}{u} + \frac{1}{v} < 1 + \frac{1}{t}$, then the quasi-Banach operator ideal $[\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]$ is ultrastable by ([37], Theorem 18.1.9.). In the case where $\frac{1}{s} := \frac{1}{t} + \frac{1}{u^*} + \frac{1}{v^*}$, the operator ideal $[\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]$ is an s-Banach ideal by ([37], Theorem 18.1.2.). We therefore see that all the results in (a) holds in the case where we take $[\mathcal{A}, \alpha]$ to be $[\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]$.

We now prove the next two results.

Theorem 2.3.7. *([37], Theorem 19.1.4) Suppose $0 < t < \infty$. Let $1 \le u, v \le \infty$ and $\frac{1}{u} + \frac{1}{v} < 1 + \frac{1}{t}$. Then*

$$[\mathcal{I}_{(t,u,v)}, \iota_{(t,u,v)}] = [\mathcal{I}_{(t,v,u)}, \iota_{(t,v,u)}]^{dual}.$$

Proof. By Definition 2.3.4, Theorem 2.3.3 and Remark 2.3.6, we have

$$\begin{aligned}
[\mathcal{I}_{(t,u,v)}, \iota_{(t,u,v)}] &= [\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]^{max} \\
&= \left([\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]^{max}\right)^{max} \quad \text{(since the rule `max' is idempotent)} \\
&= \left([\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]^{reg}\right)^{max} \\
&= \left([\mathcal{N}_{(t,v,u)}, \nu_{(t,v,u)}]^{dual}\right)^{max} \quad \text{(by Theorem 2.3.3)} \\
&= \left([\mathcal{N}_{(t,v,u)}, \nu_{(t,v,u)}]^{max}\right)^{dual} \\
&= [\mathcal{I}_{(t,v,u)}, \iota_{(t,v,u)}]^{dual}.
\end{aligned}$$

□

Theorem 2.3.8. *([37], Theorem 19.1.5) Suppose* $0 < t < \infty$*. Let* $1 \leq u, v \leq \infty$*, and* $\frac{1}{u} + \frac{1}{v} < 1 + \frac{1}{t}$*. Then*

$$[\mathcal{I}_{(t,u,v)}, \iota_{(t,u,v)}] = [\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]^{reg}.$$

Proof. Since the quasi-Banach operator ideal $[\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]$ is ultrastable, we have by Definition 2.3.4 and Remark 2.3.6 that

$$[\mathcal{I}_{(t,u,v)}, \iota_{(t,u,v)}] = [\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]^{max} = [\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]^{reg}.$$

□

2.4 Approximation property

The following concepts and results on the approximation property and the metric approximation property are found in [14], [17], [42], and [37].

Definition 2.4.1. *([17], VIII.3, Definition 1) The Banach space* X *is said to have the* approximation property *if for each compact set* $K \subseteq X$*, and for every* $\varepsilon > 0$*, there exists a continuous finite rank operator* $T : X \longrightarrow X$ *such that for all* $x \in K$*,* $\|Tx - x\| < \varepsilon$*. If in addition,* $\|T\| \leq 1$*, then* X *has the* metric approximation property*.*

Example 2.4.2. *([17], VIII.3, Example 11) Let* (Ω, Σ, μ) *be any measure space and suppose* $1 \leq p < \infty$. *Then* $L_p(\mu)$ *has the metric approximation property.*

Corollary 2.4.3. *([17], VIII.4, Corollary 2) Suppose that* X *is a reflexive Banach space. If* X *has the approximation property, then* X *also has the metric approximation property.*

Corollary 2.4.4. *([17], VIII.4, Corollary 3) Suppose that* X *is a Banach space. If the dual space* X^* *of* X *is separable and has the approximation property, then* X^* *also has the metric approximation property.*

Corollary 2.4.5. *([17], VIII.3, Corollaries 9, 5) Suppose that* X *is a Banach space. If the dual space* X^* *of* X *has the (metric) approximation property, then* X *also has the same property.*

Remark 2.4.6. *We note the following:*

(1) In ([42], second paragraph of Example 4.4), the spaces c_0, *and* ℓ_p *for* $1 \leq p < \infty$ *all have the approximation property.*

(2) Suppose that Ω *is any compact Hausdorff space. Since the space* $L_1(\mu)$ *has the metric approximation property by Example 2.4.2, it follows from ([17], last paragraph in VIII.3) that the space* $C(\Omega)$ *also has the metric approximation property by Corollary 2.4.5 since the dual of any* $C(\Omega)$*-space is isometric to some* $L_1(\mu)$ *space. In particular, each* $L_\infty(\mu)$ *space has the metric approximation property.*

The following theorem is stated in ([13], Theorem) on p.295 and is attributed to O. I. Reinov [41]:

Theorem 2.4.7. *([13], Theorem: p.295) Let* $1 \leq p \leq \infty$. *Suppose* $T \in \mathcal{L}(X, Y)$ *and that either* X^* *or* Y^{***} *has the approximation property. If* $T \in \mathcal{N}^p(X, Y^{***})$, *then* $T \in \mathcal{N}^p(X, Y)$. *In other words, under these conditions, the (Pietsch) p-nuclearity of* T^* *implies that* $T \in \mathcal{N}^p(X, Y)$.

A sharper hypothesis on the approximation property is imposed on X^{***} in [24] to improve on the above result by using Grothendieck's classics [24] (also see ([15], Corollary 1.4.9) as well as ([42], Proposition 6.4)). Actually, it is affirmed in [38] that a more general result of this ilk was already known to Grothendieck ([24], p.17) and details can be found in ([9], p298) and ([15], p.59). Most importantly, the following Lemma 2.4.8 ([38], Lemma 3) provides us with a slightly modified version which provides us with the necessary handle to hold onto for our multiple index discourse.

Lemma 2.4.8. *([38], Lemma 3) Suppose that* $1 \leq u, v < \infty$ *and* $\frac{1}{t} = \frac{1}{u} + \frac{1}{v} - 1 > 0$*. Let* X^* *have the metric approximation property. If* $T \in \mathcal{L}(X, Y)$ *with* $T \in \mathcal{N}^{reg}_{(t,u,v)}(X, Y)$*, then* $T \in \mathcal{N}_{(t,u,v)}(X, Y)$ *and*

$$\nu_{(t,u,v)}(T) = \nu^{reg}_{(t,u,v)}(T).$$

Proof. The operator ideal approach proof (as contrasted with the tensor product approach proof deduced from the proof in ([33], Corollary 3.8) with the approximation property assumption imposed on X^*) may be followed up in ([38], p.521). □

Remark 2.4.9. *Since the rule 'reg' is a hull procedure, we have*

$$[\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}] \subset [\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]^{reg},$$

so that by Lemma 2.4.8, we have,

$$[\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}] = [\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]^{reg},$$

whenever $1 \leq u, v < \infty$*,* $\frac{1}{t} = \frac{1}{u} + \frac{1}{v} - 1 > 0$ *and* X^* *has the metric approximation property.*

Chapter 3

(p, r)-compactness and $(p, r^*, 1)$-nuclear map U_A^*

This chapter is devoted to the development and proofs of some of the quasi-Banach ideal theoretic properties of U_A^* and its preadjoint U_A which will subsequently be used in the investigation of Banach spaces which enjoy certain special structural properties.

In Section 3.1, our interest is to prove the following two results: The first result involves proving that a bounded subset A of a Banach space X is relatively (p, r)-compact in X if, and only if, the evaluation map $U_A : \ell_1(A) \longrightarrow X$ is $(p; r)$-compact for $1 \leq p < \infty$ and $1 \leq r < p^*$. The second result basically involves proving that if a bounded subset A of a Banach space X is relatively (p, r)-compact, then the operator U_A is $(p, 1, r^*)$-nuclear.

Finally, in Section 3.2, we prove that for a bounded subset A of a Banach space X, the operator U_A is (p, r)-compact if, and only if, its adjoint U_A^* belongs to the injective hull of the $(p, r^*, 1)$-nuclear operators for $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ and $\frac{pr}{p+r} = 1$.

These two results in Section 3.1 and the result in Section 3.2 described above will be needed in Chapter 4 to introduce and define a Property which a Banach space may or may not have.

3.1 The operator U_A

Suppose that $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$, where $\frac{1}{p} + \frac{1}{p^*} = 1$. Let X be a Banach space. It is known as may be seen in [1] and [2] that every $(x_n) \in \ell_p(X)$ defines $\Phi_{(x_n)} \in \mathcal{L}(\ell_r, X)$ as an operator with

$$\Phi_{(x_n)}(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n x_n, \quad \text{where} \quad (\alpha_n) \in \ell_r.$$

When we consider the unit vector basis (e_n) of ℓ_{r^*} (respectively, c_0 if $r = 1$), we have

$$\Phi_{(x_n)} = \sum_{n=1}^{\infty} e_n \otimes x_n. \tag{3.1}$$

It is also known that any $T \in \mathcal{L}(X,Y)$ has a factorization of the form $T = \tilde{T}S$ where $\tilde{T} \in \mathcal{L}(X/\ker T, Y)$ is an injective operator and $S : X \longrightarrow X/\ker T$ is the quotient map for every Banach space Y.

The following propositions, Proposition 3.1.1, Proposition 3.1.2, and Proposition 3.1.3 are proved in [1].

Proposition 3.1.1. *([1], Proposition 2.12.) Let X and Y be Banach spaces. For an s-Banach operator ideal $[\mathcal{A}, \alpha]$, if $T \in \mathcal{A}(X,Y)$, then $\tilde{T} \in \mathcal{A}^{sur}(X,Y)$ with*

$$\alpha^{sur}(T) \leq \alpha^{sur}(\tilde{T}) \leq \alpha(T).$$

Proposition 3.1.2. *([1], Proposition 2.15.) Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. If X is a Banach space with $(x_n) \in \ell_p(X)$ (respectively, $(x_n) \in c_0(X)$ when $p = \infty$), then the operator $\Phi_{(x_n)} : \ell_r \longrightarrow X$ is approximable, that is, $\Phi_{(x_n)} \in \overline{\mathcal{F}}(\ell_r, X)$.*

It is then deduced from the preceding proposition that, since $\overline{\mathcal{F}} \subset \mathcal{K}$, then $\Phi_{x_n} \in \mathcal{K}(\ell_r, X)$ whenever $(x_n) \in \ell_p(X)$ (respectively, $(x_n) \in c_0$ if $p = \infty$). Moreover, the following holds:

Proposition 3.1.3. *([1], Proposition 2.16.). Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. If X is a Banach space with $(x_n) \in \ell_p(X)$ (respectively, $(x_n) \in c_0(X)$ when $p = \infty$), then the operator $\Phi_{(x_n)} : \ell_r \longrightarrow X$ is $(p,1,r^*)$-nuclear, that is, $\Phi_{(x_n)} \in \mathcal{N}_{(p,1,r^*)}$, and $\nu_{(p,1,r^*)}(\Phi_{(x_n)}) \leq \|(x_n)\|_p$.*

It is observed in ([1], chapter 4) as a key fact that the injective associate $\bar{\Phi}_{(x_n)}$ of $\Phi_{(x_n)}$ belongs to $\mathcal{N}^{sur}_{(p,1,r^*)}$. The following proposition is then deduced from Propositions 3.1.1 and 3.1.3.

Proposition 3.1.4. *([1], Proposition 2.17) Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. If X be a Banach space with $(x_n) \in \ell_p(X)$ (respectively, $(x_n) \in c_0(X)$ when $p = \infty$), then $\bar{\Phi}_{(x_n)} \in \mathcal{N}^{sur}_{(p,1,r^*)}(Z, X)$ where $Z = \ell_r / \ker \Phi_{(x_n)}$ and $\nu^{sur}_{(p,1,r^*)}(\bar{\Phi}_{(x_n)}) \leq \|(x_n)\|$.*

Suppose A is a bounded subset of X. The evaluation map U_A^* and its preadjoint U_A are defined as follows ([13], p295):

$$\begin{aligned} U_A : \ell_1(A) &\longrightarrow X \\ (\xi_a)_{a \in A} &\mapsto \sum_{a \in A} \xi_a a. \end{aligned}$$

and

$$\begin{aligned} J_A : X^* &\longrightarrow \ell_\infty(A) \\ a^* &\mapsto (\langle a, a^* \rangle)_{a \in A} \end{aligned}$$

These maps are bounded operators and notice that $U_A^* = J_A$. Note that we write U_X and J_X instead of U_{B_X}, and J_{B_X}, respectively.

Proposition 3.1.5. *Suppose X is a Banach space and let A be a bounded subset of X. Then*

$$A \subseteq U_A\big(B_{\ell_1(A)}\big) \subseteq \overline{\text{abs-conv}}(A).$$

(Here, $\overline{\text{abs-conv}}(A)$ is the closed absolute convex hull of A).

Proof. We first show that

$$A \subseteq U_A\big(B_{\ell_1(A)}\big).$$

To this end, choose $a \in A$ arbitrarily and let the 'a-unit vector' be the function $e_a : A \longrightarrow \mathbb{K}$ defined by

$$e_a(b) := \delta_{ba},$$

the Kronecker delta, where

$$\delta_{ba} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

Thus, $(\delta_{ba})_{b\in A} \in B_{\ell_1(A)}$. Hence

$$\begin{aligned} a &= 1\cdot a \\ &= \sum_{b\in A} \delta_{ba}\cdot b \\ &= U_A\big((\delta_{ba})_{b\in A}\big) \\ &\in U_A\big(B_{\ell_1(A)}\big). \end{aligned}$$

Since $a \in A$ is arbitrary, we have

$$A \subseteq U_A\big(B_{\ell_1(A)}\big). \tag{3.2}$$

We next prove that

$$U_A\big(B_{\ell_1(A)}\big) \subseteq \overline{\text{abs-conv}}(A).$$

To this end, fix $y \in U_A\big(B_{\ell_1(A)}\big)$. Then $y = U_A(\beta)$ for some $\beta = (\beta_b)_{b\in A} \in B_{\ell_1(A)}$. Since

$$\begin{aligned} U_A(\beta) &= \sum_{b\in A} \beta_b b \qquad (\text{with} \quad \sum_{b\in A} |\beta_b| \leq 1) \\ &\in \overline{\text{abs-conv}}(A), \end{aligned}$$

it follows that $y = U_A(\beta) \in \overline{\text{abs-conv}}(A)$.

Since $y \in U_A\big(B_{\ell_1(A)}\big)$ is arbitrary, we have that

$$U_A\big(B_{\ell_1(A)}\big) \subseteq \overline{\text{abs-conv}}(A). \tag{3.3}$$

By equations 3.2 and 3.3, we have

$$A \subseteq U_A\big(B_{\ell_1(A)}\big) \subseteq \overline{\text{abs-conv}}(A),$$

as required. □

Proposition 3.1.6. *([13], Proposition 3.5) Suppose $1 \leq p < \infty$. Let X be a Banach space and A be a bounded subset of X. Then the following three conditions are equivalent:*

(a) A is relatively p-compact.

(b) The bounded operator U_A is p-compact.

(c) *The bounded operator J_A is p-nuclear.*

Proof. (a)$\Rightarrow$ (b). Suppose that A is relatively p-compact. We will show that U_A is p-compact. Since A is relatively p-compact, there exists a sequence $(x_n) \in \ell_p(X)$ such that

$$A \subset p\text{-conv}(x_n).$$

Since the set p-conv(x_n) is an absolutely convex set, it is equal to its own absolutely convex hull, so that it contains $\overline{\text{abs-conv}}(A)$. Now, since

$$\begin{aligned} U_A : \ell_1(A) &\longrightarrow X \\ (\xi_a)_{a\in A} &\mapsto \sum_{a\in A} \xi_a a. \end{aligned}$$

and by the second containment in Proposition 3.1.5, we have

$$U_A(B_{\ell_1(A)}) \subseteq \overline{\text{abs-conv}}(A) \subset p\text{-conv}(x_n).$$

Thus, $U_A(B_{\ell_1(A)})$ is a relatively p-compact subset of X. Hence, $U_n(B)$ is relatively p-compact for all bounded sets $B \subset \ell_1(A) = \ell_1$ where $U_n := U_{\{x_n\}}$, and $A = \{x_n\}$, the range of (x_n) so that U_A is p-compact.

(b)$\Rightarrow$ (a). Suppose that U_A is p-compact. Then, $U_A(B_{\ell_1(A)})$ is a relatively p-compact subset of X. That is, there is a sequence $(x_n) \in \ell_p(X)$ such that

$$U_A(B_{\ell_1(A)}) \subset p\text{-conv}(x_n).$$

Since A is a bounded subset of X and by the first containment in Proposition 3.1.5, we have that

$$A \subseteq U_A(B_{\ell_1(A)}) \subset p\text{-conv}(x_n)$$

so that A is relatively p-compact.

(b)$\Rightarrow$ (c). Suppose that U_A be p-compact. Then by ([13], Corollary 3.4), the operator

$$\begin{aligned} J_A : X^* &\longrightarrow \ell_\infty(A) \\ a^* &\mapsto (\langle a, a^*\rangle)_{a\in A} \end{aligned}$$

is quasi p-nuclear. Since $\ell_\infty(A)$ is an injective space, then by ([35], Theorem 38), J_A is p-nuclear with

$$\nu_p^Q(J_A) = \nu_p(J_A).$$

(c)$\Rightarrow$ (b). Let J_A be p-nuclear. Then by ([41],Theorem 1), the operator

$$\begin{array}{rcl} U_A : \ell_1(A) & \longrightarrow & X \\ (\xi_a)_{a\in A} & \mapsto & \sum_{a\in A} \xi_a a. \end{array}$$

is right p-nuclear so that U_A is p-compact since

$$\mathcal{N}^p(X,Y) \subseteq \mathcal{K}_p(X,Y)$$

for any Banach spaces X and Y by the paragraph following ([13], Corollary 3.4). □

We note that the relative (p,r)-compactness assumption is not needed at this stage but in the next result.

Proposition 3.1.7. *Suppose that $1 \leq p < \infty$ and $1 \leq r < p^*$. If A is a bounded subset of a Banach space X, then A is relatively (p,r)-compact in X if and only if, the evaluation map $U_A : \ell_1(A) \longrightarrow X$ is (p,r)-compact.*

Proof. Suppose that $A \subset X$ is relatively (p,r)-compact. Then there is a sequence $(x_n) \in \ell_p(X)$ such that $A \subseteq (p,r)\text{-conv}(x_n)$. Since the set $(p,r)\text{-conv}(x_n)$ is an absolutely convex set, it is equal to its own absolute convex hull, and so contains $\overline{\text{abs-conv}}(A)$. We deduce from the second containment in Proposition 3.1.5 that $U_A\big(B_{\ell_1(A)}\big)$ is a relatively (p,r)-compact subset of X, and so, U_A is (p,r)-compact.
Conversely, suppose that U_A is (p,r)-compact. Then there is $(x_n) \in \ell_p(X)$ such that

$$U_A\big(B_{\ell_1(A)}\big) \subseteq (p,r)\text{-conv}(x_n).$$

By the first inclusion in Proposition 3.1.5, it holds that

$$A \subseteq (p,r)\text{-conv}(x_n).$$

This affirms that A is relatively (p,r)-compact. □

It is proved in ([1], Proposition 2.1) that the class of (p,r)-compact operators is an operator ideal as well as in ([1], Proposition 2.2) that is surjective, that is, $\mathcal{K}_{(p,r)} = \mathcal{K}_{(p,r)}^{sur}$. Thus, we have the following proposition:

Proposition 3.1.8. *([2], Theorem 3.2.) Let X and Y be Banach spaces and suppose that $1 \leq p < \infty$ and $1 \leq r \leq p^*$. Then*

$$\mathcal{K}_{(p,r)}(X,Y) = \mathcal{N}_{(p,1,r^*)}^{sur}(X,Y).$$

That is, $\mathcal{K}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}^{sur}$ as operator ideals.

Proof. Suppose $T \in \mathcal{K}_{(p,r)}(X,Y)$. Since $\bar{\Phi}_{(y_n)} \in \mathcal{N}_{(p,1,r^*)}^{sur}(Z,Y)$ where $Z = \ell_r / \ker \Phi_{(y_n)}$ and $(y_n) \in \ell_p(Y)$ by Proposition 3.1.4, it follows that $T \in \mathcal{N}_{(p,1,r^*)}^{sur}(X,Y)$ by the Ideal Property as $T = \bar{\Phi}_{(y_n)} T_{(y_n)}$ where $T_{(y_n)} \in \mathcal{L}(X,Z)$ and $\bar{\Phi}_{(y_n)}$ is the injective associate of $\Phi_{(y_n)}$. This shows that

$$\mathcal{K}_{(p,r)}(X,Y) \subset \mathcal{N}_{(p,1,r^*)}^{sur}(X,Y). \tag{3.4}$$

Conversely, to show that

$$\mathcal{N}_{(p,1,r^*)}^{sur}(X,Y) \subset \mathcal{K}_{(p,r)}(X,Y), \tag{3.5}$$

we prove that

$$\mathcal{N}_{(p,1,r^*)}(X,Y) \subset \mathcal{K}_{(p,r)}(X,Y)$$

since the operator ideal $\mathcal{K}_{(p,r)}$ is surjective by ([2], Proposition 2.2) (of the form $\mathcal{K}_{(p,r)} = \mathcal{K}_{(p,r)}^{sur}$) and $\mathcal{A}^{sur} \subset \mathcal{B}^{sur}$ whenever $\mathcal{A} \subset \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are operator ideals.

To this end, let $T \in \mathcal{N}_{(p,1,r^*)}(X,Y)$. We have

$$T = \sum_{n=1}^{\infty} \delta_n x_n^* \otimes y_n$$

by definition of the $(p,1,r^*)$-nuclear operator where $(\delta_n) \in \ell_p$ (or $(\delta_n) \in c_0$ if $p = \infty$), $(x_n^*) \in \ell_r^w(X^*)$, and $(y_n) \in \ell_\infty^w(Y)$. Without loss of generality, we take $\|(y_n)\| = 1$, and assume $\|(x_n^*)\| = 1$. We then have $(\delta_n y_n) \in \ell_p(Y)$ and

$$Tx = \sum_{n=1}^{\infty} x_n^*(x) \delta_n y_n \in \Phi_{(\delta_n y_n)}(B_{\ell_r})$$

for every $x \in B_X$. This shows that $T \in \mathcal{K}_{(p,r)}(X,Y)$, and we have

$$\mathcal{N}_{(p,1,r^*)}(X,Y) \subset \mathcal{K}_{(p,r)}(X,Y)$$

and this proves (3.5). By (3.4) and (3.5), we get

$$\mathcal{K}_{(p,r)}(X,Y) = \mathcal{N}^{sur}_{(p,1,r^*)}(X,Y).$$

□

Similarly to the operator $\Phi_{(x_n)}$ involving the bounded discrete set $A = \{x_n : n \in \mathbb{N}\}$, the range of a sequence $(x_n) \in \ell_p(X)$, we would like to give a representation of the operator $U_A : \ell_1(A) \longrightarrow X$ akin to equation 3.1, but for any bounded set (albeit on a specific domain since it will play an important role in our discourse) and show as in Proposition 3.1.3 that it is (t,u,v)-nuclear for some real numbers t, u, and v.

Proposition 3.1.9. *Suppose that $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Let A be a bounded subset of X where X is a Banach space. If A is relatively (p,r)-compact, then the operator U_A is $(p,1,r^*)$-nuclear and*

$$\nu_{(p,1,r^*)}(U_A) \leq \|(x_n)\|_p.$$

Proof. Let $A \subseteq X$ be a bounded set and recall the definition of

$$\begin{aligned} U_A : \ell_1(A) &\longrightarrow X \\ (\xi_a)_{a\in A} &\mapsto U_A\big((\xi_a)_{a\in A}\big) = \sum_{a\in A} \xi_a a. \end{aligned}$$

For a fixed $a \in A$, the 'a-unit vector' is the function $e_a : A \longrightarrow \mathbb{K}$ defined by

$$e_a(b) = \delta_{ab},$$

the Kronecker delta. Then each family $(\xi_a)_{a\in A} \in \ell_1(A)$ has the (unique) representation,

$$(\xi_a)_{a\in A} = \sum_{a\in A} \xi_a e_a.$$

Thus, for each $b \in A$, we have

$$\big[(\xi_a)_{a\in A}\big](b) = \sum_{a\in A} \xi_a e_a(b) = \xi_b = \hat{e}_b((\xi_a)_{a\in A}),$$

for some induced evaluation functional $\hat{e}_a$ on $\ell_1(A)$. Hence, fix $\xi = (\xi_a)_{a\in A} \in \ell_1(A)$ and observe that

$$\begin{aligned} U_A(\xi) = U_A\big((\xi_a)_{a\in A}\big) &= \sum_{a\in A} \xi_a a \\ &= \sum_{a\in A} \big((\xi_b)a\big)a \\ &= \sum_{a\in A} \hat{e}_a\big((\xi_b)_{b\in A}\big)a \\ &= \sum_{a\in A} \hat{e}_a(\xi)a. \end{aligned}$$

Thus,

$$U_A(\xi) = \Big[\sum_{a\in A} \hat{e}_a \otimes a\Big](\xi).$$

Since $\xi \in \ell_1(A)$ is arbitrary, it follows that U_A has an expansion (but not unique!) of the form

$$U_A = \sum_{a\in A}[\hat{e}_a \otimes a]. \tag{3.6}$$

To see that this representation is of the advertised kind, suppose that $A \subset X$ is relatively (p, r)-compact. Then there exists $(x_n) \in \ell_p(X)$ such that

$$a = \sum_{n=1}^{\infty} \alpha_n^a x_n \tag{3.7}$$

for some $(\alpha_n^a) \in B_{\ell_r}$ (respectively $(\alpha_n^a) \in B_{c_0}$ if $r = \infty$).

Substitute equation (3.7) into equation (3.6), then we have

$$\begin{aligned} U_A = \sum_{a\in A}\Big(\hat{e}_a \otimes \Big(\sum_{n=1}^{\infty} \alpha_n^a x_n\Big)\Big) &= \sum_{a\in A}\sum_{n=1}^{\infty}\Big(\hat{e}_a \otimes \alpha_n^a x_n\Big) \\ &= \sum_{a\in A}\sum_{n=1}^{\infty}\Big(\alpha_n^a \hat{e}_a \otimes x_n\Big) \\ &= \sum_{a\in A}\Big(\Big(\sum_{n=1}^{\infty} \alpha_n^a \hat{e}_a\Big) \otimes x_n\Big) \\ &= \sum_{n=1}^{\infty}\Big(\Big(\sum_{a\in A} \alpha_n^a \hat{e}_a\Big) \otimes \|x_n\|\big(\|x_n\|^{-1}x_n\big)\Big) \\ &= \sum_{n=1}^{\infty}\|x_n\|\Big(\Big(\sum_{a\in A} \alpha_n^a \hat{e}_a\Big) \otimes \big(\|x_n\|^{-1}x_n\big)\Big) \end{aligned}$$

$$= \sum_{n=1}^{\infty}(\delta_n x_n^* \otimes y_n) \tag{3.8}$$

where $\delta_n := \|x_n\|$, $y_n := \|x_n\|^{-1}x_n$, and $x_n^* := \left(\sum_{a\in A}\alpha_n^a\hat{e}_a\right)$ is convergent in $(\ell_1(A))^*$, since for every $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{a\in A}\left|\alpha_n^a\hat{e}_a((\xi_b))\right| &= \sum_{a\in A}\left|\alpha_n^a(\xi_a)\right| \\ &\leq \sum_{a\in A}\left|(\xi_a)\right| \quad (\text{since } |\alpha_n^a| \leq 1 \text{ for each } n) \\ &= \|(\xi_b)\|_1,\end{aligned}$$

and so, for every $n \in \mathbb{N}$, we have

$$\sum_{a\in A}\left\|\alpha_n^a\hat{e}_a\right\|_\infty \leq 1. \tag{3.9}$$

Since $(x_n) \in \ell_p(X)$ (respectively $(x_n) \in c_0(X)$ if $p = \infty$), it follows that $(\delta_n) \in \ell_p$ (respectively $(\delta_n) \in c_0$ if $p = \infty$) as $\sum \|\delta_n\|^p = \sum \|x_n\|^p < \infty$. Also,

$$\sup\|y_n\| = \sup\|\|x_n\|^{-1}\|x_n\|\| = 1,$$

so that $(y_n) \in \ell_\infty(X) = \ell_\infty^w(X) = \ell_{1^*}^w(X)$.

Next, since $\sum_{a\in A}\alpha_n^a\hat{e}_a \in (\ell_1(A))^*$, we claim that

$$(x_n^*) \in \ell_r^w((\ell_1(A))^*)$$

$\Big($respectively,

$$\left(\sum_{a\in A}\alpha_n^a\hat{e}_a\right)_n \in c_0(A) \subset \ell_\infty(A) = \ell_{1^*}(A) \subset (\ell_1(A))^*$$

and so

$$\left(\sum_{a\in A}\alpha_n^a\hat{e}_a\right)_n \in \ell_\infty((\ell_1(A))^*) = \ell_\infty^w((\ell_1(A))^*) = \ell_{1^*}^w((\ell_1(A))^*)$$

if $r = \infty\Big)$. To this end, fix $\varphi = (\varphi_b)_{b\in A} \in B_{\ell_1(A)}$ and observe that for each $n \in \mathbb{N}$,

$$\begin{aligned}\left|\langle\varphi, \sum_{a\in A}\alpha_n^a\hat{e}_a\rangle\right| &= \left|\sum_{a\in A}\alpha_n^a\hat{e}_a(\varphi)\right| = \left|\sum_{a\in A}\alpha_n^a\varphi_a\right| \\ &\leq \sum_{a\in A}|\alpha_n^a||\varphi_a| \\ &\leq \left(\sup_{a\in A}\left|\alpha_n^a\right|\right)\left(\sum_{a\in A}\left|\varphi_a\right|\right) \quad (\text{Hölder's inequality}) \\ &\leq \sup_{a\in A}\left|\alpha_n^a\right| \quad (\text{since } \varphi \in B_{\ell_1}(A)).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\langle\varphi,\sum_{a\in A}\alpha_n^a\hat{e}_a\rangle\right|^r &\leq \sum_{n=1}^{\infty}\left(\sup_{a\in A}\left|\alpha_n^a\right|^r\right)\\ &= \sup_{a\in A}\sum_{n=1}^{\infty}\left|\alpha_n^a\right|^r\\ &\leq 1,\end{aligned}$$

$\Big($respectively,

$$\begin{aligned}\sup_n\left|\langle\varphi,\sum_{a\in A}\alpha_n^a\hat{e}_a\rangle\right| &\leq \sup_n\sup_{a\in A}\left|\alpha_n^a\right|\\ &= \sup_{a\in A}\sup_n\left|\alpha_n^a\right|\\ &\leq 1.\Big)\end{aligned}$$

Since $\varphi\in B_{\ell_1}(A)$ is arbitrary, it follows that $(x_n^*)\in\ell_r^w(\ell_1(A)^*)$. $\Big[$Respectively, $(\sum_{a\in A}\alpha_n^a\hat{e}_a)_n\in c_0(A)\in\ell_\infty(A)=\ell_{1^*}(A)\subset(\ell_1(A)^*)$ since $\sum_{a\in A}\alpha_n^a\hat{e}_a\to 0$ as $n\to\infty$, or alternatively, since for every $n\in\mathbb{N}$ it holds that $\|\sum_{a\in A}\alpha_n^a\hat{e}_a\|_\infty\leq 1$ by equation (3.9), then if $r=\infty$, $(\sum_{a\in A}\alpha_n^a\hat{e}_a)_n$ clearly defines

$$(x_n^*)_n:=(\sum_{a\in A}\alpha_n^a\hat{e}_a)_n\in\ell_\infty((\ell_1(A))^*)=\ell_\infty^w(\ell_1(A)^*)=\ell_{1^*}^w(\ell_1(A)^*).\Big]$$

Thus, the claim is vindicated. Moreover,

$$\ell_r^w(\ell_1(A)^*)=\ell_{(r^*)^*}^w(\ell_1(A)^*)$$

if $1\leq r\leq\infty$. Thus, we have proved that the representation equation (3.8) defines U_A as a $(p,1,r^*)$-nuclear operator, so that $U_A\in\mathcal{N}_{(p,1,r^*)}(\ell_1(A),X)$. Furthermore,

$$\nu_{(p,1,r^*)}(U_A)\leq\|(x_n)\|_p.$$

□

Remark 3.1.10. *Since $\mathcal{K}_{(p,r)}=\mathcal{N}^{sur}_{(p,1,r^*)}$, the surjective hull of the s-Banach operator ideal $\mathcal{N}_{(p,1,r^*)}$ for some $s\in(0,1]$ by Proposition 3.1.8, it follows from Proposition 3.1.9 that $U_A\in\mathcal{K}_{(p,r)}$ and this reproves one direction of Proposition 3.1.7.*

Proposition 3.1.11. *Suppose that A is a bounded subset of a Banach space X. If A is relatively (p,r)-compact, then the series*

$$\sum_{n=1}^{\infty}\left[\|x_n\|\Big(\sum_{a\in A}\alpha_n^a\hat{e}_a\Big)\otimes\left(\|x_n\|^{-1}x_n\right)\right]=\sum_{n=1}^{\infty}\left[\delta_n x_n^*\otimes y_n\right]$$

in equation (3.8) converges in the operator norm of $\mathcal{L}(\ell_1(A),X)$, and so, the operator U_A is approximable, that is, $U_A\in\overline{\mathcal{F}}(\ell_1,X)$

Proof. Since A is (p,r)-compact, it follows from the proof of Proposition 3.1.8 that U_A has a representation as in (3.8), namely

$$U_A=\sum_{n=1}^{\infty}\left[\delta_n x_n^*\otimes y_n\right]$$

where $(\delta_n)=(\|x_n\|)\in\ell_p$ if $1\leq p<\infty$ (respectively, $(\delta_n)\in c_0$ if $p=\infty$), $(y_n)=(\|x_n\|^{-1}x_n)\in\ell_{1^*}^{weak}(X)$ and

$$(x_n^*)=\Big(\sum_{a\in A}\alpha_n^a\hat{e}_a\Big)\in\ell_{(r^*)^*}^{weak}((\ell_1(A))^*)$$

if $1\leq r<\infty$ (respectively,

$$(x_n^*)=\Big(\sum_{a\in A}\alpha_n^a\hat{e}_a\Big)\in\ell_{1^*}^{weak}((\ell_1(A))^*)$$

if $r=\infty$).

On setting $(x_n)_{n\leq m}:=(x_1,\ldots,x_n,0,0,\ldots)$, we may define

$$U_A^{(m)}=\sum_{n=1}^{m}\left[\delta_n x_n^*\otimes y_n\right].$$

Then,

$$\begin{aligned}\|U_A-U_A^{(m)}\| &= \|\sum_{n=m+1}^{\infty}\delta_n x_n^*\otimes y_n\|=\sup_{(\xi_b)\in B_{\ell_1(A)}}\|\sum_{n=m+1}^{\infty}\delta_n\langle(\xi_b)_{b\in A},x_n^*\rangle y_n\|\\ &= \sup_{(\xi_b)\in B_{\ell_1(A)}}\sum_{n=m+1}^{\infty}\|\delta_n\Big(\sum_{a\in A}\alpha_n^a\hat{e}_a((\xi_b)_{b\in A})\Big)y_n\|\\ &\leq \sup_{(\xi_b)\in B_{\ell_1(A)}}\sum_{n=m+1}^{\infty}|\delta_n|\Big|\sum_{a\in A}\alpha_a^n\xi_a\Big|\|y_n\|\end{aligned}$$

$$
\begin{aligned}
&= \sup_{(\xi_b)\in B_{\ell_1(A)}} \sum_{n=m+1}^{\infty} |\delta_n| \Big| \sum_{a\in A} \alpha_a^n \xi_a \Big| \quad (\text{since } \|y_n\| = 1) \\
&\leq \sup_{(\xi_b)\in B_{\ell_1(A)}} \sum_{n=m+1}^{\infty} |\delta_n| \sum_{a\in A} |\alpha_a^n||\xi_a| \\
&\leq \sup_{(\xi_b)\in B_{\ell_1(A)}} \sum_{n=m+1}^{\infty} |\delta_n| (\sup_{a\in A} |\alpha_a^n|) \sum_{a\in A} |\xi_a| \\
&\qquad (\text{Hölder's inequality over } a \in A.) \\
&= \sup_{a\in A} \sum_{n=m+1}^{\infty} |\delta_n||\alpha_a^n| \\
&\leq \begin{cases} \sup_{a\in A} \big(\sup_{n\geq m+1} |\alpha_a^n|\big)\big(\sum_{n=m+1}^{\infty} |\delta_n|\big), & p = 1, \\ \qquad (\text{Hölder's inequality}) & \\ \sup_{a\in A} \big(\sum_{n=m+1}^{\infty} |\delta_n|^p\big)^{\frac{1}{p}} \big(\sum_{n=m+1}^{\infty} |\alpha_a^n|^{p^*}\big)^{\frac{1}{p^*}}, & 1 < p < \infty,\ 1 \leq r \leq p^* \\ \sup_{a\in A} \big(\sum_{n=m+1}^{\infty} |\alpha_a^n|\big)\big(\sup_{n\geq m+1} |\delta_n|\big), & p = \infty \end{cases} \\
&\leq \begin{cases} \big(\sum_{n=m+1}^{\infty} |\delta_n|^p\big)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \sup_{n\geq m+1} |\delta_n|, & p = \infty \end{cases} \\
&= \begin{cases} \big(\sum_{n=m+1}^{\infty} \|x_n\|^p\big)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \sup_{n\geq m+1} \|x_n\|, & p = \infty \end{cases} \\
&\to 0 \text{ as } m \to \infty.
\end{aligned}
$$

Therefore, (U_A^m) converges to U_A in $\mathfrak{L}(\ell_1(A), X)$ as $m \to \infty$. □

Since $\overline{\mathfrak{F}} \subset \mathfrak{K}$, it follows at once that $U_A \in \mathfrak{K}(\ell_1(A), X)$ whenever A is a relatively (p,r)-compact subset of X, as expected, by Proposition 3.1.9.

3.2 $(p, r^*, 1)$-nuclear evaluation map U_A^*

Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ and $\frac{pr}{p+r} = 1$. In this section, we want to prove that then $U_A^* \in \mathcal{N}_{(p,r^*,1)}^{inj}$ if, and only if, $U_A \in \mathcal{K}_{(p,r)}$.

As commented in the proof of Lemma 2.4.8 in ([38], Lemma 3), the condition $\frac{1}{t} = \frac{1}{u} + \frac{1}{v} - 1 > 0$ is imposed to ensure that $\mathcal{N}_{(t,u,v)}$ is a Banach ideal, otherwise by ([37], Theorem 18.1.2) we are dealing with an s-normed ideal, where $\frac{1}{s} := \frac{1}{t} + \frac{1}{u^*} + \frac{1}{v^*}$. For the $(p, 1, r^*)$-nuclear operators

$\mathcal{N}_{(p,1,r^*)}$, Lemma 2.4.8 is applicable if we set $\frac{pr}{p+r}=1$.

We are now strengthened to settle the proof of the next proposition.

Proposition 3.2.1. *Suppose that $1\le p\le\infty$, $1\le r\le p^*$ and $\frac{pr}{p+r}=1$. Let A be a bounded subset of a Banach space X. Then the operator U_A is (p,r)-compact if, and only if, $U_A^*\in\mathcal{N}^{inj}_{(p,r^*,1)}$. In particular, U_A is p-compact if, and only if, $U_A^*\in\mathcal{N}_p$.*

Proof. Suppose that $U_A:\ell_1(A)\longrightarrow X$ is (p,r)-compact. Then

$$\begin{aligned}U_A\in\mathcal{K}_{(p,r)}(\ell_1(A),X) &= \mathcal{N}^{sur}_{(p,1,r^*)}(\ell_1(A),X) \quad \text{by ([2], Theorem 3.2)}\\ &\subset \left[\mathcal{N}^{reg}_{(p,1,r^*)}\right]^{sur}(\ell_1(A),X) \quad \text{(the rule `reg' is a hull}\\ &\quad\ \text{procedure and the rule 'sur' is a monotone procedure)}\\ &= \left[\mathcal{N}^{dual}_{(p,r^*,1)}\right]^{sur}(\ell_1(A),X) \quad \text{by ([37], Theorem 18.1.6).}\end{aligned}$$

Hence

$$U_A\circ Q_{\ell_1(A)}\in\mathcal{N}^{dual}_{(p,r^*,1)}(\ell_1(B_{\ell_1(A)}),X),$$

where

$$\ell_1(B_{\ell_1(A)})\xrightarrow{Q_{\ell_1(A)}}\ell_1(A)\xrightarrow{U_A}X.$$

Since $J_{\ell_\infty(A)}=(Q_{\ell_1(A)})^*$, it follows that

$$\begin{aligned}J_{\ell_\infty(A)}\circ U_A^*=Q^*_{\ell_1(A)}\circ U_A^* &= (U_A\circ Q_{\ell_1(A)})^*\\ &\in \mathcal{N}_{(p,r^*,1)}(X^*,\ell_\infty(B_{\ell_1(A)})),\end{aligned}$$

whence

$$U_A^*\in\mathcal{N}^{inj}_{(p,r^*,1)}(X^*,\ell_\infty(A)),$$

as was to be proved. (Observe that this direction of the proof is independent of the condition $\frac{pr}{p+r}=1$).

Conversely, suppose that $U_A^*\in\mathcal{N}^{inj}_{(p,r^*,1)}(X^*,\ell_\infty(A))$. Then

$$\begin{aligned}(U_A\circ Q_{\ell_1(A)})^* &= Q^*_{\ell_1(A)}\circ U_A^*\\ &= J_{\ell_\infty(A)}\circ U_A^*\\ &\in \mathcal{N}_{(p,r^*,1)}(X^*,\ell_\infty(B_{\ell_1(A)})).\end{aligned}$$

Therefore,

$$U_A \circ Q_{\ell_1(A)} \in \mathcal{N}^{dual}_{(p,r^*,1)}(\ell_1(B_{\ell_1(A)}), X) = \mathcal{N}^{reg}_{(p,1,r^*)}(\ell_1(B_{\ell_1(A)}), X)$$

by ([37], Theorem 18.1.6). Since $\frac{pr}{p+r} = 1$ and $\ell_\infty(B_{\ell_1(A)}) = (\ell_1(B_{\ell_1(A)}))^*$ has the metric approximation property, Lemma 2.4.8 is applicable, whence it holds that

$$U_A \circ Q_{\ell_1(A)} \in \mathcal{N}_{(p,1,r^*)}(\ell_1(B_{\ell_1(A)}), X).$$

Hence,

$$U_A \in \mathcal{N}^{sur}_{(p,1,r^*)}(\ell_1(A), X) = \mathcal{K}_{(p,r)}(\ell_1(A), X)$$

by ([2], Theorem 3.2), as was to be proved.

In conclusion, putting $r = p^*$ yields the desired particular case since by definition $(\mathcal{N}^p, \nu^p) = (\mathcal{N}_{(p,1,p)}, \nu_{(p,1,p)})$ and $(\mathcal{N}^p, \nu^p)^{sur} = (\mathcal{K}_p, \kappa_p)$ by ([13], Proposition 3.11), and so, this reclaims Proposition 3.1.6 $(b) \Leftrightarrow (c)$ as a special case. This completes the proof. $\square$

Chapter 4

On $\mathcal{C}_p^r$ Property of Banach spaces

In this chapter, we will encapsulate the results of Propositions 3.1.7, 3.1.9 and 3.2.1 (partially, the part that does not use the might of the condition $\frac{pr}{p+r} = 1$ in Lemma 2.4.8) into a Property which a Banach space X may or may not have. This property shall be known as the $\mathcal{C}_p^r$ property of Banach spaces.

In Section 4.2, we prove a characterization that a Banach space Y has the $\mathcal{C}_p^r$ Property precisely when the (p, r)-compact operators from X into Y equals the surjective hull of the dual of the $(p, r^*, 1)$-integral operators from X into Y for every Banach space X (that is, $\mathcal{K}_{(p,r)}(X, Y) = (\mathcal{I}^{dual}_{(p,1,r^*)})^{sur}(X, Y)$). Other results with regard to the $\mathcal{C}_p^r$ Property of Banach spaces were also proved, in particular, in section 4.3, we apply the results in sections 4.1 and 4.2 about the $\mathcal{C}_p^r$ Property and show that if $X^{**} \in \mathcal{C}_p^r$, then $X \in \mathcal{C}_p^r$ for every Banach space X.

4.1 On $\mathcal{C}_p^r$ Property

In this section, we introduce a property that a Banach space may or may not enjoy, and this property shall be called the $\mathcal{C}_p^r$ Property of Banach spaces.

Definition 4.1.1. *Suppose that $1 \leq p < \infty$ and $1 \leq r < p^*$. Define a class $\mathcal{C}_p^r$ of Banach* *spaces X by:*

$$\mathcal{C}_p^r = \{X : \text{a bounded subset } A \subset X \text{ is relatively } (p,r)\text{-compact if and only if } U_A^* \in \mathcal{I}_{(p,r^*,1)}^{inj}(X^*, \ell_\infty(A))\}.$$

A Banach space X is said to have the $\mathcal{C}_p^r$ Property if $X \in \mathcal{C}_p^r$.

The special case $r = p^*$ was defined by Delgado et al in ([11], Definition 2.1), namely;

Definition 4.1.2. *([11], Definition 2.1) Suppose that $1 \leq p < \infty$. Define a class $\mathcal{C}_p^{p^*} = \mathcal{C}_p$ of Banach spaces X by:*

$$\mathcal{C}_p = \{X : \text{a bounded subset } A \subset X \text{ is relatively } p\text{-compact if and only if } U_A^* \in \Pi_p(X^*, \ell_\infty(A))\}.$$

We may add: a Banach space X is said to have the $\mathcal{C}_p$ Property if $X \in \mathcal{C}_p$.

The previous special case arises from the fact that if $1 \leq p < \infty$ and $r = p^*$, then $(\mathfrak{I}_p, \iota_p)^{inj} = (\Pi_p, \pi_p)$ by ([37], Theorem 19.2.7).

The following example should serve to make the format of our definition non-vacuous in the special case $r = p^*$: it is known in this case that the (p, p^*)-compactness coincides with the p-compactness.

Example 4.1.3. *Suppose that X is a Hilbert space and let $p = 2$. Given a bounded subset A of X, A is relatively $(2,2)$-compact if and only if U_A^* is 2-summing.*

Proof. Suppose $A \subset X$ is relatively $(2,2)$-compact. Then there is a 2-summable sequence (x_n) in X with

$$A \subset \{\sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_2}\}.$$

This sequence (x_n) gives rise to a Hilbert-Schmidt operator $\phi : \ell_2 \longrightarrow X : \alpha_n \mapsto x_n$. Hence $A \subset \phi(B_{\ell_2})$. By ([16], Proposition 4.5(c)) it holds that ϕ^* is Hilbert-Schmidt too, hence 2-summing. Hence $\phi^* \in \Pi_2(X^*, \ell_2)$ by ([16], Theorem 4.10). Moreover

$$\begin{aligned} U_A^*(x^*) &= (\langle x^*, a\rangle)_{a\in A} \\ &= (\langle x^*, \phi(\alpha^a)\rangle)_{\alpha^a \in B_{\ell_2}} \quad \text{since } A \subset \phi(B_{\ell_2}). \end{aligned}$$

Hence,

$$\begin{aligned} \|U_A^*(x^*)\|_\infty &= \sup_{a\in A} \big|\langle x^*, a\rangle\big| \\ &= \sup_{\alpha^a \in B_{\ell_2}} \big|\langle x^*, \phi(\alpha^a)\rangle\big| \\ &= \sup_{\alpha^a \in B_{\ell_2}} \big|\langle \phi^* x^*, \alpha^a\rangle\big| \\ &= \|\phi^* x^*\|_2. \end{aligned} \tag{4.1}$$

Since ϕ^* is 2-summing, it follows from equation (4.1) that U_A^* is 2-summing, as was to be shown.

Conversely, suppose that A is a bounded subset of X such that $U_A^* : X^* \longrightarrow \ell_\infty(A)$ is 2-summing. Since X is a Hilbert space, it is reflexive and so it has the Radon-Nikodym Property. By ([11], Proposition 2.2), it holds that $X \in \mathcal{C}_2$. It now follows from Definition 4.1.2 that A is relatively $(2, 2)$-compact. □

4.2 Characterisation of Banach spaces with the $\mathcal{C}_p^r$ Property

Recall that the metric $d : X \times X \longrightarrow \mathbb{R}$ on a normed space X given by $d(x, y) = \|x - y\|$ defines a metric space (X, d). It defines a complete metric space on X when X is a Banach space. If we restrict this metric to the subset A of X then (A, d) is a metric space in its own right. We also recall that a metric space (X, d) is said to be *separable* if it contains a countable dense subset.

In order to provide a characterization of a Banach space X having the $\mathcal{C}_p^r$ Property, we need the following Proposition 4.2.1 and we will also recall the result in ([13], Proposition 2.1).

Proposition 4.2.1. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Let X be a Banach space and assume that A is a bounded subset of X such that $U_A^* \in \mathcal{I}_{(p,r^*,1)}^{inj}(X^*, \ell_\infty(A))$. If each countable subset of A is relatively (p,r)-compact, then A is relatively (p,r)-compact. In particular, if each countable subset of A is relatively p-compact, then A is relatively p-compact.*

Proof. Let A be a bounded subset of X and suppose that S is an arbitrary countable subset of A. Assume that S is relatively (p,r)-compact. [Let (x_n) be an enumeration of S, that is $S = \{x_n | n \in \mathbb{N}\}$, such that for some $(z_k) \in \ell_p(X)$ it holds that $x_n = \sum_{k=1}^{\infty} a_k^n z_k$, where $(a_k^n)_{k=1}^{\infty} \in B_{\ell_r}$, for all $n \in \mathbb{N}$. Then $S \subset \Phi_{(z_k)}(B_{\ell_r})$ (respectively $S \subset \Phi_{(z_k)}(B_{c_0})$ if $r = \infty$)]. We are required to show that A is relatively (p,r)-compact (respectively, (p,∞)-compact), and the relatively p-compact case will follow from letting $r = p^*$.

To this end, recall ([37], Theorem 4.7.16): given an operator ideal $\mathcal{A}$, it holds that $(\mathcal{A}^{dual})^{sur} = (\mathcal{A}^{inj})^{dual}$, hence

$$\begin{aligned} U_A \in (\mathcal{I}_{(p,r^*,1)}^{inj})^{dual}(\ell_1(A), X) &= (\mathcal{I}_{(p,r^*,1)}^{dual})^{sur}(\ell_1(A), X) \\ &= (\mathcal{I}_{(p,1,r^*)})^{sur}(\ell_1(A), X) \\ &= \mathcal{I}_{(p,1,r^*)}(\ell_1(A), X). \end{aligned}$$

Therefore U_A is a compact operator. By Proposition 3.1.5, A is a relatively compact set, and so it is separable by ([18], Theorem I.6.15). Therefore, there exists a countable subset S of A such that $\bar{A} = \bar{S}$, and by the opening paragraph in this proof, A would be relatively (p,r)-compact whenever S is relatively (p,r)-compact. □

Proposition 4.2.2. *([13], Proposition 2.1) Suppose that $1 \leq p < \infty$ and let X be a Banach space. Then the following statements are equivalent:*

(a) $X \in \mathcal{C}_p^{p^*} = \mathcal{C}_p$.

(b) For every Banach space Y, it holds that $\mathcal{K}_p(Y,X) = \Pi_p^d(Y,X)$.

(c) $\mathcal{K}_p(\ell_1(\Gamma), X) = \Pi_p^d(\ell_1(\Gamma), X)$, where Γ is any set .

(d) $\mathcal{K}_p(\ell_1, X) = \Pi_p^d(\ell_1, X)$.

Proof. Let Y be any given Banach space.
(a)$\Rightarrow$ (b). Consider $T \in \Pi_p^d(Y, X)$ and put $A = T(B_Y)$. Since

$$\begin{aligned} \|U_A^* x^*\|_\infty &= \sup_{a \in A} |\langle x^*, a\rangle| = \sup_{y \in B_Y} \big\{|\langle x^*, Ty\rangle| : a = Ty\big\} \\ &= \sup_{y \in Y} \big\{|\langle x^*, Ty\rangle| : \|y\| \leq 1\big\} = \|T^* x^*\|, \end{aligned}$$

it follows that U_A^* is p-summing, and so by hypothesis, $A = T(B_Y)$ is relatively p-compact. Therefore, $T \in \mathcal{K}_p(Y, X)$. Since $\mathcal{K}_p(Y, X) \subset \Pi_p^d(Y, X)$ by ([43], Proposition 5.3), it follows that $\mathcal{K}_p(Y, X) = \Pi_p^d(Y, X)$ as desired.

(b)$\Rightarrow$ (c). Take $Y = \ell_1(\Gamma)$, where Γ is any set in (b).

(c)$\Rightarrow$ (d). Let $\Gamma = \mathbb{N}$ in (c).

(d)$\Rightarrow$ (a). Suppose that $A \subset X$ is a bounded set such that U_A^* is p-summing. To see that A is relatively p-compact, it suffices to show that each countable subset of A is relatively p-compact by Proposition 4.2.1. Consider $\{x_n\} \subset A$ and define

$$J : (\alpha_n) \in \ell_1 \mapsto J(\alpha_n) \in \ell_1(A)$$

where

$$J(\alpha_n)(x) = \begin{cases} \alpha_n & \text{if } x = x_n \\ 0 & \text{otherwise} \end{cases}$$

From (d), $\mathcal{K}_p(\ell_1, X) = \Pi_p^d(\ell_1, X)$, it follows that $U_A \circ J$ is p-compact. Thus $\{x_n\} = \{U_A \circ J(e_n)\}$ is relatively p-compact $(e_n \in B_{\ell_1})$. $\square$

The Banach spaces which have the $\mathcal{C}_p$ Property have been characterized in Proposition 4.2.2. Theorem 4.2.3 and Corollary 4.2.5 below will improve Proposition 4.2.2 by providing a generalized characterization of Banach spaces which enjoy the $\mathcal{C}_p^r$ Property.

Theorem 4.2.3. *Suppose that $1 \leq p \leq \infty$ and $1 \leq r < p^*$. Then the following two statements are equivalent:*

(1) $Y \in \mathcal{C}_p^r$.

(2) $\mathcal{K}_{(p,r)}(X, Y) = (\mathcal{I}^{dual}_{(p,r^*,1)})^{sur}(X, Y)$ *for every Banach space* X.

Proof.
(1)$\Rightarrow$ (2) Assume that $Y \in \mathcal{C}_p^r$ and suppose that $T \in \mathcal{K}_{(p,r)}(X, Y)$. Then $T(B_X) \subset Y$ is relatively (p, r)-compact in Y. By (1), it holds that

$$U^*_{T(B_X)} : Y^* \longrightarrow \ell_\infty(T(B_X))$$

defined by $U^*_{T(B_X)}(y^*) = (\langle y^*, a\rangle)_{a \in T(B_X)}$ for every $y^* \in Y^*$ belongs to $\mathcal{I}^{inj}_{(p,r^*,1)}(Y^*, \ell_\infty(T(B_X)))$. Consider $\ell_\infty(B_X \cup T(B_X))$ and observe that each $f \in \ell_\infty(T(B_X))$ is the restriction to $T(B_X)$ of some function $F : (B_X \cup T(B_X)) \longrightarrow \mathbb{K}$, where $\mathbb{K}$ is the scalar field, defined by

$$F(z) = \begin{cases} 0, & z \in B_X \\ f(z), & z \in T(B_X). \end{cases}$$

Hence, $F \in \ell_\infty(B_X \cup T(B_X))$. The mapping $\ell_\infty(T(B_X)) \hookrightarrow \ell_\infty(B_X \cup T(B_X)) : f \mapsto F$ clearly defines an inclusion map of $\ell_\infty(T(B_X))$ into $\ell_\infty(B_X \cup T(B_X))$. It is clearly well-defined, since if F_0 is another function, $F_0 \in \ell_\infty(B_X \cup T(B_X))$ such that $f = F_0|_{(T(B_X))}$, $F_0(B_X) = 0$ and $f \mapsto F_0$, then $F = F_0$. In the same spirit, for each $g \in \ell_\infty(B_X)$, $\ell_\infty(B_X) \hookrightarrow \ell_\infty(B_X \cup T(B_X)) : g \mapsto G$ is a well-defined inclusion map where $g = G|_{B_X}$ for some $G \in \ell_\infty(B_X \cup T(B_X))$ defined by

$$G(w) = \begin{cases} g(w), & w \in B_X \\ 0, & w \in T(B_X). \end{cases}$$

Furthermore, when $U^*_{T(B_X)}$ and $U^*_{B_X}$ are considered as mappings into $\ell_\infty(B_X \cup T(B_X))$, it holds that for every $y^* \in Y^*$,

$$\begin{aligned} U^*_{T(B_X)}(y^*) &= (\langle y^*, a\rangle) \in \ell_\infty(B_X \cup T(B_X)) \quad (a \in T(B_X) \Rightarrow a = Tx, \text{ for some } x \in B_X) \\ &= (\langle y^*, Tx\rangle)_{x\in B_X} \in \ell_\infty(B_X \cup T(B_X)) \\ &= (\langle T^*y^*, x\rangle)_{x\in B_X} \in \ell_\infty(B_X \cup T(B_X)) \\ &= U^*_{B_X}(T^*y^*) \\ &= U^*_{B_X} \circ T^*(y^*). \end{aligned}$$

Hence, $U^*_{T(B_X)} = U^*_{B_X} \circ T^*$ as mappings into $\ell_\infty(B_X \cup T(B_X))$. Since $U^*_{T(B_X)} \in \mathcal{I}^{inj}_{(p,r^*,1)}(Y^*, \ell_\infty(T(B_X))$, it follows that so is $U^*_{B_X} \circ T^*$, and hence $T^* \in (\mathcal{I}^{inj}_{(p,r^*,1)})^{inj}(Y^*, X^*) = \mathcal{I}^{inj}_{(p,r^*,1)}(Y^*, X^*)$ since the rule 'inj' is idempotent. Therefore, $T \in (\mathcal{I}^{inj}_{(p,r^*,1)})^{dual}(X, Y) = (\mathcal{I}^{dual}_{(p,r^*,1)})^{sur}(X, Y)$.
Conversely, suppose that $T \in (\mathcal{I}^{dual}_{(p,r^*,1)})^{sur}(X, Y)$. Then $T^* \in \mathcal{I}^{inj}_{(p,r^*,1)}(Y^*, X^*)$. Hence, $U^*_{T(B_X)} = U^*_{B_X} \circ T^* \in \mathcal{I}^{inj}_{(p,r^*,1)}(Y^*, \ell_\infty(B_X \cup T(B_X)))$. Therefore, $U^*_{T(B_X)} \in \mathcal{I}^{inj}_{(p,r^*,1)}(Y^*, \ell_\infty(T(B_X)))$. Since $Y \in \mathcal{C}_p^r$ by (1), it follows from Definition 4.1.1 that $T(B_X)$ is relatively (p, r)-compact. Hence, $T \in \mathcal{K}_{(p,r)}(X, Y)$. This completes the proof that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) Fix a Banach space Y and suppose that $\mathcal{K}_{(p,r)}(X, Y) = (\mathcal{I}^{dual}_{(p,r^*,1)})^{sur}(X, Y)$ for every Banach space X. If A is a relatively (p, r)-compact subset of Y, then $U_A : \ell_1(A) \longrightarrow Y$ is (p, r)-compact by Proposition 3.1.7. Hence, $U_A \in (\mathcal{I}^{dual}_{(p,r^*,1)})^{sur}(\ell_1(A), Y)$. Thus,

$$U_A \circ Q_{B_{\ell_1(A)}} \in (\mathcal{I}^{dual}_{(p,r^*,1)})(\ell_1(B_{\ell_1}(A)), Y),$$

and so

$$\begin{aligned} J_{B_{\ell_1(A)}} \circ U^*_A &= Q^*_{B_{\ell_1(A)}} \circ U^*_A \\ &= (U_A \circ Q_{B_{\ell_1(A)}})^* \in \mathcal{I}_{(p,r^*,1)}(Y^*, \ell_\infty(B_{\ell_1(A)})). \end{aligned}$$

Hence, $U^*_A \in \mathcal{I}^{inj}_{(p,r^*,1)}(Y^*, \ell_\infty(A))$.
On the other hand, let A be a bounded subset of Y and $U^*_A \in \mathcal{I}^{inj}_{(p,r^*,1)}(Y^*, \ell_\infty(A))$. Then $U_A \in (\mathcal{I}^{inj}_{(p,r^*,1)})^{dual}(\ell_1(A), Y) = (\mathcal{I}^{dual}_{(p,r^*,1)})^{sur}(\ell_1(A), Y)$. Hence, U_A is (p, r)-compact by (2),

whence it follows that A is relatively (p,r)-compact in Y by Proposition 3.1.7. Therefore, $Y \in \mathcal{C}_p^r$ by Definition 4.1.1. This concludes that (2) $\Rightarrow$ (1). $\square$

Remark 4.2.4. *(1) In the particular case when $r = p^*$ in Theorem 4.2.3, it holds that $Y \in \mathcal{C}_p = \mathcal{C}_p^{p^*}$ if and only if for every Banach space X,*

$$\begin{aligned}
\mathcal{K}_p(X,Y) &= (\mathcal{I}_{(p,p,1)}^{dual})^{sur}(X,Y) \\
&= (\mathcal{I}_{(p,1,p)})^{sur}(X,Y) \\
&= (\mathcal{N}_{(p,1,p)}^{reg})^{sur}(X,Y) \\
&= (\mathcal{N}_{(p,p,1)}^{dual})^{sur}(X,Y) \\
&= (\mathcal{N}_{(p,p,1)}^{inj})^{dual}(X,Y) \\
&= (\mathcal{QN}_p)^{dual}(X,Y),
\end{aligned}$$

an expected special case of ([13], Theorem 3.8). Or starting with ([11], Proposition 2.1), we have that $Y \in \mathcal{C}_p = \mathcal{C}_p^{p^*}$ if and only if for every Banach space X,

$$\begin{aligned}
\mathfrak{K}_p(X,Y) &= \Pi_p^{dual}(X,Y) \\
&= (\mathcal{I}_p^{inj})^{dual}(X,Y) \\
&= (\mathcal{I}_p^{dual})^{sur}(X,Y) \\
&= (\mathcal{I}_{(p,p,1)}^{dual})^{sur}(X,Y).
\end{aligned}$$

(2) In general, under the equivalence with $\mathcal{C}_p^r$, we get $Y \in \mathcal{C}_p^r$ if and only if $\mathcal{K}_{(p,r)}(X,Y) = \left(\mathcal{N}_{(p,r^*,1)}^{inj}\right)^{dual}(X,Y)$ for every Banach space X and so using $\mathcal{J}$ instead of $\mathcal{N}$ in Theorem 4.2.3 is an improvement and clarifies why the characterization of Banach spaces with the $\mathcal{C}_p^r$ Property is not a restatement of ([1], Theorem 4.1) (see also ([2], Theorem 3.2)).

Corollary 4.2.5. *Suppose that $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Then the following statements are equivalent:*

(1) $X \in \mathcal{C}_p^r$.

(2) $\mathcal{K}_{(p,r)}(\ell_1(\Gamma), X) = (\mathcal{I}^{dual}_{(p,r^*,1)})^{sur}(\ell_1(\Gamma), X)$ *for any set* Γ.

(3) $\mathcal{K}_{(p,r)}(\ell_1, X) = (\mathcal{I}^{dual}_{(p,r^*,1)})^{sur}(\ell_1, X)$.

Proof.
(1) $\Rightarrow$ (2) This follows from Theorem 4.2.3 ((1) $\Rightarrow$ (2)) with $X = \ell_1(\Gamma)$ and $Y = X$.

(2) $\Rightarrow$ (3) This follows by replacing Γ with $\mathbb{N}$ in the previous case.

(3) $\Rightarrow$ (1) Suppose that $A \subset X$ is a bounded set such that $U_A^* \in \mathcal{I}^{inj}_{(p,r^*,1)}(X^*, \ell_\infty(A))$. To realize that the subset A of X is relatively (p,r)-compact, it is sufficient to prove that each countable subset of A is relatively (p,r)-compact by Proposition 4.2.1. To this end, consider a subset $\{x_n\}$ of A and define $J : \ell_1 \longrightarrow \ell_1(A) : (\alpha_n) \mapsto J(\alpha_n)$, where

$$J(\alpha_n)(x) = \begin{cases} \alpha_n, & \text{if } x = x_n \\ 0, & \text{otherwise.} \end{cases}$$

Consider

$$U_A \circ J : \ell_1 \longrightarrow X.$$

Then $(U_A \circ J)^* = J^* \circ U_A^*$, where

$$J^* \circ U_A^* : X^* \longrightarrow \ell_\infty$$

belongs to $\mathcal{I}^{inj}_{(p,r^*,1)}(X^*, \ell_\infty)$ by the ideal property. Hence,

$$\begin{aligned} U_A \circ J \in (\mathcal{I}^{inj}_{(p,r^*,1)})^{dual}(\ell_1, X) &= (\mathcal{I}^{dual}_{(p,r^*,1)})^{sur}(\ell_1, X) \\ &= \mathcal{K}_{(p,r)} \text{ (by condition (3)).} \end{aligned}$$

Therefore, $U_A \circ J$ is (p,r)-compact from ℓ_1 to X. Since

$$J(e_n)(x) = \begin{cases} 1, & \text{if } x = x_n \\ 0, & \text{otherwise.} \end{cases} = e_{x_n}(x)$$

for every $x \in A$, it follows that $J(e_n) = e_{x_n}$, and so,

$$(U_A \circ J)(e_n) = U_A(e_{x_n}) = \sum_{x \in A} e_{x_n}(x) x = x_n.$$

Since $\{e_n : n \in \mathbb{N}\} \subset B_{\ell_1}$, it follows that $\{x_n\} = \{U_A \circ J(e_n)\}$ is a relatively (p,r)-compact set.

On the other hand, if $A \subset X$ is relatively (p,r)-compact, then it follows from Proposition 3.1.7 that $U_A \in \mathcal{K}_{(p,r)}(\ell_1(A), X)$. Consider, for a countable subset $\{x_n\}$ of A as before, the operator $J : \ell_1 \longrightarrow \ell_1(A)$ defined in the foregoing paragraph. Then $U_A \circ J \in \mathcal{K}_{(p,r)}(\ell_1, X)$ by the ideal property. Hence, $U_A \circ J \in \left(\mathcal{I}^{inj}_{(p,r^*,1)}\right)^{dual}(\ell_1, X)$ by condition (3), and so $J^* \circ U_A^* = (U_A \circ J)^* \in \mathcal{I}^{inj}_{(p,r^*,1)}(X^*, \ell_\infty)$. Thus, $U_A^* \in \left(\mathcal{I}^{inj}_{(p,r^*,1)}\right)^{inj}(X^*, \ell_\infty(A)) = (\mathcal{I}^{inj}_{(p,r^*,1)})(X^*, \ell_\infty(A))$ since the rule 'inj' is idempotent. Therefore, $X \in \mathcal{C}_p^r$ and so condition (1) holds. This concludes the proof. □

If $0 < t \leq \infty$, and $1 \leq u, v \leq \infty$, we have

$$\begin{aligned} \mathcal{N}^{inj}_{(t,u,v)} &\subseteq (\mathcal{N}^{inj}_{(t,u,v)})^{max} \\ &= (\mathcal{N}^{max}_{(t,u,v)})^{inj} \\ &= \mathcal{I}^{inj}_{(t,u,v)}. \end{aligned} \tag{4.2}$$

In particular, when $v = 1$ it follows from Propositions 3.1.7 and 3.2.1 that the following result holds.

Corollary 4.2.6. *Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. If X is a Banach space, then a bounded set A in X is a relatively (p,r)-compact set if and only if $X \in \mathcal{C}_p^r$.*

Proof. The necessity is the only direction that needs proof. By Propositions 3.1.7 and 3.2.1, it holds that $U_A^* \in \mathcal{N}^{inj}_{(p,r^*,1)}(X^*, \ell_\infty(A))$, whence by equation (4.2) it follows that $U_A^* \in \mathcal{I}^{inj}_{(p,r^*,1)}(X^*, \ell_\infty(A))$. □

4.3 An application related to the $\mathcal{C}_p^r$ Property

In this section, we shall apply the results in the previous two sections about the $\mathcal{C}_p^r$ Property and show that if $X^{**} \in \mathcal{C}_p^r$, then $X \in \mathcal{C}_p^r$. We first need the following result.

Theorem 4.3.1. *Suppose that $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ and $\frac{pr}{p+r} = 1$. If a subset A of a Banach space X is relatively (p,r)-compact in X^{**}, then A is relatively (p,r)-compact in X. In particular, an operator $T \in \mathcal{L}(X,Y)$ is (p,r)-compact if and only if $T^{**} \in \mathcal{L}(X^{**},Y^{**})$ is (p,r)-compact.*

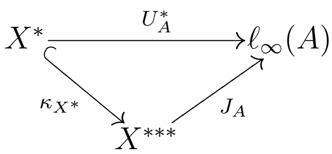
Proof. By considering A as a subset of X^{**}, Proposition 3.1.7 and Proposition 3.2.1 assure us that the map

$$\begin{aligned} J_A : X^{***} &\longrightarrow \ell_\infty(A) \\ x^{***} &\longmapsto (\langle a, x^{***}\rangle)_{a\in A}. \end{aligned}$$

belongs to $\mathcal{N}^{inj}_{(p,r^*,1)}(X^{***}, \ell_\infty(A))$. Hence, so does

$$\begin{aligned} U_A^* = J_A|_{X^*} : X^* &\longrightarrow \ell_\infty(A) \\ x^* &\longmapsto (\langle a, x^*\rangle)_{a\in A}. \end{aligned}$$

by the ideal property, since A is a subset of X; just look at the following commutative diagram: $U_A^* = J_A \circ \kappa_{X^*} \in \mathcal{N}^{inj}_{(p,r^*,1)}(X^*, \ell_\infty(A))$



Therefore, by Propositions 3.2.1 and 3.1.7 (again, and in this order!), $A \subset X$ is relatively (p,r)-compact.

Next, we prove that for any Banach spaces X and Y, it holds that $T \in \mathcal{K}_{(p,r)}(X,Y)$ if and only $T^{**} \in \mathcal{K}_{(p,r)}(X^{**},Y^{**})$. To this end, suppose that T^{**} is (p,r)-compact. Then T^{**} is a *posteriori* weakly compact; hence it follows from Gantmacher's Theorem that so is T by Proposition 1.2.2, and so, $T^{**}(X^{**}) \subset Y$. Moreover, $T^{**}|_X = T$. Since $T^{**}(B_{X^{**}})$ is a relatively (p,r)-compact subset of Y^{**}, it follows that $T(B_X) = T^{**}|_X(B_X)$ is a relatively (p,r)-compact subset of Y^{**} too when B_X is considered as a subset of $B_{X^{**}}$. Since $T(B_X)$ is a subset of Y, it follows from the first part proved above that $T(B_X)$ is a relatively (p,r)-compact subset of Y. Therefore, T is (p,r)-compact too.

On the other hand, suppose that T is (p,r)-compact. Then $T(B_X)$ is a relatively (p,r)-compact subset of Y. Moreover, a *posteriori* weakly compact, so that $T^{**}(X^{**}) \subset Y$ by Gantmacher's Theorem (see the foregoing paragraph for reference) and $T^{**}|_X = T$. Let x_0^{**} be an arbitrary element of $B_{X^{**}}$. Then, by the weak*-density of B_X in $B_{X^{**}}$, there exists a net $(x_\delta^0) \subset B_X$ such that

$$x_0^{**} = \sigma(X^{**}, X^*) - \lim_\delta x_\delta^0.$$

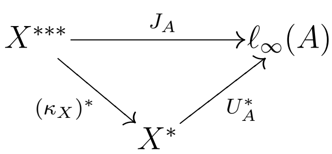
Since T^{**} is $\sigma(X^{**}, X^*) - \sigma(Y, Y^*)$-continuous, it follows that

$$\begin{aligned} T^{**}(x_0^{**}) &= \sigma(Y, Y^*) - \lim_\delta T^{**} x_\delta^0 \\ &= \sigma(Y, Y^*) - \lim_\delta T x_\delta^0 \end{aligned}$$

and so $T^{**}(x_0^{**}) \in \overline{(p,r)\text{-conv}(y_n)}^{\sigma(Y,Y^*)}$ for some sequence $(y_n) \in \ell_p(Y)$ such that $T(B_X) \subset (p,r)\text{-conv}(y_n)$. Since $(p,r)\text{-conv}(y_n)$ is convex, its weak and norm closures coincide, thanks to Mazur's Theorem. Moreover, $(p,r)\text{-conv}(y_n)$ is weakly compact, and so is norm-closed if $1 < p \leq \infty$. Thus, it follows that $T^{**}(x_0^{**}) \in (p,r)$ - $\text{conv}(y_n)$ if $1 < p \leq \infty$, respectively, $T^{**}(x_0^{**}) \in \overline{1 \text{ - conv}(y_n)}^{\|\cdot\|}$: here $1\text{-conv}(y_n) := (1,\infty)\text{-conv}(y_n) = \Phi_{(y_n)}(B_{c_0})$, where $(y_n) \in \ell_1(Y)$. Since $x_0^{**} \in B_{X^{**}}$ is arbitrary, it follows that $T^{**}(B_{X^{**}}) \subset (p,r)$- $\text{conv}(y_n)$ if $1 < p \leq \infty$, respectively, $T^{**}(B_{X^{**}}) \subset \overline{1 \text{ - conv}(y_n)}^{\|\cdot\|} = \overline{\{\sum_n \alpha_n y_n : (\alpha) \in B_{c_0}\}}^{\|\cdot\|} = \{\sum_n \alpha_n y_n : (\alpha) \in B_{\ell_\infty}\} = \Phi_{(y_n)}(B_{\ell_\infty})$, a closed and weakly compact set (see [13], Remark 3.3) about the fact that both definitions of the 1-convex hulls yield the same notions of 1-compactness. Therefore, we may conclude that T^{**} is (p,r)-compact too by ([1], Definition 3.7), as was to be proved. □

Proposition 4.3.2. *Suppose that* $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ *and* $\frac{pr}{p+r} = 1$. *If* $X^{**} \in \mathcal{C}_p^r$, *then* $X \in \mathcal{C}_p^r$.

Proof. Let A be a bounded subset of X and suppose $U_A^* \in \mathcal{I}_{(p,r^*,1)}^{inj}(X^*, \ell_\infty(A))$. To show that A is relatively (p,r)-compact in X, it suffices to show that $\kappa_X(A)$ is relatively (p,r)-compact in X^{**} by Theorem 4.3.1. To this end, consider the following commutative diagram;



where $\kappa_X : X \longrightarrow X^{**}$ is the canonical embedding, $U_A : \ell_1(A) \longrightarrow X$, and

$$\begin{aligned} J_A = U_A^* \circ (\kappa_X)^* : X^{***} &\longrightarrow \ell_\infty(A) \\ x^{***} &\longmapsto (\langle \kappa_X(a), x^{***} \rangle)_{a \in A}. \end{aligned}$$

Therefore,

$$J_A(x^{***}) = (\langle \kappa_X(a), x^{***} \rangle)_{\kappa_X(a) \in \kappa_X(A)} = U_{\kappa_X(A)}^*(x^{***})$$

for every $x^{***} \in X^{***}$ so that $J_A = U_{\kappa_X(A)}^*$. Since A is a subset of X and $U_A^* \in \mathcal{I}_{(p,r^*,1)}^{inj}(X^*, \ell_\infty(A))$, it follows from the ideal property that $J_A = U_A^* \circ (\kappa_X)^* \in \mathcal{I}_{(p,r^*,1)}^{inj}(X^{***}, \ell_\infty(A))$. Hence, $U_{\kappa_X(A)}^* \in \mathcal{I}_{(p,r^*,1)}^{inj}(X^{***}, \ell_\infty(A))$. Since $X^{**} \in \mathcal{C}_p^r$ by hypothesis, it follows that $\kappa_{X^*}(A)$ is relatively (p, r)-compact in X^{**}, as desired.

On the other hand, that the relative (p, r)-compactness of $A \subset X$ implies that

$$U_A^* \in \mathcal{I}_{(p,r^*,1)}^{inj}(X^*, \ell_\infty(A))$$

follows from Propositions 3.1.7 and 3.2.1 and the fact that the maximum hull is a hull procedure. This concludes the proof that $X \in \mathcal{C}_p^r$. □

Chapter 5

On mid (p, r)-compact operators

In this chapter, we study the notion of mid (p, r)-compact sets and operators for $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. We begin by introducing and defining the mid (p, r)-compact subsets of a Banach space X and the mid (p, r)-compact operators between Banach spaces X and Y. We will also introduce and study the set $\mathcal{K}^{\text{mid}}_{(p,r)}(X, Y)$ of mid (p, r)-compact operators between Banach spaces X and Y. We prove that the ideal $(\mathcal{K}^{\text{mid}}_{(p,r)}(X, Y), \kappa^{\text{mid}}_{(p,r)}(\cdot))$ is a quasi-Banach operator ideal.

In section 5.2, we introduce and define the concept of (p, r)-limited sets. We prove that every mid (p, r)-compact subset of a Banach space is (p, r)-limited. Other results with regard to the mid (p, r)-compact sets and operators and the (p, r)-limited will also be proved.

5.1 Mid (p, r)-compact operators

Let $1 \leq p < \infty$. Given a Banach space X, recall the Banach spaces $\ell_p(X)$ and $\ell_p^w(X)$ of, respectively, absolutely and weakly p-summable X-valued sequences equipped with their usual norms. Consider the vector space

$$\ell_p^{\text{mid}}(X) := \{(x_j)_{j=1}^\infty \in \ell_p^w(X) | ((x_n^*(x_j))_{j=1}^\infty)_{n=1}^\infty \in \ell_p(\ell_p) \text{ whenever } (x_n^*)_{n=1}^\infty \in \ell_p^w(X^*)\}$$

under pointwise operations, equipped with the norm

$$\|(x_j)_{j=1}^{\infty}\|_{mid,p} := \sup_{(x_n^*)_{n=1}^{\infty}\in B_{\ell_p^w(X^*)}} \Big\{ \Big(\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} |x_n^*(x_j)|^p \Big)^{1/p} \Big\},$$

under which it is a Banach space [5].

If $p=\infty$, then ℓ_p, ℓ_p^w and ℓ_p^{mid} are replaced by c_0, c_0^w and c_0^{mid}, and $\ell_p(\ell_p)$ takes the form $c_0(c_0)$. The norm on $c_0^{mid}(X)$ takes the form

$$\|(x_j)_{j=1}^{\infty}\|_{mid,\infty} \quad := \sup_{(x_n^*)_{n=1}^{\infty}\in B_{c_0^w(X^*)}} \Big\{ \sup_n \sup_j |x_n^*(x_j)| \Big\}. \tag{5.1}$$

The norm that the mid p-summable sequence space $\ell_p^{mid}(X)$ inherits from $\ell_p^w(X)$ does not, in general, complete the mid p-summable sequence space, and the above norm is the appropriate one on $\ell_p^{mid}(X)$.

The symbol $X \overset{1}{\hookrightarrow} Y$ means that X is a linear subspace of Y and $\|x\|_Y \le \|x\|_X$ for every $x \in X$. Moreover, it is shown in [5] that

$$\ell_p(X) \overset{1}{\hookrightarrow} \ell_p^{mid}(X) \overset{1}{\hookrightarrow} \ell_p^w(X). \tag{5.2}$$

The definition of the (p,r)-convex hull of $(x_n)\in \ell_p^w(X)$ is extended in the following way (see [1], Definition 7.3).

Definition 5.1.1. *Let $1\le p\le\infty$ and $1\le r\le p^*$. Let $(x_n)\in\ell_p^w(X)$. If $r\ne\infty$, the (p,r)-convex hull of (x_n) is given by*

$$(p,r)\text{-conv}(x_n) = \Big\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_r} \Big\}.$$

If $r=\infty$ and $(x_n)\in\ell_1^u(X)$, the sequence space of unconditionally 1-summable sequences, we define

$$1\text{-conv}(x_n) := (1,\infty)\text{-conv}(x_n) = \Big\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_\infty} \Big\}.$$

If $r=\infty$ and $(x_n)\in\ell_1^w(X)$, then define

$$1\text{-co}(x_n) := (1,\infty)\text{-co}(x_n) = \Big\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{c_0} \Big\}.$$

Given $(x_n) \in \ell_p^w(X)$, the operator $\Phi_{(x_n)} : \ell_r \longrightarrow X$ ($\Phi_{(x_n)} : c_0 \longrightarrow X$ when $r = \infty$) is also well defined and $(p,r)\text{-conv}(x_n) = \Phi_{(x_n)}(B_{\ell_r})$ if $r \neq \infty$, while

$$1\text{-co}(x_n) = (1,\infty)\text{-co}(x_n) = \Phi_{(x_n)}(B_{c_0}).$$

However, $\Phi_{(x_n)}$ need not be compact, unless $(x_n) \in \ell_p^u(X)$, the Banach space of unconditionally p-summable sequences ([9], §§8.2 and 8.3) and ([20], Theorem 1.4) (also see ([1], §7.3). As described in ([1], p.60), the space $\ell_p^u(X)$ was defined as the (closed) subspace of $\ell_p^w(X)$, consisting of $(x_n) \in \ell_p^w(X)$ such that $(x_n) = \lim_{N\to\infty}(x_1, \ldots, x_N, 0, 0, \ldots,) \in \ell_p^w(X)$. It is known that $\ell_\infty^u(X) = c_0(X)$ as Banach spaces (details may be found in [20], p.351) and ([9], 8.2).

We recall the following definition from ([1], Definition 7.4).

Definition 5.1.2. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. A subset K of a Banach space X is said to be relatively unconditionally (p,r)-compact if $K \subset (p,r)\text{-conv}(x_n)$ for some $(x_n) \in \ell_p^u(X)$. It is said that K is relatively weakly (p,r)-compact if $K \subset (p,r)\text{-conv}(x_n)$ ($K \subset 1\text{-co}(x_n)$ when $r = \infty$) for some $(x_n) \in \ell_p^w(X)$ ($(x_n) \in c_0^w(X)$ when $p = \infty$).*

Concerning the operators, we obtain the following ([1], Definition 7.5) .

Definition 5.1.3. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Given Banach spaces X and Y, an operator $T \in \mathcal{L}(X,Y)$ is said to be unconditionally (respectively, weakly) (p,r)-compact if $T(B_X)$ is a relatively unconditionally (respectively, weakly) (p,r)-compact subset of Y. Denote by $\mathcal{U}_{(p,r)}$ and $\mathcal{W}_{(p,r)}$ the classes of all unconditionally and weakly (p,r)-compact operators acting between arbitrary Banach spaces.*

By ([5], Example 1.7), the Banach spaces $\ell_p^u(X)$ and $\ell_p^{\text{mid}}(X)$ are generally not comparable. In particular, $\ell_p^{\text{mid}}(X)$ is not contained in $\ell_p^u(X)$ despite the fact that both sequence spaces satisfy

$$\ell_p(X) \subset \ell_p^u(X), \ell_p^{\text{mid}}(X) \subset \ell_p^w(X).$$

Our main concern in this section is to expound a theory on *mid (p,r)-compact sets and operators*. This we do by defining and introducing the *relatively mid (p,r)-compact sets* in X by defining the (p,r)-convex hull of $(x_n) \in \ell_p^{mid}(X)$ as in Definition 5.1.1 above. Alternatively, by replacing $\ell_p(X)$ (as in ([1], Definition 3.2)) with $\ell_p^{\text{mid}}(X)$.

Definition 5.1.4. *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ and $(x_j) \in \ell_p^{\text{mid}}(X)$. Then call*

$$(p,r)\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_r} \right\}$$

the (p,r)-convex hull of (x_n).

The operator $\Phi_{(x_j)} : \ell_r \longrightarrow X$ $(\Phi_{(x_j)}) : c_0 \longrightarrow X$ when $r = \infty)$ defined by

$$\Phi_{(x_j)}(\alpha_j) = \sum_{j=1}^{\infty} \alpha_j x_j$$

is bounded. Let $(x_n) \in \ell_p^{\text{mid}}(X)$. Fix $\varphi \in B_{X^*}$ arbitrarily and note that $(x_n^*) := (\varphi, 0, 0, \ldots) \in B_{\ell_p^w(X^*)}$. So if $(\alpha_j) \in B_{\ell_r}$, then

$$\begin{aligned}
|\varphi(\Phi_{(x_j)}(\alpha_j))|^p &= |\sum_{j=1}^{\infty} \alpha_j \varphi(x_j)|^p \\
&\leq (\sum_{j=1}^{\infty} |\alpha_j||\varphi(x_j)|)^p \\
&\leq \Big((\sum_{j=1}^{\infty} |\alpha_j|^r)^{1/r} (\sum_{j=1}^{\infty} |\varphi(x_j)|^{r^*})^{1/r^*} \Big)^p \\
&\leq \Big((\sum_{j=1}^{\infty} |\alpha_j|^r)^{1/r} (\sum_{j=1}^{\infty} |\varphi(x_j)|^{p})^{1/p} \Big)^p \\
&\leq \sum_{j=1}^{\infty} |\varphi(x_j)|^p \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |x_n^*(x_j)|^p \\
&\leq (\|(x_j)_{j=1}^{\infty}\|_{\text{mid},p})^p.
\end{aligned}$$

Hence

$$\|\Phi_{(x_j)}(\alpha_j)\| \leq \|(x_j)_{j=1}^{\infty}\|_{\text{mid},p},$$

and so,

$$\|\Phi_{(x_j)}\| \leq \|(x_j)_{j=1}^{\infty}\|_{\text{mid},p}. \tag{5.3}$$

Coincidence of the mid (p,r)-compactness with the previous cases occurs in the following extreme cases which are stated in [5] and originally come from ([45], Proposition 3.1 and Theorem 4.5).

Theorem 5.1.5. *([5], Theorem 1.2) Let X be a Banach space and $1 \leq p < \infty$. Then*

(i) $\ell_p^{\mathrm{mid}}(X) = \ell_p^w(X)$ *if and only if* $\Pi_p(X, \ell_p) = \mathcal{L}(X, \ell_p)$.

(ii) $\ell_p^{\mathrm{mid}}(X) = \ell_p(X)$ *if and only if X is a subspace of $L_p(\mu)$ for some measure μ.*

The Banach spaces X for which Theorem 5.1.5 holds are examples of what the authors of [5] have termed *weak mid p-spaces* and *strong mid p-spaces* : A Banach space X is said to be a *weak mid p-space* if $\ell_p^{\mathrm{mid}}(X) = \ell_p^w(X)$, and it is said to be a *strong mid p-space* if $\ell_p^{\mathrm{mid}}(X) = \ell_p(X)$.

Definition 5.1.6. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. A subset K of X is said to be relatively mid (p,r)-compact if $K \subset (p,r)\text{-conv}(x_n)$ for some $(x_n) \in \ell_p^{\mathrm{mid}}(X)$.*

Therefore our relatively mid (p,r)-compact sets are relatively (p,r)-compact (respectively, relatively weakly (p,r)-compact) whenever X is a strong mid p-space (respectively, weak mid p-space) by the definition of the foregoing chapters and the weakly (p,r)-compact sets treated in ([1], Chap 7).

Definition 5.1.7. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Given Banach spaces X and Y, an operator $T : X \longrightarrow Y$ is said to be mid (p,r)*-compact *if $T(B_X)$ is a relatively mid (p,r)*-compact *subset of Y. Denote by $\mathcal{K}_{(p,r)}^{\mathrm{mid}}(X,Y)$ the set of mid (p,r)*-compact *operators from X to Y.*

Let X and Y be Banach spaces, $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$ and $T \in \mathcal{K}_{(p,r)}^{\mathrm{mid}}(X,Y)$. Then the quantity

$$\begin{aligned} \kappa_{(p,r)}^{\mathrm{mid}}(T) \quad = \quad & \inf\{\|(y_n)\|_{\mathrm{mid},p} : (y_n) \in \ell_p^{\mathrm{mid}}(Y)\,((y_n) \in c_0^{\mathrm{mid}}(Y)\,\text{when}\,p=\infty), \\ & T(B_X) \subset \{\sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_r}\}\} \end{aligned} \tag{5.4}$$

defines, as will be shown below, a complete quasi-norm on $\mathcal{K}_{(p,r)}^{\mathrm{mid}}(X,Y)$. Recall from equation 5.2 that $(y_n) \in \ell_p(Y)$ implies that $(y_n) \in \ell_p^{mid}(Y)$ and $\|(y_n)\|_{\mathrm{mid},p} \leq \|(y_n)\|_p$. Also recall from

([32], Theorem 4.6) that, if $T \in \mathcal{K}_{(p,r)}(X,Y)$, then

$$\kappa_{(p,r)}(T) = \inf\{\|(y_n)\|_p : (y_n) \in \ell_p(Y)\,((y_n) \in c_0(Y)\text{ when }p=\infty), \\ T(B_X) \subset \{\sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_r}\}\}$$

The class $\mathcal{K}^{\text{mid}}_{(p,r)}$ is an operator ideal and the proof follows a similar style to that of $\mathcal{K}_{(p,r)}$ (see ([1], Proposition 3.8) and [2] and uses the definition of $\ell_p^{\text{mid}}(Y)$ as we show next.

Proposition 5.1.8. *Let $1 \le p \le \infty$ and $1 \le r \le p^*$. The class $\mathcal{K}^{\text{mid}}_{(p,r)}$ of mid (p,r)-compact operators is an operator ideal.*

Proof. Let $S,T \in \mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$. Then there exist $(x_n),(y_n) \in \ell_p^{\text{mid}}(Y)$ $((x_n),(y_n) \in c_0^{\text{mid}}(Y)$ when $p=\infty$) such that $S(B_X) \subset \Phi_{(x_n)}(B_{\ell_r})$ and $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$. Let $c \in \mathbb{K}$ and define

$$z_n = \begin{cases} 2^{1/r} c x_{(n+1)/2}, & n \text{ odd} \\ 2^{1/r} y_{n/2}, & n \text{ even.}\end{cases}$$

Then for every $(y_n^*) \in B_{\ell_p^w(Y^*)}$ we have

$$\begin{aligned}\sum_{n=1}^\infty\sum_{k=1}^\infty |y_n^*(z_k)|^p &= \sum_{n=1}^\infty\sum_{k=1}^\infty |y_n^*(z_{2k-1})|^p + \sum_{n=1}^\infty\sum_{k=1}^\infty |y_n^*(z_{2k})|^p \\ &= \sum_{n=1}^\infty\sum_{k=1}^\infty |y_n^*(2^{1/r} c x_k)|^p + \sum_{n=1}^\infty\sum_{k=1}^\infty |y_n^*(2^{1/r} y_k)|^p \\ &= 2^{p/r}|c|^p \sum_{n=1}^\infty\sum_{k=1}^\infty |y_n^*(x_k)|^p + 2^{p/r}\sum_{n=1}^\infty\sum_{k=1}^\infty |y_n^*(y_k)|^p \\ &\le 2^{p/r}|c|^p(\|(x_k)_{k=1}^\infty\|_{\text{mid},p})^p + 2^{p/r}(\|(y_k)_{k=1}^\infty\|_{\text{mid},p})^p \\ &= 2^{p/r}\left(|c|^p(\|(x_k)_{k=1}^\infty\|_{\text{mid},p})^p + (\|(y_k)_{k=1}^\infty\|_{\text{mid},p})^p\right),\end{aligned}$$

so that

$$(\sum_{n=1}^\infty\sum_{k=1}^\infty |y_n^*(z_k)|^p)^{1/p} \le 2^{1/r}\left(|c|^p(\|(x_k)_{k=1}^\infty\|_{\text{mid},p})^p + (\|(y_k)_{k=1}^\infty\|_{\text{mid},p})^p\right)^{1/p} < \infty.$$

This proves that $(z_n) \in \ell_p^{\text{mid}}(Y)$ ($(z_n) \in c_0(Y)$ after the appropriate adjustments when $p=\infty$).

Let $w \in (cS+T)(B_X)$. Then there exist $x \in B_X$ such that $w = cS(x)+T(x)$. Hence, for this x, there exist $(a_n),(b_n) \in B_{\ell_r}$ such that

$$S(x) = \Phi_{(x_n)}(a_n) = \sum_{n=1}^\infty a_n x_n \quad T(x) = \Phi_{(y_n)}(b_n) = \sum_{n=1}^\infty b_n y_n$$

respectively.

Define

$$c_n = \begin{cases} 2^{-1/r}a_{(n+1)/2}, & n \text{ odd} \\ 2^{-1/r}b_{n/2}, & n \text{ even.} \end{cases}$$

Then $(c_n) \in B_{\ell_r}$. For,

$$\begin{aligned}
\sum_{n=1}^{\infty}|c_n|^r &= \sum_{n=1}^{\infty}|c_{2n-1}|^r + \sum_{n=1}^{\infty}|c_{2n}|^r \\
&= \sum_{n=1}^{\infty}|2^{-1/r}a_n|^r + \sum_{n=1}^{\infty}|2^{-1/r}b_n|^r \\
&= 2^{-1}\Big(\sum_{n=1}^{\infty}|a_n|^r + \sum_{n=1}^{\infty}|b_n|^r\Big) \\
&\le 2^{-1}\Big(\|(a_n)\|_r^r + \|(b_n)\|_r^r\Big) \\
&\le 2^{-1}(2) = 1,
\end{aligned}$$

so that $(\sum_{n=1}^{\infty}|c_n|^r)^{1/r} \le 1$, as claimed.

So

$$\begin{aligned}
\Phi_{(z_n)}(c_n) &= \sum_{n=1}^{\infty}c_n z_n \\
&= \sum_{n=1}^{\infty}c_{2n-1}z_{2n-1} + \sum_{n=1}^{\infty}c_{2n}z_{2n} \\
&= \sum_{n=1}^{\infty}2^{-1/r}a_n 2^{1/r}cx_n + \sum_{n=1}^{\infty}2^{-1/r}b_n 2^{1/r}y_n \\
&= \sum_{n=1}^{\infty}2^{-1/r}a_n 2^{1/r}cx_n + \sum_{n=1}^{\infty}2^{-1/r}b_n 2^{1/r}y_n \\
&= c\sum_{n=1}^{\infty}a_n x_n + \sum_{n=1}^{\infty}b_n y_n
\end{aligned}$$

$$\begin{aligned}
&= c\Phi_{(x_n)}(a_n) + \Phi_{(y_n)}(b_n) \\
&= cS(x) + T(x) \\
&= w.
\end{aligned}$$

This proves that $w \in \Phi_{(z_n)}(B_{\ell_r})$. Therefore $(cS+T)(B_X) \subset \Phi_{(z_n)}(B_{\ell_r})$, so that $(cS+T) \in \mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$, as such showing that $\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$.

On defining $(z_n) \in \ell_p^{\text{mid}}(Y)$ by $z_1 = \|x^*\|y, z_2 = z_3 = \ldots = 0$, it is clear that, if $(x_n^*) \in \ell_p^w(Y^*)$, then

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|x_n^*(z_k)|^p = (\sum_{n=1}^{\infty}|x_n^*(z_1)|^p \leq \|z_1\|^p(\|(x_n^*)\|_p^w)^p,$$

whence

$$\|(z_n)\|_{\text{mid},p} \leq \|z_1\| = \|\|x^*\|y\| = \|x^*\|\|y\|.$$

Furthermore, given $x \in B_X$, it holds that

$$x^* \otimes y(x) = x^*(x)y = \frac{x^*(x)}{\|x^*\|}\|x^*\|y = \sum_{n=1}^{\infty} a_n z_n \in \Phi_{(z_n)}(B_{\ell_r}),$$

with $a_1 = \frac{x^*(x)}{\|x^*\|}, a_2 = a_3 = \ldots = 0$, and so $(a_n) \in B_{\ell_r}$. Therefore

$$x^* \otimes y(B_X) \subset \Phi_{(z_n)}(B_{\ell_r}),$$

and so $x^* \otimes y \in \mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$; that is, the space $\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$ contains finite rank operators.

The ideal property also holds. That is, if $A \in \mathcal{L}(Z,X)$ and $B \in \mathcal{L}(Y,W)$, then $BTA \in \mathcal{K}^{\text{mid}}_{(p,r)}(Z,W)$. For, let $z \in B_Z$. Then $\frac{Az}{\|A\|} \in B_X$. So there exists $(t_n) \in B_{\ell_r}$ such that

$$T\Big(\frac{Az}{\|A\|}\Big) = \Phi_{(y_n)}(t_n).$$

Hence

$$\begin{aligned} B\Big(T\Big(\frac{Az}{\|A\|}\Big)\Big) &= B(\Phi_{(y_n)}(t_n)) \\ &= B(\sum_{n=1}^{\infty} t_n y_n) \\ &= \sum_{n=1}^{\infty} t_n B y_n, \end{aligned}$$

and so,

$$(BTA)z = \sum_{n=1}^{\infty} t_n\|A\|By_n = \Phi_{(w_n)}(t_n),$$

where $(w_n) := (\|A\|By_n) \in \ell_p^{\text{mid}}(Y)$ since for every $(y_n^*) \in \ell_p^w(Y^*)$,

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |y_n^*(\|A\|By_k)|^p &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \|A\|^p|y_n^*(By_k)|^p \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \|A\|^p|B^*y_n^*(y_k)|^p \\
&\leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \|A\|^p\|B^*\|^p\,|y_n^*(y_k)|^p \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \|A\|^p\|B\|^p\,|y_n^*(y_k)|^p \\
&= \|A\|^p\|B\|^p \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |y_n^*(y_k)|^p < \infty.
\end{aligned}$$

□

Next, recall the definition of a quasi-normed operator ideal (also see ([9], 9.3)). We show that $(\mathcal{K}_{(p,r)}^{\text{mid}}(X,Y), \kappa_{(p,r)}^{\text{mid}}(\cdot))$ is a quasi-normed linear space.

Proposition 5.1.9. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. The linear space $\mathcal{K}_{(p,r)}^{\text{mid}}$ of mid (p,r)-compact operators is a quasi-normed operator ideal.*

Proof. The restriction of $\kappa_{(p,r)}^{\text{mid}}(\cdot)$ to $\mathcal{K}_{(p,r)}^{\text{mid}}(X,Y)$ is a quasi-norm. For consider $T \in (\mathcal{K}_{(p,r)}^{\text{mid}}(X,Y), \kappa_{(p,r)}^{\text{mid}}(\cdot))$ such that $\kappa_{(p,r)}^{\text{mid}}(T) = 0$, and fix $\epsilon > 0$. Then there exists $(y_n) \in \ell_p^{\text{mid}}(Y)$ such that $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$ and

$$\|(y_n)\|_{\text{mid},p} < \epsilon. \tag{5.5}$$

Fix $x \in B_X$ and choose $\alpha^x = (\alpha_n^x) \in B_{\ell_r}$ so that $Tx = \sum_{n=1}^{\infty} \alpha_n^x y_n$. Observe that, if $\varphi \in B_{Y^*}$, then $f = (f_1, f_2, \ldots, f_k, \ldots,) := (\varphi, 0, 0, \ldots) \in B_{\ell_p^w(Y^*)}$. Moreover,

$$\begin{aligned}
\langle Tx, \varphi\rangle &= \sum_{n=1}^{\infty} \alpha_n^x \langle y_n, \varphi\rangle \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \alpha_n^x \langle y_n, f_k\rangle,
\end{aligned}$$

whence

$$\begin{aligned}
|\langle Tx, \varphi\rangle| &= |\sum_{n=1}^{\infty} \alpha_n^x \langle y_n, \varphi\rangle| \\
&\leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\alpha_n^x||\langle y_n, f_k\rangle| \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |\alpha_n^x|\ |\langle y_n, f_k\rangle| \\
&\leq \sum_{k=1}^{\infty} \Big(\sum_{n=1}^{\infty} |\alpha_n^x|^r\Big)^{1/r} \Big(\sum_{n=1}^{\infty} |\langle y_n, f_k\rangle|^{r^*}\Big)^{1/r^*} \\
&\leq \sum_{k=1}^{\infty} \Big(\sum_{n=1}^{\infty} |\langle y_n, f_k\rangle|^{r^*}\Big)^{1/r^*} \quad (\text{since } \alpha^x \in B_{\ell_r}) \\
&= \Big(\sum_{n=1}^{\infty} |\langle y_n, \varphi\rangle|^{r^*}\Big)^{1/r^*} \\
&\leq \Big(\sum_{n=1}^{\infty} |\langle y_n, \varphi\rangle|^{p}\Big)^{1/p} \quad (\text{since } r^* \geq p, \text{ and so } \ell_p \subset \ell_{r^*}) \\
&\leq \|(y_n)\|^w \quad (\text{by equation 5.2}) \\
&< \epsilon. \quad (\text{by equation 5.5}) \qquad (5.6)
\end{aligned}$$

Hence, $\|Tx\| \leq \epsilon$ (by equation 5.6), and so

$$\|T\| \leq \epsilon. \qquad (5.7)$$

Letting ϵ tend to zero, we conclude from equation 5.7 that $\|T\| = 0$. Since $\|\cdot\|$ is a norm on $\mathcal{L}$, it follows that $T = 0$, as was to be proved.

Now, fix $c \in \mathbb{K}$. Since $cT \in \mathcal{K}_{(p,r)}^{\text{mid}}(X, Y)$, as seen above in Proposition 5.1.8, it follows that there exists $(y_n) \in \ell_p^{\text{mid}}(Y)$ such that $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r}) = \{\sum_{n=1}^{\infty} \alpha_n y_n : \alpha \in B_{\ell_r}\}$. Fix $c \neq 0$ in $\mathbb{K}$ and for every $\bar{x} \in B_X$, choose $\alpha^x \in B_{\ell_r}$ such that $Tx = \sum_{n=1}^{\infty} \alpha_n^x y_n$. Then $cTx = \sum_{n=1}^{\infty} \alpha_n^x c y_n$. Since $(cy_n) \in \ell_p^{\text{mid}}(Y)$, it follows that

$$\begin{aligned}
\kappa_{(p,r)}^{\text{mid}}(cT) &\leq \|(cy_n)\|_{\text{mid},p} \\
&= |c|\|(y_n)\|_{\text{mid},p} \quad (\text{since } \|\cdot\|_{\text{mid},p} \text{ is a norm}) \qquad (5.8)
\end{aligned}$$

Hence

$$\kappa_{(p,r)}^{\text{mid}}(cT) \leq |c|\kappa_{(p,r)}^{\text{mid}}(T). \qquad (5.9)$$

On the other hand, fix $\epsilon > 0$. Since $cT \in \mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$, choose $(z_n) \in \ell_p^{\text{mid}}(Y)$ such that

$$\kappa^{\text{mid}}_{(p,r)}(cT) > (1+\epsilon)\|(z_n)\|_{\text{mid},p}.$$

Then

$$\begin{aligned} \kappa^{\text{mid}}_{(p,r)}(cT) &> (1+\epsilon)\|(z_n)\|_{\text{mid},p} \\ &= (1+\epsilon)\|c(\frac{1}{c}z_n)\|_{\text{mid},p} \\ &= (1+\epsilon)|c|\|(\frac{1}{c}z_n)\|_{\text{mid},p} \quad (\text{since } \|\cdot\|_{\text{mid},p} \text{ is a norm}) \end{aligned} \tag{5.10}$$

But $(cT)(B_X) \subset \{\sum_{n=1}^{\infty}\alpha_n z_n : (\alpha_n) \in B_{\ell_r}\}$ implies that $(T)(B_X) \subset \{\sum_{n=1}^{\infty}\frac{\alpha_n}{c} z_n : (\alpha_n) \in B_{\ell_r}\}$ with $(\frac{\alpha_n}{c}) \in \ell_r$ (which is a linear space). Therefore

$$\kappa^{\text{mid}}_{(p,r)}(T) \le \|(\frac{1}{c}z_n)\|_{\text{mid},p}.$$

This together with equation 5.10 implies that

$$\kappa^{\text{mid}}_{(p,r)}(cT) \;\ge\; (1+\epsilon)|c|\kappa^{\text{mid}}_{(p,r)}(T).$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\kappa^{\text{mid}}_{(p,r)}(cT) \;\ge\; |c|\kappa^{\text{mid}}_{(p,r)}(T). \tag{5.11}$$

From equations 5.11 and 5.9 it holds that

$$\kappa^{\text{mid}}_{(p,r)}(cT) \;=\; |c|\kappa^{\text{mid}}_{(p,r)}(T), \tag{5.12}$$

as was to be shown.

Let $S, T \in \mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$ and choose $(x_n),(y_n) \in \ell_p^{\text{mid}}(Y)$ such that $S(B_X) \subset \Phi_{(x_n)}(B_{\ell_r})$ and $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_r})$. Fix $x \in B_X$. As above (in the proof of Proposition 5.1.8) with $c = 1$, we can define $(z_n) \in \ell_p^{\text{mid}}(Y)$ and $(c_n) \in B_{\ell_r}$ (with appropriate adjustments when $p = \infty$) such that

$$(S+T)x = \Phi_{(z_n)}(c_n).$$

Then

$$
\begin{aligned}
& \kappa^{\text{mid}}_{(p,r)}(S+T) \leq \|(z_n)\|_{\text{mid},p} \\
= & \sup_{(w_n^*)\in B_{\ell_p^{\text{mid}}(Y^*)}} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(z_k)|^p\right)^{1/p} \\
= & \sup_{(w_n^*)\in B_{\ell_p^{\text{mid}}(Y^*)}} \left(\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(z_{2k-1})|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(z_{2k})|^p\right)^{1/p}\right) \\
= & \sup_{(w_n^*)\in B_{\ell_p^{\text{mid}}(Y^*)}} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(2^{1/r}x_k)|^p\right)^{1/p} + \sup_{(w_n^*)\in B_{\ell_p^{\text{mid}}(Y^*)}} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(2^{1/r}y_k)|^p\right)^{1/p} \\
= & 2^{1/r}\sup_{(w_n^*)\in B_{\ell_p^{\text{mid}}(Y^*)}} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(x_k)|^p\right)^{1/p} + 2^{1/r}\sup_{(w_n^*)\in B_{\ell_p^{\text{mid}}(Y^*)}} \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|w_n^*(y_k)|^p\right)^{1/p} \\
= & 2^{1/r}(\|(x_k)_{k=1}^{\infty}\|_{\text{mid},p}) + 2^{1/r}(\|(y_k)_{k=1}^{\infty}\|_{\text{mid},p}) \\
= & 2^{1/r}(\|(x_k)_{k=1}^{\infty}\|_{\text{mid},p}) + (\|(y_k)_{k=1}^{\infty}\|_{\text{mid},p}). \qquad (5.13)
\end{aligned}
$$

On taking the infimum on both sides, first over all the (x_n) defining S and then over all the (y_n) defining T we obtain

$$\kappa^{\text{mid}}_{(p,r)}(S+T) \quad \leq \quad 2^{1/r}\left(\kappa^{\text{mid}}_{(p,r)}(S) + \kappa^{\text{mid}}_{(p,r)}(T)\right)$$

This proves that $(\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y), \kappa^{\text{mid}}_{(p,r)}(\cdot))$ is indeed a quasi-normed linear space.

Next we show that $(\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y), \kappa^{\text{mid}}_{(p,r)}(\cdot))$ is actually a quasi-normed operator ideal. To this end, recall from Proposition 5.1.8 that $\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$ admits finite rank operators: Given $x \in B_X$ and $y \in B_Y$, it holds that $x^* \otimes y \in \mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$ with $\kappa^{\text{mid}}_{(p,r)}(x^* \otimes y) \leq 1$ (see the representation in the proof of Proposition 5.1.8). In particular, since $\mathbb{K}^* = \mathbb{K}$, and for all $\alpha \in \mathbb{K}$ it holds that

$$
\begin{aligned}
id_{\mathbb{K}}(\alpha) = \alpha \quad &= \quad 1 \cdot \alpha \cdot 1 \\
&= \quad 1 \otimes 1(\alpha) \quad (1 \in \mathbb{K}^*, 1 \in \mathbb{K}),
\end{aligned}
$$

we have

$$id_{\mathbb{K}} = 1 \otimes 1 \in \mathcal{K}^{\text{mid}}_{(p,r)}(X,Y),$$

whence

$$\kappa^{\text{mid}}_{(p,r)}(id_{\mathbb{K}}) \quad \leq \quad 1 \tag{5.14}$$

(Alternatively, as quasi-normed spaces,

$$\mathcal{K}_{(p,r)}(X,Y) \hookrightarrow \mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$$

and

$$\kappa^{\text{mid}}_{(p,r)}(T) \leq \kappa_{(p,r)}(T).$$

Since $id_{\mathbb{K}} \in \mathcal{K}_{(p,r)}$ with $\kappa_{(p,r)}(id_{\mathbb{K}}) = 1$, it follows that

$$\kappa^{\text{mid}}_{(p,r)}(id_{\mathbb{K}}) \leq \kappa_{(p,r)}(id_{\mathbb{K}}) = 1.\Big)$$

On the other hand fix $\epsilon > 0$. Then there exists $(z_n) \in S_{\ell_p^{\text{mid}}(Y)}$ such that

$$\|(z_n)\|_{\text{mid},p} < \kappa^{\text{mid}}_{(p,r)}(id_{\mathbb{K}}) + \epsilon. \tag{5.15}$$

Fix $\varphi \in B_{Y^*}$ arbitrarily and observe that $f_n := (\varphi, 0, \ldots) \in B_{\ell_p^w(Y^*)}$ where $f_1 = \varphi$ and $f_n = 0$ for all $n \geq 2$. Hence,

$$\begin{aligned}\left(\sum_{k=1}^{\infty} |\langle \varphi, z_k\rangle|^p\right)^{1/p} &= \left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |\langle f_n, z_k\rangle|^p\right)^{1/p} \\ &\leq \|(z_n)\|_{\text{mid},p} < \kappa^{\text{mid}}_{(p,r)}(id_{\mathbb{K}}) + \epsilon \quad \text{(by equation 5.15)}.\end{aligned}$$

Taking the supremum over φ in the previous inequality gives

$$1 = \|(z_n)\|_{\ell_p^w} \leq \kappa^{\text{mid}}_{(p,r)}(id_{\mathbb{K}}) + \epsilon.$$

Letting ϵ tend to 0 leads to

$$1 \quad \leq \quad \kappa^{\text{mid}}_{(p,r)}(id_{\mathbb{K}}) \tag{5.16}$$

By equation 5.14 and equation 5.16 we have

$$\|id_{\mathbb{K}}\|_{\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)} = 1,$$

as was to be shown.

Lastly, it follows (again) from Proposition 5.1.8 that, if $T \in \mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$, $A \in \mathcal{L}(Z,X)$ and $B \in \mathcal{L}(Y,W)$, then $BTA \in \mathcal{K}^{\text{mid}}_{(p,r)}(Z,W)$, where $(\|A\|By_n) \in \ell_p^{\text{mid}}(Y)$ defines BTA whenever $(y_n) \in \ell_p^{\text{mid}}(Y)$ defines T. So

$$\begin{aligned}\kappa^{\text{mid}}_{(p,r)}(BTA) &\leq \|(\|A\|By_n)\|_{\text{mid},p} \\ &= \sup_{(y_n^*)\in B_{\ell_p^w(Y^*)}} \Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |y_n^*(\|A\|By_k)|^p\Big)^{1/p} \quad \text{(by equation 5.3)} \\ &\leq \|A\|\|B\|\Big(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |y_n^*(y_k)|^p\Big)^{1/p}.\end{aligned}$$

On taking the infimum on both sides of the preceding inequality over the (y_n)'s in $\ell_p^{\text{mid}}(Y)$ that define T we obtain

$$\kappa^{\text{mid}}_{(p,r)}(BTA) \;\leq\; \|A\|\kappa^{\text{mid}}_{(p,r)}(T)\|B\|,$$

as was to be proved, and this affirms the claim that $(\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y), \kappa^{\text{mid}}_{(p,r)}(\cdot))$ is a quasi-normed operator ideal.

□

That $(\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y), \kappa^{\text{mid}}_{(p,r)}(\cdot))$ is a quasi-Banach operator ideal will follow from proving that it is an s-Banach operator ideal where $s = \frac{pr}{p+r}$ is obtained from the indices of the s-Banach ideal $\mathcal{N}_{(p,1,r^*)}$. Indeed, this definition of s leads to $0 < s \leq 1$.

Proposition 5.1.10. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Then $(\mathcal{K}^{\text{mid}}_{(p,r)}, \kappa^{\text{mid}}_{(p,r)}(\cdot))$ is an s-Banach operator ideal.*

Proof. Assume that $(T_n) \subset \mathfrak{K}^{\text{mid}}_{(p,r)}(X,Y)$ is a sequence such that $\sum_{n=1}^{\infty} \kappa^{\text{mid}}_{(p,r)}(T_n)^s < \infty$. Let $U := \sum_{n=1}^{\infty} T_n$ be a formal sum.

Given the ideal quasi-norm $\kappa^{\text{mid}}_{(p,r)}(\cdot)$ with the universal constant $c := 2^{\frac{1}{s}-1}$, a classical renorming process yields an equivalent ideal s-norm

$$|||T|||_{\mathfrak{K}^{\text{mid}}_{(p,r)}(X,Y)} := \inf\{\Big(\sum_{i=1}^{n} \kappa^{\text{mid}}_{(p,r)}(T_i)^s\Big)^{1/s} : T := \sum_{i=1}^{n} T_i, T_1, \ldots, T_n \in \mathfrak{K}^{\text{mid}}_{(p,r)}(X,Y)\}$$

(see [37], 6.2.5)). Using this s-norm yields

$$\begin{aligned} |||\sum_{k=1}^{n} T_k|||^s_{\mathfrak{K}^{\mathrm{mid}}_{(p,r)}(X,Y)} &\leq \sum_{k=1}^{n} \kappa^{\mathrm{mid}}_{(p,r)}(T_k)^s \\ &\leq \sum_{k=1}^{\infty} \kappa^{\mathrm{mid}}_{(p,r)}(T_k)^s \end{aligned}$$

for each $n \in \mathbb{N}$. On taking the $|||\cdot|||_{\mathfrak{K}^{\mathrm{mid}}_{(p,r)}(X,Y)}$-limit on both sides as n tends to ∞, yields $|||\sum_{k=1}^{\infty} T_k|||^s_{\mathfrak{K}^{\mathrm{mid}}_{(p,r)}(X,Y)} < \infty$, whence $U = \sum_{n=1}^{\infty} T_n \in \mathfrak{K}^{\mathrm{mid}}_{(p,r)}(X,Y)$, by the equivalence of the s-norm $|||\cdot|||_{\mathfrak{K}^{\mathrm{mid}}_{(p,r)}(X,Y)}$ with the quasinorm $\kappa^{\mathrm{mid}}_{(p,r)}(\cdot)$, as was to be proved.

Since for each $x \in B_X$ and $y \in B_Y$ it holds that $\kappa^{\mathrm{mid}}_{(p,r)}(x^* \otimes y) = \|x^*\|\,\|y\|$, for any quasi-Banach operator ideal ([37], 6.1.5), and the ideal property is also satisfied as seen in the proof of Proposition 5.1.8, it follows that $(\mathfrak{K}^{\mathrm{mid}}_{(p,r)}(X,Y), \kappa^{\mathrm{mid}}_{(p,r)}(\cdot))$ is indeed an s-Banach operator ideal. □

Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ and $(x_j) \in \ell^{\mathrm{mid}}_p(X)$. Let X be a Banach space. Recall from the earlier discussion that $(e_n) \in S_{\ell^w_r(\ell^*_r)}$, where we considered the unit vectors $e_n \in \ell_{r^*} \subset (\ell_r)^*$ as coordinate functionals for ℓ_r. The operator $\Phi_{(x_j)} : \ell_r \longrightarrow X$ ($\Phi_{(x_j)} : c_0 \longrightarrow X$ when $r = \infty$) defined by

$$\Phi_{(x_j)}(\alpha_j) = \sum_{j=1}^{\infty} \alpha_j x_j$$

admits a representation

$$\Phi_{(x_j)} = \sum_{j=1}^{\infty} e_j \otimes x_j \tag{5.17}$$

Moreover, $\Phi_{(x_j)}$ is approximable:

Proposition 5.1.11. *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ and $(x_j) \in \ell^{\mathrm{mid}}_p(X)$. Let X be a Banach space. Then $\Phi_{(x_j)}$ with the representation in equation 5.17 is approximable. Hence, $\Phi_{(x_j)} \in \mathcal{K}(\ell_r, X)$.*

Proof. Let $(x_j)_{j \leq m} := (x_1, \ldots, x_m, 0, 0, \ldots)$ and define

$$\Phi_{(x_j)_{j\leq m}} = \sum_{j=1}^{m} e_j \otimes x_j.$$

Then

$$
\begin{aligned}
\|\Phi_{(x_j)} - \Phi_{(x_j)_{j\le m}}\| &= \sup_{(a_j)\in B_{\ell_r}} \|\sum_{j=m+1}^{\infty} a_j x_j\| \\
&= \sup_{(a_j)\in B_{\ell_r}} \sup_{x^*\in B_{X^*}} |\sum_{j=m+1}^{\infty} a_j x^*(x_j)| \\
&= \sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} |\sum_{j=m+1}^{\infty} a_j x^*(x_j)| \\
&\le \sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \sum_{j=m+1}^{\infty} |a_j||x^*(x_j)| \\
&\le \begin{cases}
\sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \left(\sum_{j=m+1}^{\infty} |a_j|^{p^*}\right)^{1/p^*} \left(\sum_{j=m+1}^{\infty} |x^*(x_j)|^p\right)^{1/p}, \\
1 < p < \infty,\ 1 \le r \le p^* \\
\sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \left(\sup_{j\ge m+1} |a_j|\right)\left(\sum_{j=m+1}^{\infty} |x^*(x_j)|\right), \\
p = 1 \\
\sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \left(\sum_{j=m+1}^{\infty} |a_j|\right)\left(\sup_{j\ge m+1} |x^*(x_j)|\right), \\
p = \infty
\end{cases} \\
&\le \begin{cases}
\sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \left(\sum_{j=m+1}^{\infty} |a_j|^{r}\right)^{1/r} \left(\sum_{j=m+1}^{\infty} |x^*(x_j)|^p\right)^{1/p}, \\
1 < p < \infty,\ 1 \le r \le p^* \\
\sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \left(\sup_{j\ge m+1} |a_j|\right)\left(\sum_{j=m+1}^{\infty} |x^*(x_j)|\right), \\
p = 1,\ r = p^* \\
\sup_{x^*\in B_{X^*}} \sup_{(a_j)\in B_{\ell_r}} \left(\sum_{j=m+1}^{\infty} |a_j|\right)\left(\sup_{j\ge m+1} |x^*(x_j)|\right), \\
p = \infty,\ r = p^*
\end{cases} \\
&\le \begin{cases}
\sup_{x^*\in B_{X^*}} \left(\sum_{j=m+1}^{\infty} |x^*(x_j)|^p\right)^{1/p}, & 1 \le p < \infty,\ 1 \le r \le p^* \\
\sup_{x^*\in B_{X^*}} \left(\sup_{j\ge m+1} |x^*(x_j)|\right), & p = \infty
\end{cases} \\
&= \begin{cases}
\sup_{(z_n^*)\in B_{\ell_p^w(X^*)}} \left(\sum_{n=1}^{\infty} \sum_{j=m+1}^{\infty} |z_n^*(x_j)|^p\right)^{1/p}, \\
1 \le p < \infty,\ 1 \le r \le p^* \quad (z_1^* := x^*, z_n^* := 0, n \ne 1) \\
\sup_{x^*\in B_{X^*}} \left(\sup_{n\ge 1} \sup_{j\ge m+1} |z_n^*(x_j)|\right), \\
p = \infty \quad (z_1^* := x^*, z_n^* := 0, n \ne 1)
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \begin{cases} \sup_{(x_n^*)\in B_{\ell_p^w(X^*)}} \left(\sum_{n=1}^{\infty} \sum_{j=m+1}^{\infty} |x_n^*(x_j)|^p \right)^{1/p}, \\ 1 \leq p < \infty,\ 1 \leq r \leq p^* \\ \sup_{x^*\in B_{X^*}} \left(\sup_{n\geq 1} \sup_{j\geq m+1} |x_n^*(x_j)| \right), \\ p = \infty \end{cases} \\
&= \begin{cases} \|(x_j)_{j=1}^{\infty} - (x_j)_{j=1}^{m}\|_{\mathrm{mid}\, p}, & 1 \leq p < \infty \\ \|(x_j)_{j=1}^{\infty} - (x_j)_{j=1}^{m}\|_{\mathrm{mid}\, \infty}, & p = \infty \end{cases} \\
&\overset{m}{\to} \ 0.
\end{aligned}
$$

This proves that $\Phi_{(x_j)_{j\leq m}}$ converges to $\Phi_{(x_j)}$ in the operator norm as $m \to \infty$, whence $\Phi_{(x_j)} \in \mathcal{K}(\ell_r, X)$ whenever $(x_j) \in \ell_p^{\mathrm{mid}}(X)$ ($(x_j) \in c_0^{\mathrm{mid}}(X)$ when $p = \infty$). (This reproves its being bounded (equation 5.3)). □

Let $(f_n) \in B_{\ell_p^w(X^*)}$. Then

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |f_n(x_k)|^p = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |f_n(x_k)|^p \leq (\|(x_j)\|_{\mathrm{mid},p})^p. \tag{5.18}$$

Define

$$(\sum_{k=1}^{\infty} |f_n(x_k)|^p)^{1/p} = \alpha_n$$

for each $n \in \mathbb{N}$. Then $(\alpha_n) \in \ell_p$. Moreover, for each $n \in \mathbb{N}$ it holds that

$$e_n \otimes x_n = \alpha_n e_n \otimes \alpha_n^{-1} x_n.$$

Note that

$$\|(\alpha_j^{-1}x_j)_{j=1}^{\infty}\|_{mid,\infty} = \sup_n \sup_k |\alpha_k^{-1} f_n(x_k)| \leq 1.$$

By extending equation 5.2 to the case of $p = \infty$ we may conclude that

$$\|(\alpha_j^{-1}x_j)_{j=1}^{\infty}\|_{\infty}^{w} \leq \|(\alpha_j^{-1}x_j)_{j=1}^{\infty}\|_{mid,\infty} = \sup_n \sup_k |\alpha_k^{-1} f_n(x_k)| \leq 1.$$

So $(\alpha_j^{-1}x_j)_{j=1}^{\infty} \in \ell_{1^*}^{w}(X) = \ell_{1^*}^{\mathrm{mid}}(X)$. Therefore $\Phi_{(x_j)} \in \mathcal{N}_{(p,1,r^*)}^{\mathrm{mid}}(\ell_r, X)$ and, clearly,

$$\nu_{(p,1,r^*)}^{\mathrm{mid}}(\Phi_{(x_j)}) \leq \|(\alpha_n)\|_p \leq \|(x_j)\|_{\mathrm{mid},p}$$

by equation 5.18.

If $(y_j) \in \ell_p^{\mathrm{mid}}(Y)$ for a given Banach space Y, $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$, the operator $\Phi_{(y_j)}$ admits a factorization

$$\Phi_{(y_j)} = \overline{\Phi}_{(y_j)}\, q,$$

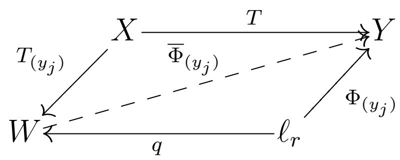
where $q : \ell_r \longrightarrow W := \ell_r/N(\Phi_{(y_j)})$ is the linear quotient mapping and $\overline{\Phi}_{(y_j)} : W \longrightarrow Y$ is the linear injective associate of $\Phi_{(y_j)}$. This kind of factorization is a property of every operator in $\mathcal{L}(X, Y)$ for all Banach spaces X and Y (see [1], p.19). Furthermore, suppose that $T(B_X) \subset \Phi_{(x_j)}(B_{\ell_r})$ so that for each $x \in B_X$ there exists $\alpha \in B_{\ell_r}$ such that $Tx = \Phi_{(x_j)}\alpha$. It is known that each such x in X determines an $\alpha \in \ell_r$ uniquely modulo the null space $N(\Phi_{(x_j)})$ of $\Phi_{(x_j)}$ such that $\|\alpha\|_r \leq \|x\|$. This was first observed and introduced by the authors of [43] for $r = p^*$ and then later improved on in [1] and [2] for any $1 \leq r \leq p^*$. Since the involved factorizations are basically algebraic, the foregoing also hold in our setting of $(y_j) \in \ell_p^{\mathrm{mid}}(Y)$. This, *said and done*, we define

$$\begin{aligned} T_{(y_j)} : X &\longrightarrow W \\ x &\longmapsto q\alpha \end{aligned}$$

where $\alpha \in \ell_r$ satisfies $\|\alpha\|_r \leq \|x\|$ and $Tx = \Phi_{(y_j)}\alpha$. If $\|x\| \leq 1$, then $\|T_{(y_j)}x\| \leq \|q\| = 1$. Following the dictum from [43], we call the injective operator $T_{(y_n)}$ the *essential quotient* of T with respect to (y_j). Hence

$$T \;=\; \overline{\Phi}_{(y_j)} T_{(y_j)} \tag{5.19}$$

where $T_{(y_j)} \in \mathcal{L}(X, W)$, $\|T_{(y_j)}\| \leq 1$, and $N(T_{(y_j)}) = N(T)$ and (as in [43]) we shall say that T is quotiented in ℓ_r by $(y_j) \in \ell_p^{\mathrm{mid}}(Y)$. Then each of the two triangles in the following diagram is commutative:



Moreover, the injective associate $\overline{\Phi}_{(x_j)}$ of $\Phi_{(x_j)}$ belongs to $\mathcal{N}_{(p,1,r^*)}^{\mathrm{mid}^{sur}}(W, Y)$ and the proof is similar in approach to that in ([2], Proposition 3.1) when $(x_j) \in \ell_p(X)$.

Given Banach spaces X and Y, $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$, fix $T \in \mathcal{K}^{\text{mid}}_{(p,r)}(X, Y)$. Then the operator T has a natural factorization similar to the case of $\mathcal{K}_{(p,r)}(X, Y)$ ([1], pp.33 – 34).

Definition 5.1.12. *([5], Definition 2.1) Given Banach spaces X and Y, an operator $T \in \mathcal{L}(X, Y)$ is said to be absolutely mid p-summing if $(T(x_j))_{j=1}^{\infty} \in \ell_p(Y)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_p^{\text{mid}}(X)$. It is said to be weakly mid p-summing if $(T(x_j))_{j=1}^{\infty} \in \ell_p^{mid}(Y)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_p^{w}(X)$.*

We now show that the operator

$$\begin{aligned} \Phi^*_{(x_j)}: \; X^* &\longrightarrow (\ell_r)^* \\ x^* &\longmapsto (\langle x^*, x_j \rangle)_{j=1}^{\infty} \end{aligned}$$

is weakly mid p-summing whenever $(x_j) \in \ell_p^{\text{mid}}(X)$

Proposition 5.1.13. *Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ and $(x_j) \in \ell_p^{\text{mid}}(X)$. Consider the operator $\Phi_{(x_j)} : \ell_r \longrightarrow X$ ($\Phi_{(x_j)} : c_0 \longrightarrow X$ when $r = \infty$) defined by*

$$\Phi_{(x_j)}(\alpha_j) = \sum_{j=1}^{\infty} \alpha_j x_j.$$

Then the operator

$$\begin{aligned} \Phi^*_{(x_j)}: \; X^* &\longrightarrow (\ell_r)^* \\ x^* &\longmapsto (\langle x^*, x_j \rangle)_{j=1}^{\infty} \end{aligned}$$

is weakly mid p-summing.

Proof. Let $(x_j) \in \ell_p^{\text{mid}}(X)$ and consider the linear operator $\Phi_{(x_j)} : \ell_r \longrightarrow X$ ($\Phi_{(x_j)} : c_0 \longrightarrow X$ when $r = \infty$) defined by

$$\Phi_{(x_j)}(\alpha_j) = \sum_{j=1}^{\infty} \alpha_j x_j.$$

Let $(x_n^*) \in \ell_p^w(X^*)$. Fix $\varphi = (g_n)$ where $(g_n) \in B_{\ell_p^w((\ell_r)^{**})}$ is arbitrary. Then

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|\langle g_n, \Phi^*_{(x_j)}(x_k^*)\rangle|^p &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|\langle \Phi^{**}_{(x_j)}(g_n), x_k^*\rangle|^p \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\|\Phi^{**}_{(x_j)}(g_n)\|^p\,|\langle \frac{\Phi^{**}_{(x_j)}(g_n)}{\|\Phi^{**}_{(x_j)}(g_n)\|}, x_k^*\rangle|^p \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sup_{\phi\in B_{(\ell_r)^*}}|\langle \phi, \Phi^{**}_{(x_j)}(g_n)\rangle|^p|\langle \frac{\Phi^{**}_{(x_j)}(g_n)}{\|\Phi^{**}_{(x_j)}(g_n)\|}, x_k^*\rangle|^p \\
&= \sum_{n=1}^{\infty}\sup_{\phi\in B_{(\ell_r)^*}}|\langle \phi, \Phi^{**}_{(x_j)}(g_n)\rangle|^p\sum_{k=1}^{\infty}|\langle \frac{\Phi^{**}_{(x_j)}(g_n)}{\|\Phi^{**}_{(x_j)}(g_n)\|}, x_k^*\rangle|^p \\
&\le (\|(x_k^*)\|_p^w)^p\sup_{\phi\in B_{(\ell_r)^*}}\sum_{n=1}^{\infty}|\langle \phi, \Phi^{**}_{(x_j)}(g_n)\rangle|^p \\
&= (\|(x_k^*)\|_p^w)^p(\|(\Phi^{**}_{(x_j)}(g_n))\|_p^w)^p \\
&= (\|(x_k^*)\|_p^w)^p(\|\widehat{\Phi^{**}_{(x_j)}}((g_n)_{n=1}^{\infty})\|_p^w)^p \\
&\le (\|(x_k^*)\|_p^w)^p\|\Phi^{**}_{(x_j)}\|^p \quad (\text{ since } (g_n)\in B_{\ell_p^w((\ell_r)^{**})}\ ) \\
&= (\|(x_k^*)\|_p^w)^p\|\Phi_{(x_j)}\|^p \\
&\le (\|(x_k^*)\|_p^w)^p(\|(x_j)_{j=1}^{\infty}\|_{\mathrm{mid},p})^p \quad (\text{using equation } 5.3).
\end{aligned}$$

Therefore $\Phi^*_{(x_j)}$ is weakly mid p-summing by Definition 5.1.12. □

Concerning the operator $\Phi_{(x_n)_{n=1}^{\infty}}$ we take note of the following: By the example cited above, namely ([5], Example 1.7), and since $(e_k) \in \ell_2^w(c_0) = \ell_2^{\mathrm{mid}}(c_0)$, we may consider

$$\begin{aligned}
\Phi_{(e_n)}: c_0 &\longrightarrow c_0 \\
(a_n) &\longmapsto \sum_{n=1}^{\infty} a_n e_n = (a_n).
\end{aligned}$$

Moreover, $(\Phi_{(e_n)}(e_k))_{k=1}^{\infty} = (e_k)_{k=1}^{\infty} \notin \ell_2(c_0)$, so that $\Phi_{(e_n)}$ is not absolutely mid 2-summing by Definition 5.1.12.

Furthermore, since $(e_n) \in \ell_1^{\mathrm{mid}}(\ell_1)$ and $\|(e_j)\|_{\mathrm{mid},1} = 1$, consider

$$\begin{aligned}
\Phi_{(e_n)}: c_0 &\longrightarrow \ell_1 \\
(a_n) &\longmapsto \sum_{n=1}^{\infty} a_n e_n = (a_n).
\end{aligned}$$

and recall that $(e_k) \in \ell_1^w(c_0)$. But $((\Phi_{(e_n)}(e_k))_{k=1}^\infty = (e_k)_{k=1}^\infty \notin \ell_1^{\text{mid}}(\ell_1)$ since $\ell_1^{\text{mid}}(\ell_1) = \ell_1(\ell_1)$, by Theorem 5.1.5(ii) and $(e_k)_{k=1}^\infty$ is not strongly ℓ_1-summable. This proves that $\Phi_{(e_n)}$ is not weakly mid 1-summing.

Now, observe that

$$\begin{aligned} \Phi_{(e_n)} : c_0 &\longrightarrow c_0 \\ (a_n) &\longmapsto \sum_{n=1}^\infty a_n e_n = (a_n) \end{aligned}$$

gives rise to

$$\begin{aligned} \Phi_{(e_n)}^* : \ell_1 &\longrightarrow \ell_1 \\ \varphi &\longmapsto (\langle \varphi, e_n \rangle)_{n=1}^\infty. \end{aligned}$$

Note that $(e_k) \in \ell_1^{\text{mid}}(\ell_1)$ and $\|(e_k)\|_{\text{mid},1} = 1$, as before, and $(\Phi_{(e_n)}^*(e_k))_{k=1}^\infty = ((\langle e_k, e_n \rangle)_{n=1}^\infty)_{k=1}^\infty = (e_k)_{k=1}^\infty \notin \ell_1(\ell_1)$. This shows that $(\Phi_{(e_n)}^*)$ is not absolutely mid 1-summing.

5.2 On (p, r)-limited Sets

Recall that a nonempty subset A of a Banach space X is called *limited* in X if for every weak*-null sequence (f_n) in X^* (that is, $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in X$), it holds that $f_n \to 0$ uniformly on A. Alternatively ([45]), for every weak*-null sequence (f_n) in X^*, there exists $(\alpha_n) \in c_0$ such that $|f_n(x)| \leq \alpha_n$ for all $x \in A$ and all $n \in \mathbb{N}$. It is known that every compact set is limited. The converse is however false and this was initially mistakenly thought to be true by Gelfand ([22]) but was later refuted by Phillips ([40]) who produced an example of non-compact limited set (see [45], §2, second paragraph).

Recently, the concept of a limited set has been extended to the p-level in [45]: A subset A of X is said to be *p-limited* in X $(1 \leq p < \infty)$ if for every weak*-p-summable sequence (f_n) in X^* (that is, $\sum_{n=1}^\infty |f_n(x)|^p < \infty$ for all $x \in X$), there exists an $(\alpha_n) \in \ell_p$ such that $|f_n(x)| \leq \alpha_n$ for all $x \in A$ and $n \in \mathbb{N}$. Subsequently, the following result was proved in ([45], Proposition 2.1).

Proposition 5.2.1. *Let $1 \leq p < \infty$ and X be a Banach space. Then every p-compact subset of X is p-limited.*

Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. In analogy with the concept of (p,r)-compact operators, we'll extend the concept of a limited set to the (p,r)-level, where the (p,p^*)-level is the p-level mentioned above.

Definition 5.2.2. *Let $1 \leq p < \infty$, $1 \leq r \leq p^*$, and X and Y be Banach spaces. A subset A of X is said to be (p,r)-limited in X if for every weak*-p-summable sequence (f_n) in X^* (that is, $\sum_{n=1}^{\infty} |f_n(x)|^p < \infty$ for all x in X), there exists $(\alpha_n) \in \ell_{r^*}$ such that $|f_n(x)| \leq \alpha_n$ for $x \in A$ and $n \in \mathbb{N}$. An operator $T \in \mathfrak{L}(X,Y)$ is said to be (p,r)-limited if $T(B_X)$ is (p,r)-limited in Y.*

It follows from the paragraph preceding ([5], Proposition 2.12) that the weakly mid p-summing operators defined in Proposition 5.1.12 are precisely the sequentially p-limited operators defined in ([45], Definition 4.1). These are the (p,p^*)-limited operators in terms of Definition 5.2.2. It is to be noted, though, that by virtue of how they are defined the absolutely (respectively, weakly) mid (p,r)-summing operators (in [5]) are not related to the (p,r)-limited operators given in Definition 5.2.2.

Now, Proposition 5.2.1 can be extended as follows.

Theorem 5.2.3. *Let $1 \leq p < \infty$, $1 \leq r \leq p^*$ and X be a Banach space. Then every mid (p,r)-compact subset of X is (p,r)-limited.*

Proof. Let $K \subset X$ be a mid (p,r)-compact set and choose $(f_n) \in \ell_p^{w^*}(X^*)$ arbitrarily. Then there exists $(x_n) \in \ell_p^{\mathrm{mid}}(X)$ such that $K \subset \Phi_{(x_n)}(B_{\ell_r})$. Furthermore,

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|f_n(x_k)|^p &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\|(f_n)\|_p^{w^*})^p \left|\frac{f_n}{\|(f_n)\|_p^{w^*}}(x_k)\right|^p \\ &\leq (\|(f_n)\|_p^{w^*}\ \|(x_k)\|_{\mathrm{mid},p})^p \quad (\text{ since } \ell_p^{w^*}(X^*) = \ell_p^{w}(X^*) \text{ isometrically}).\end{aligned}$$

Since $\ell_p^{w^*}(X^*) \subset \ell_{r^*}^{w^*}(X^*)$, it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|f_n(x_k)|^{r^*} &\leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|f_n(x_k)|^p \\ &\leq (\|(f_n)\|_p^{w^*}\ \|(x_k)\|_{\mathrm{mid},p})^p\end{aligned}$$

by the previous inequality.

Let

$$(\sum_{k=1}^{\infty} |f_n(x_k)|^{r^*})^{1/r^*} = \alpha_n$$

for all n. Then $(\alpha_n) \in \ell_{r^*}$.

Fix $z \in K$ arbitrarily. Then $z = \sum_{k=1}^{\infty} \gamma_k x_k$ for some $(\gamma_k) \in B_{\ell_r}$. Now, for each $n \in \mathbb{N}$ it holds that

$$\begin{aligned}
|f_n(z)| &= |\sum_{k=1}^{\infty} \gamma_k f_n(x_k)| \\
&\leq \sum_{k=1}^{\infty} |\gamma_k|\,|f_n(x_k)| \\
&\leq (\sum_{k=1}^{\infty} |\gamma_k|^r)^{1/r} (\sum_{k=1}^{\infty} |f_n(x_k)|^{r^*})^{1/r^*} \\
&\leq (\sum_{k=1}^{\infty} |f_n(x_k)|^{r^*})^{1/r^*} \\
&= \alpha_n.
\end{aligned}$$

Therefore K is (p, r)-limited, as was to be proven. □

Corollary 5.2.4. *Let $1 \leq p < \infty, 1 \leq r \leq p^*$ and X be a Banach space. Suppose that $(x_n) \in \ell_p^{\mathrm{mid}}(X)$. Then $\Phi_{(x_n)}(B_{\ell_r})$ is a (p, r)-limited set. In particular, $\Phi_{(x_n)}$ is (p, r)-limited.*

Proof. If $(x_n) \in \ell_p^{mid}(X)$, then $\Phi_{(x_n)}(B_{\ell_r})$ is a mid (p, r)-compact set in X. □

Theorem 5.2.5. *Let X and Y be Banach spaces, $1 \leq p < \infty$, and $1 \leq r \leq p^*$. If $T \in \mathcal{K}_{(p,r)}^{\mathrm{mid}}(X, Y)$, then T is (p, r)-limited.*

Proof. This follows from Theorem 5.2.3 since $T(B_X)$ is a mid (p, r)-limited set in Y. □

Another property of (p, r)-limited sets that generalizes the p-level one in ([45], Proposition 2.2) is as follows:

Proposition 5.2.6. *Let $1 \leq p < \infty, 1 \leq r \leq p^*$ and X be a Banach space. Given any two subsets A and B of X it holds that* *(a) If B is (p,r)-limited and $A \subset B$, then A is also (p,r)-limited.*

(b) If A is (p,r)-limited, then $\bar{A}$ is (p,r)-limited.

(c) If A and B are (p,r)-limited, so are $A \cup B$, $A+B$ and $A \cap B$.

(d) If A is (p,r)-limited and $T \in \mathcal{L}(X,Y)$, then $T(A)$ is (p,r)-limited.

Proof. (a) Let $(f_n) \in \ell_p^{w^*}(X^*)$. Since B is (p,r)-limited, there exists $(\alpha_n) \in \ell_{r^*}$ such that $|f_n(x)| \leq \alpha_n$ for every $x \in B$. Fix $a \in A$ arbitrarily. Since $A \subset B$, it follows that $a \in B$ so that $|f_n(a)| \leq \alpha_n$ for every $n \in \mathbb{N}$. This proves that A is (p,r)-limited, as was to be proven.

(b) Let $(f_n) \in \ell_p^{w^*}(X^*)$ and fix $x \in \bar{A}$ arbitrarily. Then there exists (x_k) in A such that $x_k \to x$ as $k \to \infty$. Thus, for each $n \in \mathbb{N}$ it holds that $f_n(x_k) \to f_n(x)$. Since A is (p,r)-limited, it follows that there is $(\alpha_n) \in \ell_{r^*}$ such that for each fixed $n \in \mathbb{N}$ $|f_n(x_k)| \leq \alpha_n$ for every $k \in \mathbb{N}$. Hence, for each fixed $n \in \mathbb{N}$, $|f_n(x)| \leq \alpha_n$, so that $\bar{A}$ is (p,r)-limited, as was to be shown.

(c) Let $(f_n) \in \ell_p^{w^*}(X^*)$. Since A is (p,r)-limited, there exists $(\alpha_n) \in \ell_{r^*}$ such that $|f_n(a)| \leq \alpha_n$ for every $a \in A$ and $n \in \mathbb{N}$. Since B is (p,r)-limited, there exists $(\beta_n) \in \ell_{r^*}$ such that $|f_n(b)| \leq \beta_n$ for every $b \in B$ and $n \in \mathbb{N}$. Define (γ_n) by

$$\gamma_n = \begin{cases} \alpha_n & , n \text{ odd} \\ \\ \beta_n & , n \text{ even}. \end{cases}$$

Then $(\gamma_n) \in \ell_{r^*}$ since

$$\begin{aligned} \sum_{n=1}^{\infty} |\gamma_n|^{r^*} &= \sum_{n=1}^{\infty} |\gamma_{2n-1}|^{r^*} + \sum_{n=1}^{\infty} |\gamma_{2n}|^{r^*} = \sum_{n=1}^{\infty} |\alpha_{2n-1}|^{r^*} + \sum_{n=1}^{\infty} |\beta_{2n}|^{r^*} \\ &\leq \sum_{n=1}^{\infty} |\alpha_{2n-1}|^{r^*} + \sum_{n=1}^{\infty} |\alpha_{2n}|^{r^*} + \sum_{n=1}^{\infty} |\beta_{2n-1}|^{r^*} + \sum_{n=1}^{\infty} |\beta_{2n}|^{r^*} \\ &= \|(\alpha_n)\|_{r^*}^{r^*} + \|(\beta_n)\|_{r^*}^{r^*} \\ &< \infty. \end{aligned}$$

Fix $z \in A \cup B$ arbitrarily. Then we either have $z \in A$ or $z \in B$. If $z \in A$ then by the preceding paragraphs it holds that for every $n \in \mathbb{N}$,

$$|f_{2n-1}(z)| \leq \alpha_{2n-1} = \gamma_{2n-1}.$$

If $z \in B$ then by the preceding paragraphs it holds that for every $n \in \mathbb{N}$,

$$|f_{2n}(z)| \leq \beta_{2n} = \gamma_{2n}.$$

Hence, it does holds

$$|f_n(z)| \leq \gamma_n$$

for every $z \in A \cup B$ and $n \in \mathbb{N}$. This proves that $A \cup B$ is (p, r)-limited.

Next, let $(f_n) \in \ell_p^{w^*}(X^*)$ and fix $z \in (A + B)$ arbitrarily. Then $z = a + b$ for some $a \in A$ and $b \in B$. Since A is (p, r)-limited, there exists $(\alpha_n) \in \ell_{r^*}$ such that $|f_n(a)| \leq \alpha_n$ for every n. Since B is (p, r)-limited, there exists $(\beta_n) \in \ell_{r^*}$ such that $|f_n(b)| \leq \beta_n$ for every n. Let $\gamma_n := \alpha_n + \beta_n$ for each $n \in \mathbb{N}$. Then $(\gamma_n) \in \ell_{r^*}$ too. Moreover, it holds that for every $n \in \mathbb{N}$,

$$|f_n(z)| = |f_n(a) + f_n(b)| \leq |f_n(a)| + |f_n(b)| \leq \alpha_n + \beta_n = \gamma_n.$$

This proves that $A + B$ is (p, r)-limited.

That $A \cap B$ is (p, r)-limited follows from (a) by considering $A \cap B \subset A$ or $A \cap B \subset B$ whenever A and B are (p, r)-limited.

(d) Let $(f_n) \in \ell_p^{w^*}(Y^*)$. Fix $y \in T(A)$ arbitrarily where $T \in \mathcal{L}(X, Y)$. Then there exists $x \in A$ such that $y = T(x)$. Define $g_n \in X^*$ by $g_n = T^* f_n$ for each $n \in \mathbb{N}$. Then $(g_n) \in \ell_p^{w^*}(X^*)$ since for every $x \in X$ it follows that

$$\sum_{n=1}^{\infty} |g_n(x)|^p = \sum_{n=1}^{\infty} |T^* f_n(x)|^p = \sum_{n=1}^{\infty} |f_n(Tx)|^p = \sum_{n=1}^{\infty} |f_n(y)|^p < \infty.$$

Since A is (p, r)-limited, there exists $(\alpha_n) \in \ell_{r^*}$ such that $|g_n(z)| \leq \alpha_n$ for every $z \in A$ and $n \in \mathbb{N}$. With this $(\alpha_n) \in \ell_{r^*}$,

$$|f_n(y)| = |f_n(T(x))| = |T^* f_n(x)| = |g_n(x)| \leq \alpha_n$$

for every $n \in \mathbb{N}$. This proves that $T(A)$ is (p, r)-limited whenever A is (p, r)-limited.

□

Chapter 6

Conclusion and future work

In this thesis, the main object was to extend the results in [11] and [5] to multiple indexes for $1 \leq p \leq \infty$, and $1 \leq r \leq p^*$, where p^* is the conjugate index of p and the objectives were successfully achieved.

In chapter 4, we introduced and studied a compactness property which a Banach space may or may not have. This compactness property was denoted by $\mathcal{C}_p^r$ and it is the class of all Banach spaces X such that X belongs to $\mathcal{C}_p^r$ if for every bounded subset A of X, A is relatively (p,r)-compact if, and only if, U_A^* belongs to the injective hull of the $(p,r^*,1)$-integral operators where U_A^* is the adjoint of the operator $U_A : \ell_1(A) \longrightarrow X$. The case where $r = p^*$ was introduced in [11] and was denoted by $\mathcal{C}_p$. We investigated the relationship between the (p,r)-compactness of sets and the $\mathcal{C}_p^r$ Property of Banach spaces. We provided a necessary and sufficient condition under which a Banach space may enjoy this property, that is, $Y \in \mathcal{C}_p^r$ if and only if $\mathcal{K}_{(p,r)}(X,Y) = (\mathcal{I}^{dual}_{(p,1,r^*)})^{sur}(X,Y)$ where X and Y are Banach spaces. We also provided an application related to the $\mathcal{C}_p^r$ Property by proving that $X \in \mathcal{C}_p^r$ whenever $X^{**} \in \mathcal{C}_p^r$.

In ([11], Proposition 2.2), it was shown that;

(1) If X^{**} has the Radon-Nikodym Property, then $X \in \mathcal{C}_p$. In particular, every reflexive Banach space belongs to $\mathcal{C}_p$.

(2) If $X^{**} \in \mathcal{C}_p$, then $X \in \mathcal{C}_p$.

(3) c_0, $\ell_\infty \notin \mathcal{C}_p$.

(4) If μ is a finite measure space, then $L_1(\mu) \notin \mathcal{C}_p$.

In Corollaries 2.5 and 2.6 in [11], it was shown that every separable dual space belongs to $\mathcal{C}_p$ and for any set Γ, $\ell_1(\Gamma)$ belongs to $\mathcal{C}_p$, respectively.
With regard to our case, in Proposition 4.3.2, we proved that condition (2) of ([11], Proposition 2.2) can be extended to our multiple index situation, that is, $X \in \mathcal{C}_p^r$ whenever $X^{**} \in \mathcal{C}_p^r$. However, the other three conditions of Proposition 2.2, Corollaries 2.5 and 2.6 in [11] were not looked into with regard to the multiple index discourse in this thesis.
For $1 \leq p < \infty$, let $1 \leq r < p^*$ and $\frac{pr}{p+r} = 1$. The following questions arises from ([11], Proposition 2.2) and ([11], Corollary 2.6):

(1) If X^{**} has the Radon-Nikodym Property, does X belong to $\mathcal{C}_p^r$? In particular, does every reflexive Banach space belong to $\mathcal{C}_p^r$?

(2) Do the spaces c_0 and ℓ_∞ belong to $\mathcal{C}_p^r$?

(3) If μ is a finite measure space, does $L_1(\mu)$ belong to $\mathcal{C}_p^r$?

(4) Finally, for any set Γ, does $\ell_1(\Gamma)$ belong to $\mathcal{C}_p^r$?

In chapter 5, we extended the theory of mid p-compact sets and operators to mid (p,r)-compact sets and operators as well as the p-limited sets to (p,r)-limited sets for $1 \leq p \leq \infty$, and $1 \leq r \leq p^*$. We achieved this by first introducing and studying the relatively mid (p,r)-compact subsets of a Banach space X and the (p,r)-limited subsets A of X. The set of mid (p,r)-compact operators between Banach spaces X and Y was denoted by $\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$. One key result proved was that the ideal $(\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y), \kappa^{\text{mid}}_{(p,r)}(\cdot))$ is a quasi-Banach operator ideal. Finally, we proved that every mid (p,r)-compact subset of X is (p,r)-limited and that the set $\mathcal{K}^{\text{mid}}_{(p,r)}(X,Y)$ consists of (p,r)-limited sets. Unfortunately, the relationship between the mid (p,r)-compact operators and the (r,p,p^*)-nuclear (or (r,p,p^*)-integral) operators were not looked into or discussed in this thesis for $1 \leq p \leq \infty$, $1 \leq r \leq p^*$ and $\frac{1}{p} + \frac{1}{p^*} = 1$.

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