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Teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study

By

Lancelot Sibanengi Makandidze

Submitted in fulfilment of the requirements for the degree

PHILOSOPHIAE DOCTOR

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at the

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Supervisor: Dr RD Sekao

Co-supervisors: Prof. UI Ogbonnaya
Dr M Leshota

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DECLARATION

I, Lancelot Sibanengi Makandidze (21833428) declare that the thesis titled **Teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study**, which I hereby submit for the degree **PHILOSOPHIAE DOCTOR** at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

……………………………

Lancelot Sibanengi Makandidze

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The author declares that he has observed the ethical standards required in terms of the University of Pretoria's Code of Ethics for Researchers and the Policy guidelines for responsible research.

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Month Year: November 2024

DEDICATION

I dedicate this study to my father and mother who have produced two doctors in their lifetime thus far.

ACKNOWLEDGEMENTS

For having made this breakthrough in my life, I would like to convey my heartfelt thankfulness to the following:

- The Lord, who: is my Shepherd, I shall not want; makes me lie down in green pastures; guides me along the right paths for His name's sake; is with me even though I walk through the darkest valley; prepares a table before me in the presence of my enemies; has His goodness and love that follow me all the days of my life.
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- My parents and siblings who always believed in my potentialities.
- The humble and committed teachers and learners who participated in my study.

ABSTRACT

This interpretive qualitative case study aimed to explore teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study. Six participating teachers' development of mathematical knowledge for teaching trigonometric functions to Grade 11 learners was explored during their collaborative lesson planning, lesson presentation and observation, and post-lesson reflection stages of the South African Lesson Study model. The Mathematical Knowledge for Teaching Framework was used as the lens for the study. Data were collected using observations, document analysis and semi-structured interviews. The study revealed that teachers developed their mathematical knowledge: (1) through collaborative lesson planning discussions (that incorporated all the six knowledge domains) of trigonometric functions content, teaching strategies (or instructional approaches) and learner learning (or thinking), (2) during lesson presentation and observation (that incorporated three knowledge domains) by observing the layout and sequencing of trigonometric functions content and ideas, and the handling of learners' responses/thinking (oral and written), (3) through post-lesson reflection conversations (that incorporated all the six knowledge domains) critiquing the achievement of objectives, refining and re-aligning previous and future lessons. The other finding was that teachers reported gaining content and pedagogic skills in trigonometric functions through participating in Lesson Study. Teachers also viewed Lesson Study as an intervention to address poor performance in trigonometric functions. From these findings of my study, I recommend that Lesson Study be adopted as a teacher development model in South Africa's mathematics education and be used as an intervention to address learners' problems with trigonometric functions and other problematic topics.

Keywords: trigonometric functions, Lesson Study, mathematical knowledge for teaching, knowledge domains, knowledge development

Language editor

CERTIFICATE OF EDITING

28425 Hillside Extension Hillside, Mucheke, Masvingo, Zimbabwe +263 772 973 022
jairosgonye@gmail.com
jgonye@gzu.ac.zw

TO WHOM IT MAY CONCERN

Re: Confirmation of Editing of Lancelot Sibanengi Makandidze's

(Reg Number : 21833428) Doctoral Thesis

This is to certify that I, Jairos Gonye (National Identity Number 27-049723-X-27), have language-edited and proof-read **Lancelot Sibanengi Makandidze**'s Doctoral thesis with the title **Teachers' development of mathematical knowledge for teaching trigonometric functions throughLesson Study** to be submitted to the **University of Pretoria' s Faculty of Education** in terms of the requirements for the attainment of a **Doctor of Philosophy .**

My qualifications are as follows: PHDA (English) (University of Venda); MA (English) (University of Zimbabwe); BA (Hons) (English) (University of Zimbabwe) and Graduate Certificate in Education (Grad.CE) (University of Zimbabwe).

Thank you

Jairos Gonye

Date: 26/10/2024

LIST OF ACRONYMS AND ABBREVIATIONS

ATP	Annual Teaching Plan
CAPS	Curriculum and Assessment Policy Statement
CCK	Common Content Knowledge
CK	Content Knowledge
DBE	Department of Basic Education
DCES	Deputy Chief Education Specialist
DH	Departmental Head
DoE	Department of Education
HOD	Head of Department
HCK	Horizon Content Knowledge
FET	Further Education and Training
ISPFTED	Integrated Strategic Planning Framework for Teacher Education and Development
JICA	Japan International Cooperation Agency
JITT	Just in Time Training
KCC	Knowledge of Content and Curriculum
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
LS	Lesson Study
MDoE	Mpumalanga Department of Education
MKT	Mathematical Knowledge for Teaching
MSSI	Mpumalanga Secondary Science Initiative
NSC	National Senior Certificates
PCK	Pedagogical Content Knowledge
PLC	Professional Learning Communities
PD	Professional Development
RQ	Research Question
SAT	School Assessment Team
SBA	School based assessment
SCK	Specialised Content Knowledge
SES	Senior Education Specialist

SMK	Subject Matter Knowledge
UK	United Kingdom
USA	United States of America

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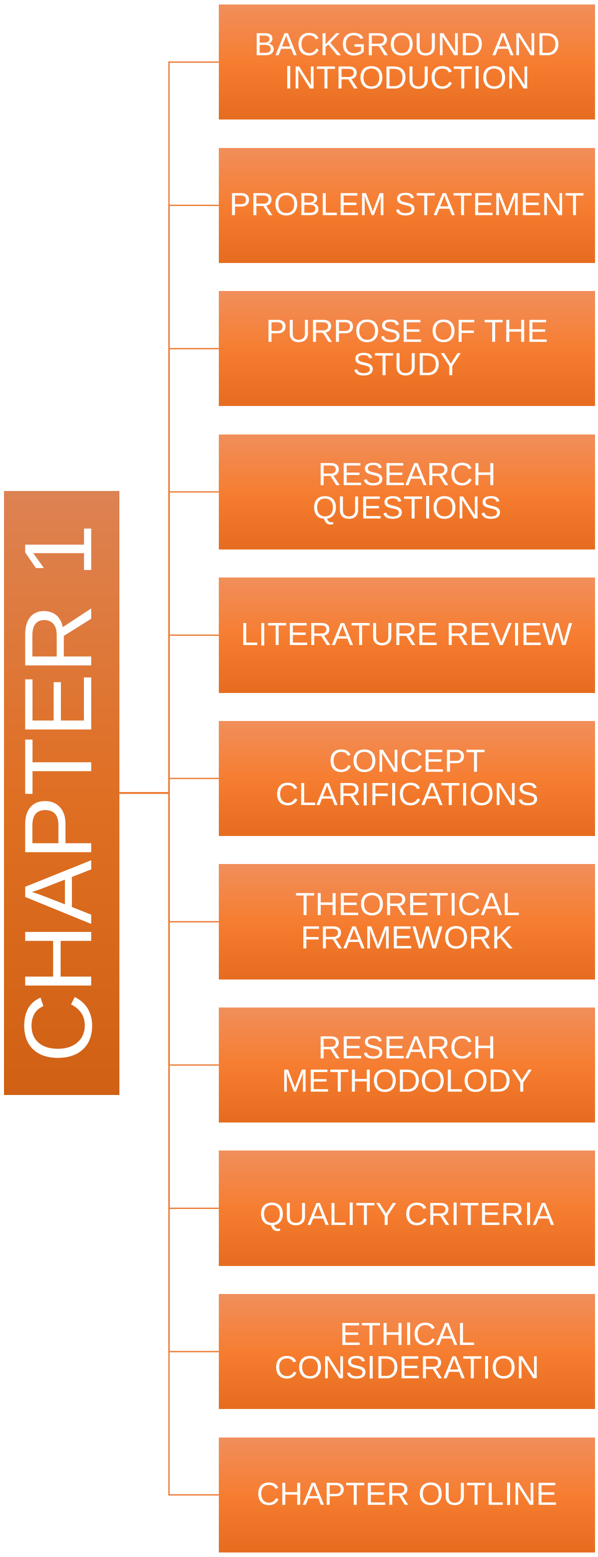
1. CHAPTER 1 – INTRODUCTION



1.1 BACKGROUND AND INTRODUCTION

Trigonometric functions are important in schools and university curricula (Moore, 2010). In fact, Bornstein (2017) reports that transformations in trigonometric functions are applied widely in science and engineering. These two fields are in demand in the South African industry, and many South African learners aspire to pursue them. However, candidates' poor performance in trigonometric functions has been a concern to examiners and Mathematics education in South Africa for more than a decade. The Curriculum and Assessment Policy Statement (CAPS) prescribes the knowledge and skills that learners should be able to demonstrate regarding trigonometric functions in Grades 10 to 12. These include investigating the effects of the parameters a, k, p and q on $y = sin\,x$; $y = \cos x$; and $y = \tan x$; and drawing graphs defined by $y = a\sin k(x+p)+q$; $y = a\cos k(x+p)+q$ and $y = a\tan k(x+p)+q$ at most two parameters at a time. Despite this curriculum requirement, the yearly National Senior Certificates (NSC) Diagnostic reports, particularly those from 2015 to 2023, have continually reported gross as well as persistent misconceptions and/or common errors in the topic. Learners have mostly shown problems in interpreting the effects of parameters on trigonometric functions, especially when required to draw and/or describe the functions after transformation (DBE, 2015a, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023). Such problems are depicted in the decline of the average performance in trigonometric functions from 42% in 2015 to 34% in 2023 (DBE, 2015a, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023).

The consistent decline in performance in trigonometric functions could mean that the current professional development programmes, including Just in Time Training (JITT) (see explanation in Chapter 2, section 2.2), are not effective enough to address the challenge. There are various textbooks in schools that attempt to treat the topic, but the problem persists. Perhaps, teachers might not be using the books in a systematic manner (Leshota, 2015), or that the textbooks do not effectively address the topic. This may be an excuse used by teachers. However, it is the teachers' responsibility to find requisite resources and use them properly. The situation calls for teachers' improved instruction on trigonometric functions, which may in turn, impact positively on the performance by learners.

Stigler and Hiebert (2009) assert that written recommendations alone, like those on trigonometric functions in the NSC Diagnostic reports, cannot improve such situations. There is, therefore, the need for effective intervention in the form of a study (Sekao (2011).Sekao (2011) holds the view that diagnosis should always be followed by intervention so as to improve teaching and learning of mathematics in South Africa. It is against this background that I intend to employ Lesson Study (LS), which will simultaneously serve as an intervention and a professional development model (Elipane, 2012; Ogegbo et al., 2019; Riales, 2011; Sekao, 2023) to explore teachers' development of mathematical knowledge for teaching trigonometric functions.

1.2 PROBLEM STATEMENT

According to synthesis by Moore (2010), earlier studies reveal that both learners and teachers have difficulties in understanding trigonometric functions. This is further supported by Maknun et al. (2019) who report that in-service and pre-service teachers as well as learners are struggling to understand the topic. Maknun et al. (2019)

contend that learners struggle to interpret trigonometric functions while teachers are struggling to stress and explain trigonometry as a function. In a global context, it appears nothing has changed in the past number of years as researchers still find that learners and teachers have difficulties with trigonometric functions. In the South African context, the continuous poor performance by learners in trigonometric functions, as reported in the NSC diagnostic reports, could be an indication that South African teachers and learners have challenges with this topic (DBE, 2015a, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023). Additionally, during school visits or support sessions for teachers held in the past six years, I, as the researcher, have observed that the topic is just done in one period or two, which is far insufficient time to cover so many concepts/aspects with conceptual understanding. The work assigned, as seen in the learners' exercise books, is too little, showing incomplete dots and plots (point by point) drawings of trigonometric functions, with little to no attempt on sketches that show effects of parameters. There are also instances where teachers omit the trigonometric functions in their planning and teaching. Even fellow senior education specialists (SES) from other districts have reported similar challenges during provincial gatherings. This could be an indication that teachers are not confident enough to teach trigonometric functions, possibly because they lack the mathematical knowledge for teaching the topic. Researchers like Bornstein (2020), Moore (2010), and Weber (2005) have reported on the challenges surrounding the topic as well as on the scarcity of research studies on the teaching and learning of trigonometric functions. The situation calls for more research (like the current one) that might bring tangible solutions to teachers, curriculum developers and teacher education institutions, particularly in mathematics education.

Currently, the nature of teacher professional development for South African teachers is not effective enough to address the challenges in the teaching and learning of trigonometric functions. This is evidenced by the yearly decline of performance in the topic. In-service teachers in South Africa are developed in content through training sessions such as JITT, which is a teacher development programme being enforced by the Department of Basic Education (DBE). The sessions are held face-to-face for a maximum of two days, in venues outside the school environment. During these sessions teachers are trained on the topics that they would be teaching to learners in the next fortnight or longer. District Mathematics SESs, Provincial Mathematics Deputy Chief Education Specialists (DCESs) and or lead teachers, are involved as the trainers and organisers. The training sessions are mainly characterised by the following processes and procedures: a pre-test; presentation of content by trainers; question and answer sessions related to the topic(s) presented, and a post-test. The teachers are then released to go back to the classroom to deliver the subject matter received during the training.

The JITT is typical of a top-down expert-led approach where an expert/trainer transmits his or her own 'best pedagogy' to passive individual teachers (Rappleye & Komatsu, 2017). This is unlike in Lesson Study (LS). Rappleye and Komatsu (2017) note that the trainer is not critiqued, thereby creating the impression that the trainer has the best pre-existing perfect pedagogy that works in all cases. Rather, individual teachers should be allowed to share their experiences in order to jointly construct pedagogical skills that are suited to the specific conditions they find themselves in. It, therefore, follows that development in content knowledge and pedagogical skills in South Africa should be subject to revision as new experiences and situations emerge

(Rappleye & Komatsu, 2017). This can only be done through mutual interaction of experts/trainers (SESs, DCES, and lead teachers) and teachers in a LS context. Many education systems have abandoned the top-down expert-led approach, and South Africa should consider pursuing the same route. The following researchers have studied the teaching and/or learning of trigonometric functions over years: Arnold (2006); Bekene Bedada and Machaba (2022); Carlsen (2018); Cetin (2015); Demir and Heck (2013); Detiana et al. (2020); de Villiers and Jugmohan (2012); Gholami (2022a); Karavakou and Kynigos (2019); Kepceoglu (2016); Martín-Fernández et al. (2019); Naidoo and Govender (2014); Siyepu (2015); Ogbonnaya and Mogari (2014); Orhani (2022); and Weber (2005). None of these studies have been conducted in the context of LS unlike the one that I conducted.

1.3 PURPOSE OF THE STUDY

The purpose of this qualitative study is to explore teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study. To pursue this purpose, I will explore four key issues: (i) teachers' development of mathematical knowledge for teaching trigonometric functions in the collaborative lesson planning stage, (ii) evolvement of teachers' mathematical knowledge for teaching trigonometric functions during lesson presentation, (iii) honing of mathematical knowledge for teaching trigonometric functions through reflective practice, and (iv) teachers' perspectives on the use of Lesson Study in the teaching of mathematics.

1.4 RESEARCH QUESTIONS

The study was conducted to answer the main research question (RQ): How do teachers develop mathematical knowledge for teaching trigonometric functions through Lesson Study? The following sub-questions guided the study to answer the main question. I also briefly explain the purpose of each research question.

RQ1: How do teachers develop mathematical knowledge for teaching trigonometric functions in the collaborative lesson planning stage? Through this RQ, I intended to explore teachers' collaborative activities that revealed development/growth in content knowledge (CK) and pedagogical content knowledge (PCK) during lesson planning.

RQ2: How does teachers' mathematical knowledge for teaching trigonometric functions evolve during the research lesson presentation? In this RQ, I focused on the teacher(s)' classroom activities and interactions that signalled the unfolding of teachers' development of mathematical knowledge.

RQ3: How do teachers' reflective practice hone their mathematical knowledge for teaching trigonometric functions? In this RQ, I sought to delve into collaborative dialogical discussions and activities that suggested sharpening of CK and PCK amongst participating teachers.

RQ4 What are the teachers' perspectives on the use of Lesson Study in developing mathematical knowledge for teaching trigonometric functions? This RQ's main purpose was to triangulate observations and document analysis through gleaning teachers' perspectives on their participation and experience in the LS. When teachers undergo LS their mathematical knowledge for teaching development could be affected. They could find their participation to be either beneficial or challenging. Hence my quest to explore their perspectives on the participation.

1.5 LITERATURE REVIEW

In this section, I prelude the literature review that is fully presented in Chapter 2 by outlining the main sections and briefly shedding light on each. The literature review of my study consists of the following sections: Setting the scene: Landscape of teacher development in South Africa; Origins of Lesson Study as a teacher development model; Lesson Study in South Africa; Teaching and learning of trigonometric functions; Teachers' perspectives in the use of Lesson Study in the teaching of mathematics.

Setting the scene: Landscape of Teacher Development in South Africa

South Africa, like any other country is striving to develop its mathematics teachers to improve instruction and learner achievement. Currently, South Africa has professional learning communities (PLCs) and just-in-time training (JITT) as its models of teacher development for mathematics teachers (DBE, 2011b, 2015b). PLCs seem to be thriving in selected districts in the country but are still marred by traditional training workshops where senior educational specialists (SESs) are dominating the sessions (Chauraya & Barmby, 2022). JITT is typical top-down, expert-led, and/or facilitator-led approach in which teachers are passive individual receptors of knowledge (Gersten et al., 2010; Rappleye & Komatsu, 2017; Van Heerden, 2022). Currently, JITT is the one that is enforced by the Department of Basic Education (DBE) where mathematics teachers are trained on topics just before they teach them according to the Annual Teaching Plan (ATP).

Origins of Lesson Study as a teacher development model

Lesson Study (LS) (which is called *jugyou kenkyuu* in Japanese) is a teacher development model that was coined in Japan where a group of teachers collaboratively work on a topic that is problematic to learners to improve the teaching and learning of that topic (Riales, 2011; Sekao, 2023; Stigler & Hiebert, 2009). The LS model is characterised by collaborative activities that essentially involve planning a lesson, followed by one teacher presenting it while the rest are observing, and thereafter reflecting on the lesson to inform further improvements. Research reports that such collaborative activities generate mathematical knowledge for teaching (Sudejamnong et al., 2014). Fujii (2019) avers that LS started pervading international education systems following the publication of the book *The Teaching Gap,* by Stigler and Hiebert in 1999.

The United States of America (USA) is one of the first countries to adopt and adapt the Japanese LS. Individual states in the USA contextualise LS in different ways. One of the most common LS models is the one that was tailored by Lewis et al. (2022b) which has the following stages: study, plan, teach, and reflect. The other model practiced in the USA Pacific Northwest region has four stages namely study and plan; teach, observe, and debrief; revise and reteach; and reflect and report.

The United Kingdom (UK) thrives with LS model which has 12 stages that are categorised into three cycles. LS in Netherlands was introduced in 2009 and its model has the following six stages: choose a research theme and formulate research question(s); study classroom and student material and design the research lesson; teach and observe the research lesson; discuss and evaluate the research lesson based on student data; revise, re-teach and discuss the research lesson; and reflect and share experiences and results (disseminate).

In Singapore, a LS model with three stages: planning session, research lesson, and post-discussion session is practised. Zambia has a LS model comprising the following eight stages: (1) Defining Problems; (2) Planning a lesson; (3) Conducting the lesson; (4) Reviewing the lesson; (5) Planning a lesson again; (6) Conducting revised lesson; (7) Reviewing the lesson again; and (8) Compiling learning. Malawi introduced LS in 2016 and its stages are similar to those of Zambia with minor adaptations.

Lesson Study in South Africa

LS in South Africa was introduced in 1999 by the Japan International Cooperation Agency (JICA) in the province of Mpumalanga in collaboration with the University of Pretoria (Ono & Ferreira, 2010). The project targeted mathematics and science teachers with the aim of improving teaching skills as well as subject knowledge. The Mpumalanga Department of Education mediated and trained teachers on LS through workshops, but the intended progress of the project suffered a brief hiatus. Currently, South Africa has a prevalent LS model that has the following five stages: diagnostic analysis; collaborative lesson planning; lesson presentation and observation; post lesson reflection; and lesson improvement (Sekao & Engelbrecht, 2021). My study is based on this model.

Teaching and learning of trigonometric functions.

Trigonometric functions (also known as circular functions) are functions that give the relationship between the angles and sides of a triangle (Buyjus, 2022). Trigonometric functions are found in secondary schools' and tertiary institutions' curricula. In the South African context, at secondary level, trigonometric functions are stipulated according to grades in the Curriculum and Assessment Policy Statement (CAPS) Grades 10-12 (DBE, 2011a). The trigonometric functions are taught in Grades 10 and 11 and examined in the final Grade 12 examination. There is scarcity of research in the teaching strategies of trigonometric functions (Zeng, 2019). The synthesis of literature that I have done identified the following ways of teaching trigonometric functions: a combination of unit circle and technology, and approaches that include amongst other things, knowledge in mathematics. Ogbonnaya and Mogari (2014) report that there is a strong relationship between learners' achievement and teachers' content knowledge in trigonometric functions. While trigonometric functions are important in the mathematics curriculum, many scholars and researchers agree that trigonometric functions are problematic to learners, and novice, pre-and in-service teachers (Demir & Heck, 2013; Gholami, 2022b; Maknun et al., 2019; Martín-Fernández et al., 2019; Nizeyimana et al., 2023; Santiago, 2024; Sintema, 2020; Tatira, 2021; Weber, 2005; Zeng, 2019).

Understanding the affordances of LS from participating teachers' perspectives has the potential to lure other teachers and convince stakeholders to support the practice (Druken, 2023; Fernandez, 2005). However, Druken (2023) reports that such studies are scarce. Druken (2023) asserts that instilling LS in a country needs resources and support from the ministry at large. LS is still in its infancy in South Africa, hence there is a need to mobilise support from the DBE in the form of resources (to train teachers) and time. Fauskanger et al. (2022) investigated mathematics teachers' perspectives on the use of LS in the teaching and learning of mathematics. The study reports that teachers valued and practised traditional methods of teaching before participating in LS. After participating in LS, teachers had new perspectives that valued the importance of learner-centred teaching and discovery learning.

1.6 CONCEPT CLARIFICATION

CAPS: is a policy statement that takes the form of a syllabus.

ATP: is an extract from CAPS that resembles a pacesetter in that it stipulates the dates and number of days that should be spent teaching a topic.

Learner: A learner, according to the South African Schools Act, refers to "any person receiving education or obliged to receive education" (DoE, 1996, p. 6). In general, a learner is known as a pupil or a student in other countries.

1.7 THEORETICAL FRAMEWORK

A theoretical framework is perceived as a researcher's lens to understand a phenomenon (Lempriere, 2019). In my study, I employed Mathematical Knowledge for Teaching (MKT) framework. Sahidin et al. (2019) assert that the MKT framework supports mathematics education in research and practice to improve teaching and learning. The MKT framework was developed from Shulman's idea of pedagogical content knowledge (PCK) and was empirically tested by Ball et al. (2008). Hill et al. (2005) define MKT as the mathematical knowledge that is used to discharge the work of teaching mathematics. The MKT comprises two knowledge bases, namely; subject matter knowledge (SMK) or content knowledge (CK) (mastering the material/matter) and pedagogical content knowledge (PCK) (delivery of material to learners or pedagogic mastery) (Sahidin et al., 2019). SMK houses three domains, namely, Common Content Knowledge (CCK), Specialised Content Knowledge (SCK) and horizon content knowledge (HCK).The PCK base also embodies three as follows: Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC). Since researchers like Sudejamnong et al. (2014) report that MKT is generated in LS environments, I used tenets of the MKT domains as my observation criteria during (data collection) teachers' collaborative activities. I further used the domains as codes during data analysis.

1.8 RESEARCH METHODOLOGY

The research methodology of my study follows Saunders et al. (2023) research onion comprising of six layers as follows: research philosophy; research approach, methodological choice; research strategy; time horizon; techniques and procedures. In this section, I briefly present each of the layers and the full details are found in Chapter 4.

Research philosophy

A research philosophy, which is also known as a research paradigm, is a system of beliefs and assumptions about developing new knowledge in a particular field (Saunders et al., 2023). Employing a research philosophy instils rigour in the study and ensures that all the elements of research are consistent, congruent, and compatible with one another (Ling & Ling, 2016). The research paradigm informing my study is the interpretivist paradigm which is characterised by subjectivity and use of multiple data collection methods (in this case observations, document analysis and interviews) (Ling & Ling, 2016; Willis et al., 2007).

Research approach

Saunders et al. (2023) view research approach as concerned with theory testing or theory building. For Saunders et al. (2023), therefore, research approach can assume a form of inductive approach, deductive approach or abductive approach. Qualitative studies are compatible with inductive approaches and deductive approaches (Creswell & Poth, 2018; Tracy, 2020). In my study, I applied INDUCTIVE-Deductive reasoning because it enabled me to come up with conclusions based on evidence and reasoning (Miessler, 2020). My study is more inductive than deductive (hence INDUCTIVE-Deductive) because it relied more on data collection to understand how teachers develop mathematical knowledge for teaching trigonometric functions in a LS context. Melnikovas (2018) perceives the inductive approach as a process that involves starting with observation and data collection, moving to description and analysis to form a theory.

Methodological choice

Melnikovas (2018) and Saunders et al. (2023) purports that methodological choice seeks to determine the use of quantitative and qualitative methods or various mixtures of both. I opted for a qualitative research method because I needed to explore and deeply understand how teachers developed mathematical knowledge for teaching trigonometric functions through LS. According to the concurrence of Creswell and Poth (2018), Merriam and Grenier (2019), and Tracy (2020), qualitative studies have the following characteristics: a natural setting, researcher as a key instrument; multiple methods of data collection, reasoning inductively and deductively, participants' multiple perspectives and meanings, context-dependent, emergent design, reflexivity, and holistic account. These characteristics appear to be encompassing all Saunders et al. (2023)'s onion layers, thereby rendering congruency and compatibility to the elements of my entire study.

Research strategy

What Saunders et al. (2023) refer to as research strategy is commonly known as research design while it is viewed as an approach by Creswell and Poth (2018) and Willig (2022). Yin (2018) notes that case studies are common in social sciences (like mathematics education). A case study can either be made up of a single case or multiple cases (Yin, 2018). My study is a qualitative single case study. Kumar (2014) contends that the case selected in a case study becomes the basis of a thorough, holistic, and in-depth exploration of the aspect(s) that a researcher wants to find out about. The case of my study is Further Education and Training (FET) mathematics teachers from a single school whose development of mathematical knowledge for teaching trigonometric functions through LS was explored.

Time horizon

Time horizon refers to the duration of a study which could be either cross-sectional or longitudinal (Melnikovas, 2018). My current study is cross-sectional in nature. Melnikovas (2018) defines a cross-sectional study as a study that is conducted within a short time.

Techniques and procedures.

Techniques and procedures occupy the innermost layer of Saunders et al. (2023)'s research onion which is concerned with sampling, data collection, and data analysis procedures. In my study, I purposefully sampled six teachers. Purposeful sampling, which is common in qualitative research, is defined as choosing a meaningful sample that fits the parameters of the project's research questions and goals (Kumar, 2014; Tracy, 2013). I collected data through observations, document analysis, and interviews. The data that I collected assisted me in finding possible solutions and backing up my arguments in the teachers' development of mathematical knowledge for teaching trigonometric functions through LS (Rizzolo, 2023). My data analysis was mainly based on and guided by a conglomerate of Braun and Clarke (2006) six phases of thematic analysis and the embedded technique postulated (Adu, 2023). Initial MKT domain codes (and indicators) emanating deductively from the MKT theoretical framework by Ball et al. (2008) as adapted from Ní Shúilleabháin and Clivaz (2017) and Clivaz and Ni Shuilleabhain (2019) served as the basis of data analysis in this study (see Appendix A). Nowell et al. (2017) posit that such a pre-existing coding framework provides a detailed analysis of aspects/indicators of the data that a researcher is interested in exploring.

1.9 QUALITY CRITERIA

In my study quality criteria was accomplished through trustworthiness, which comprises four indicators: credibility, transferability, dependability, and confirmability. I explain each one of them briefly:

- Credibility - The credibility of a qualitative study entails the truthfulness of findings on the phenomenon being explored (Billups, 2021; Stahl & King, 2020). The credibility of data on teachers' use of LS to develop mathematical knowledge for teaching trigonometric functions was established through triangulation, where I used document analysis, observations, and interviews as methods of data collection.
- Transferability - Trochim and Donnelly (2007) view transferability as the degree to which the results of a qualitative study can be transferred to other contexts or settings. I attained transferability through thick description where I

extensively and thoroughly described the process that I employed for the study so that other researchers may follow and replicate it (Billups, 2021; Stahl & King, 2020).

- Dependability - Dependability is concerned with whether the same results would be obtained if the same thing could be observed twice (Trochim & Donnelly, 2007). I achieved this by keeping an extensive and detailed record of the study process for others to replicate to ascertain the level of dependability.
- Confirmability - Billups (2021) perceives confirmability as the element of trustworthiness that is concerned with the accuracy of the findings of a phenomenon. In my study, I achieved confirmability through audit trails and reflexivity. Audit trails are reflected in the way that I clarified the processes and procedures followed to complete my study. This enables readers of my study and other researchers to replicate it and compare the results (Billups, 2021; Kumar, 2014). I attained reflexivity by stating my experiences in conducting the study as well as disclosing possible biases (Billups, 2021; Creswell & Poth, 2018)

1.10 ETHICAL CONSIDERATIONS

I addressed ethical issues as follows:

- University of Pretoria – I applied for ethical approval from the Ethics Committee in the Faculty of Education to collect data. After I collected data, I applied for the Ethics Clearance Certificate from the Ethics Committee to confirm that I collected data in an ethical manner.

- Informed consent – I secured informed consent from participants (teachers, and learners through their parents) in writing, where I stated clearly that their participation was voluntary and that they were free to opt out any time with no consequences.
- Anonymity/privacy – I used pseudonyms for the participating teachers by naming them Teacher A, Teacher B, Teacher C, Teacher D, Teacher E, and Teacher F. The faces of both learners and teachers who are in pictures were blurred to hide their identities.
- Confidentiality - Data collected from the participants in the form of lesson plans, video-recorded observations and interviews were kept confidential and used for research purposes only.

1.11 CHAPTER OUTLINE

This study consists of six chapters as follows:

Chapter 1: Introduction

In Chapter 1, I presented an introduction to the study that comprised the background and introduction, problem statement, purpose of the study, research questions, literature review, concept clarification, theoretical framework, research methodology, quality criteria, and ethical considerations.

Chapter 2: Literature Review

In this chapter, I started by exploring literature on the landscape of teacher development in South Africa to set the tone. This was followed by deliberations on the origins of Lesson Study that paved way for the discussion on globalisation of Lesson Study. I discussed Lesson Study models in United States of America, United Kingdom, Netherlands, Singapore, Zambia, Malawi, and South Africa. I then delved into the teaching and learning of trigonometric functions under the following subheadings definition and curricula of trigonometric functions, effective teaching of trigonometric functions, learners' conceptual difficulties in trigonometric functions, and observed attempt by South Africa to address the challenges in trigonometric functions. I ended the chapter by exploring teachers' perspectives on the use of Lesson Study in the teaching of mathematics.

Chapter 3: Theoretical Framework

In chapter 3, I detailed the Mathematical Knowledge for Teaching (MKT) framework and its applicability to my study. The aspects of the MKT framework include Common Content Knowledge (CCK), Specialised Content Knowledge (SCK), Horizon Content Knowledge (HCK), Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC).

Chapter 4: Methodology

In chapter 4, I presented the methodology according to the research onion metaphor advocated by Saunders et al. (2023). The layers of the research onion include research philosophy, research approach, methodological choice, research strategy, time horizon, techniques and procedures.

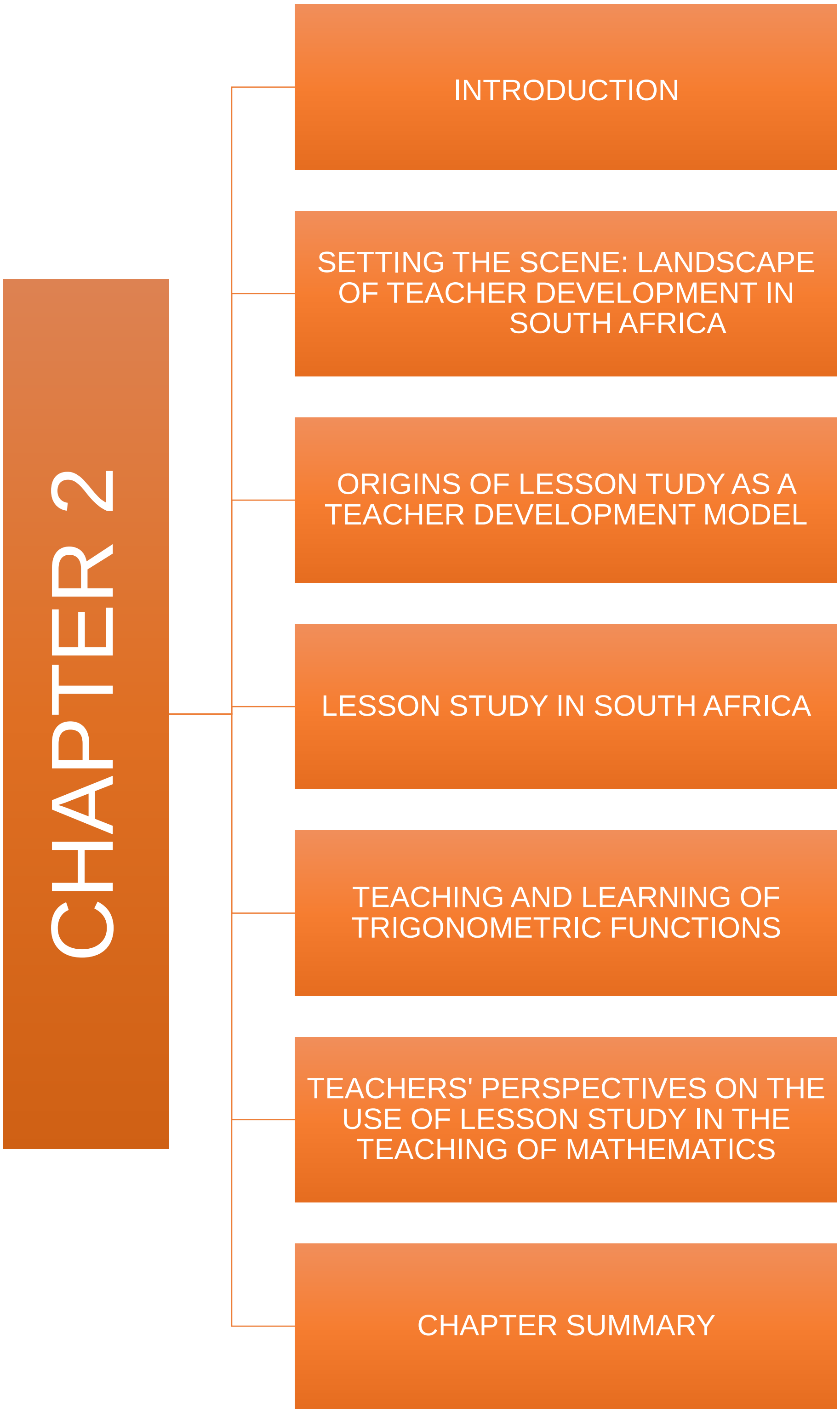
Chapter 5: Findings

In chapter 5, I presented findings of data from observations, document analysis and interviews. The findings chapter is organised according to the following subheadings development of mathematical knowledge for teaching trigonometric functions in collaborative lesson planning, evolvement of mathematical knowledge for teaching trigonometric functions in research lesson presentation, honing of mathematical knowledge for teaching trigonometric functions through reflective practice, consolidation of the findings regarding teachers' development of mathematical knowledge for teaching trigonometric functions in a lesson study context, and teachers' perspectives on using lesson study in developing mathematical knowledge for teaching trigonometric functions.

Chapter 6 – Discussion, Recommendations and Conclusions

In chapter 6, I discussed the findings and answered the research questions. The discussions were organised according to the following subheadings: development of mathematical knowledge for teaching trigonometric functions in collaborative lesson planning, evolvement of mathematical knowledge for teaching trigonometric functions in research lesson presentation, honing of mathematical knowledge for teaching trigonometric functions through reflective practice, consolidation of the findings regarding teachers' development of mathematical knowledge for teaching trigonometric functions in a lesson study context, and teachers' perspectives on using lesson study in developing mathematical knowledge for teaching trigonometric functions. I highlighted the implications while discussing the findings. Furthermore, I answered the main research question under the subheading teachers' development of mathematical knowledge for teaching trigonometric functions through lesson study. I also presented the utility of the theoretical framework, contributions of the study, limitations of the study, recommendations, reflexivity, and conclusions.

2. CHAPTER 2: LITERATURE REVIEW



2.1 INTRODUCTION

Trigonometric functions are important in mathematics, science, and technology industries. This has seen the topic pervading all secondary school and tertiary education curricula. Unfortunately, learners and teachers struggle with the topic (Demir & Heck, 2013; Gholami, 2022b; Loce, 2021; Maknun et al., 2019; Martín-Fernández et al., 2019). Gholami (2022b) reports that teachers are currently employing traditional methods in teaching concepts that promote memorisation by learners. Perhaps engaging in an effective teacher development model like Lesson Study (LS) could alleviate the situation.

In this chapter, I reviewed the relevant literature to gain more insights into the problem that I was exploring. I reviewed and explored relevant literature by focusing on the following headings: Setting the scene: Landscape of Teacher Development in South Africa; Origins of Lesson Study as a teacher development model; Lesson Study in South Africa; Teaching and learning of trigonometric functions; Teachers' perspectives on the use of Lesson Study in the teaching of mathematics.

2.2 SETTING THE SCENE: LANDSCAPE OF TEACHER DEVELOPMENT IN SOUTH AFRICA

Like with a few other African education systems, South Africa's education system reflects a feeling that newly qualified teachers are insufficiently prepared in content and pedagogy whereas senior teachers need continuous development because of learner performance that remains below expectations (Bett, 2016; Venketsamy, 2022).

This has resulted in South Africa embarking on professional teacher development programmes (employer-driven) that are intended to improve learners' achievement (De Clercq & Shalem, 2014; Shalem & De Clercq, 2019). The South African mathematics education professional teacher development landscape across the country is characterised by professional learning communities (PLCs) and just-in-time training (JITT) (DBE, 2011b, 2015b).

PLCs are common practices engaged in by teachers, school management teams (SMTs) and subject specialists where they identify their professional developmental needs, and establish activities that will enable them to develop one another (DBE, 2011b). The Department of Basic Education (DBE) advocated for the institution of PLCs through the Integrated Strategic Planning Framework for Teacher Education and Development (ISPFTED) in South Africa 2011-2025 (DBE, 2011b, 2015b). The DBE was informed by research reports that PLCs could improve the quality of teaching, learning and morale when practised well (Chauraya & Barmby, 2022; DBE, 2015b). The ISPFTED reveals that the DBE anticipates establishing PLCs at the school level (DBE, 2011b, 2015b).

In the district where I work, PLCs do exist, and the common term used is cluster. I will use the terms PLC and cluster interchangeably. These clusters were formed according to the proximity of schools. Each cluster has a structure consisting of a chairperson, a deputy chairperson, and a secretary. The structures are responsible for chairing meetings and coordinating common practice activities in the teaching and learning of mathematics. WhatsApp groups have been formed per cluster where announcements and sharing of content resources are done. Teachers within a cluster set common school-based assessments (SBAs) such as tests and examinations. Again, teachers gather in those groups after every school term to post-moderate the SBAs. In so doing, teachers develop the skills of examining, pre- and post-moderating, and typing question papers and solutions (memos). Senior Education Specialists (SES) also use these groupings to mediate the Annual Teaching Plans (ATPs) and SBAs at the beginning of each year. Content training by SESs is sometimes done within each cluster during planned afternoons. Chauraya and Barmby (2022) view such sessions as traditional workshops because the SES is the source of knowledge. In support of this view, Langford et al. (2022) argue that teacher-led PLCs breed quality professional development more than the traditional top-down training sessions for teachers.

The DBE clearly articulated on PLCs through its ISPFTED. However, in most districts in South Africa PLCs are dysfunctional or non-existent. Where they exist, their operations are far below expectations because of varied challenges. One reason could be that teachers are taking them as extra loads on their plethora of duties. The other reason could be the geographical locations of schools, especially in rural setups where they are far apart. From my experience, the success of PLCs in South Africa depends on the commitment of teachers, which is in line with the DBE (2015b) insisting that the PLCs should be teacher-driven. In my district, the practice of teachers engaging in discussing problematic topics (like trigonometric functions) on their own is still lacking. In Singapore, the number of schools practising LS leapt from 59 (in 2009) to 112 (in 2010) after the introduction of PLCs (Lee & Lim-Ratnam, 2014). It would appear PLCs and LS are compatible with each other. Perhaps the introduction of LS in the already existing PLCs in my district and the rest of South Africa would boost and reinforce the growth of these two institutions of common practice.

Just In Time Training (JITT) is one of the teacher professional development (PD) approaches that South Africa has adopted for teachers in Grades R to 12 (Fleisch, 2016; Fleisch & Alsofrom, 2022; Hazell et al., 2019; Shalem & De Clercq, 2019). Van Heerden (2022) sees JITT as "an approach to organisational or individual learning that encourages need-related training to be on hand exactly when and in the format that is needed by the learner". Calleja et al. (2021) report that JITT is a new teacher PD borrowed from game-based learning. Subsequently, Riel (2000) views JITT as an industrial approach type of training that takes teachers as passive learners. From my understanding, the JITT approach values providing knowledge to teachers a few days before it is needed by learners in the classroom.

In-service teachers in South Africa are developed in content through JITT sessions, which is the teacher development programme being enforced by the Department of Basic Education. The sessions are held either face-to-face or on-line, outside the school environment. The face-to-face sessions can either be held at the district level after school hours (14:30 to 16:30) or at the provincial level for a maximum of two days (Saturday and Sunday). During afternoon sessions (at district level) teachers are trained on the topic(s) that they will be teaching to learners in the following fortnight. Currently, the provincial-level training is dominant. Teachers gather at the beginning of a term at a residential camp (either in a hotel or a lodge) and the trainer(s) present content for the whole term. The on-line sessions are also held either in the afternoons (for a maximum of two hours to cover content for two weeks) or on Saturdays (09:00 to 16:00 to cover content for the whole term). For instance, the Grade 11 teachers (at the beginning of Term 3) (are) could be trained on Trigonometric functions, Trigonometry of 2D shapes, Statistics, Probability and Finance, which are Term 3 topics according to the ATP. This means that content knowledge (on the topics concerned) is made available to teachers at the moment before that knowledge is needed in the classroom (Calleja et al., 2021; O'Donnel, 2017). District Mathematics Senior Education Specialists (SES) like me, Provincial Mathematics Deputy Chief Education Specialists (DCES) and/or lead teachers are involved as trainers and organisers. The training is mainly characterised by the following processes and procedures: a pre-test; presentation of content by trainers; hinting by the trainer(s) on how the topic(s) could be taught to learners; question and answer sessions related to the topic(s) presented; a post-test. The teachers are then released to go back to the classrooms to deliver the subject matter received during the training. The notion held is that the trained teachers have a sense of ownership and it is anticipated that they will teach the content with confidence (O'Donnel, 2017). The purpose of the JITT is to address teachers' content knowledge gaps before they teach the topics concerned (Fleisch, 2016). The trainers (some of whom are my colleagues) and teachers are concerned that the time is too short to crumple and teach volumes of content within a short time. This concern has also been noted by De Clercq and Shalem (2014). Bett (2016) also views such an approach as thrusting content upon teachers. It is recommended that the duration of training sessions be long enough to cover the necessary content (Ogbonnaya & Mogari, 2014).

The JITT practised in South Africa is typical of a top-down expert-led approach where an expert or trainer (SES, DCES or lead teacher) transmits his or her own 'best pedagogy' to passive individual teachers (Gersten et al., 2010; Rappleye & Komatsu, 2017). Moreover, JITT is viewed as a one size fit all type of training approach and it is reported that the teachers being trained in the South African landscape are furnished with expert-designed handouts for classroom use (De Clercq & Shalem, 2014; Furiwai & Singh-Pillay, 2020). In addition, Van Heerden (2022) argues that JITT is a facilitator-led training intervention whose successes rely on the trainer/presenter. Unlike in the Lesson Study, Rappleye and Komatsu (2017) note that the trainer is not critiqued thereby creating the impression that the trainer has the best pre-existing perfect content and pedagogy that works in all cases. Rather, individual teachers should be allowed to share their experiences to jointly construct pedagogical skills that are suited to the specific conditions they find themselves in. From the ongoing synthesis of the literature, I view JITT as an approach that treats teachers like empty vessels that should be filled with knowledge (Cenoz & Gorter, 2020). It, therefore, follows that development in content knowledge and pedagogical skills in South Africa should be subject to revision as new experiences and situations emerge (Rappleye & Komatsu, 2017). This can only be done through the mutual interaction of experts/trainers (District Mathematics SESs, Provincial Mathematics DCESs, and lead teachers) and teachers in an LS context. Many education systems have abandoned the top-down expert-led approach and South Africa ought to follow suit. Research reports that professional development programmes that are collaborative and conducted in school/classroom contexts coupled with reflection, are more effective than those that are expert/trainer-dominated and off-site (Bannister, 2018; Bett, 2016; Calleja et al., 2021; DBE, 2015b; Stoll et al., 2012).

Currently, the nature of teacher professional development for South African teachers is not effective enough to address the challenges faced in the teaching and learning of trigonometric functions. Most South African teachers have attended JITT yearly for more than ten years but their results in mathematics (particularly in trigonometric functions) have not improved. This is evidenced by the yearly decline in performance in the topic. Lesson Study (LS) is a unique and potent type of teacher PD approach compared to those practised in many countries' mathematics education systems (Chokshi & Fernandez, 2004; Fujii, 2019; Kager et al., 2022; Perry & Lewis, 2009). I hold that LS might provide a context in which teachers could develop robust mathematical knowledge for teaching trigonometric functions.

The teacher development literature arena depicts two models/approaches. On one hand, there is a top-down and expert-led traditional approach where teachers are a passive audience receiving instructions and illustrations from the trainer (Gersten et al., 2010; Langford et al., 2022). On the other hand, there is a teacher-led and practice-embedded approach that is characterised by collaborative and interactive discussions (Gersten et al., 2010; Langford et al., 2022). The teacher-led and practice-embedded approach is preferred and sought in modern education systems because it promotes teacher learning by incorporating collaborative activities that involve teachers teaching learners in the presence of other teachers, and reflection, and this identifies with LS (Braseth, 2024; Choy et al., 2021; Gibbons et al., 2021).

2.3 ORIGINS OF LESSON STUDY AS A TEACHER DEVELOPMENT MODEL

LS, according to Riales (2011), is a practice that originates in Japan where a group of teachers collaboratively work on a topic that is problematic to learners in order to improve the teaching and learning of that topic. Collaborative work involves planning a lesson, one teacher presenting it and the rest observing, and thereafter analysing, and discussing the strengths and weaknesses of the lesson for future improvements.

Chinnappan and Cheah (2012) note that teacher collaboration is a key feature of LS. A study by Sudejamnong et al. (2014) also reports that mathematical knowledge for teaching (MKT) is developed or generated during these collaborative activities. All these collective efforts are done with the syllabus and learners' thinking in mind (Riales, 2011; Sekao, 2023). Again, Stigler and Hiebert (2009) see LS (called *jugyou kenkyuu* in Japanese) as a component of continuous school-based teacher professional development (known in Japanese as *kounaikenshuu*) where groups of teachers meet regularly over periods ranging from several months to a year to work on the design, implementation, testing, and improvement of one or several 'research lessons' (*kenkyuu jugyou*). The product is tagged a research lesson since it is planned collaboratively to meet diagnosed learner needs in an action research context (Sekao, 2023). Furthermore, Riales (2011) contends that teachers' participation in LS contributes to their professional development. My current study, which aims to explore teachers' development of mathematical knowledge for teaching trigonometric functions through LS, is aligned with this contention.

Different levels of LS are distinguished which are determined by how the education system is organised. For instance, in South Africa there one of the layers of the : school-based, circuit-based, district-based, and provincial/national-based (Sekao, 2023). The types of LS are informed by the structure of the Department of Basic Education of South Africa. Schools report to the circuit, and the circuit reports to the district while the district reports to the province which is answerable to the national office.

School-based LS, which is called school-wide LS by Lewis et al. (2022a), is LS which is embraced and practised by the whole school staff, LS teams and individual classroom teachers. The school adopts LS as a vision thereby supporting LS teams where teachers tackle problematic topics in their respective subject departments (Lewis et al., 2022a; Sekao, 2023). The content and teaching skills learned by teachers in LS teams' activities would be applied to the classroom (Lewis et al., 2022a; Sekao, 2023). I hold that school-based LS promotes the honing of instructional skills in all subjects in the school, which in turn supports learners' understanding.

Circuit-based LS is formed by LS teams from different schools within a circuit clustering together per subject area (Mxenge & Bertram, 2023; Sekao, 2023). In my district, the teachers have formulated PLCs within circuits which could serve as an anchor for circuit-based LS. Teachers in circuit-based LS are anticipated to collaboratively plan, teach, reflect, and improve lessons on challenging topics (Sekao, 2023). I contend that circuit-based LS could be beneficial to small schools that might have one or two teachers per subject area. Teachers from such schools would meet more teachers teaching their subject area during circuit-based LS.

As Sekao (2023) puts it, district-based LS is when teachers from schools and or circuits in a district practise LS in a seminar setting. During district-based LS, teachers share their teaching experiences on a particular problematic topic and or tackle the topic in a LS context (Sekao, 2023). The teachers in the seminar might organise a class where one teacher presents the planned lesson while others observe followed by reflection and improvement (Sekao, 2023). The district-based LS has the potential to enrich subject matter knowledge and teaching skills since there would be many teachers from different schools.

Provincial-based LS is a collaboration of delegates from different districts within a province and national-based LS comprises teachers from various provinces of the country (Sekao, 2023). According to Sekao (2023), both provincial-based LS and national-based LS are held in conference contexts but are not held often because of time and budgetary constraints. LS would flourish in South Africa if all types of LS remain functional. I argue that the functionality could be assured by strengthening school-based LS.

2.3.1 Globalisation of Lesson Study

Fujii (2019) reports that LS started to spread to other countries outside Japan after Stigler and Hiebert published *The Teaching Gap* in 1999. Although LS originated in Japan, the processes and practices followed in pursuit of this teacher professional development are varied within the country. Stigler and Hiebert (2009) contend that the overarching process in Japan is made up of the following eight steps:

Step 1 - Defining the problem.

Step 2 - Planning the lesson.

Step 3 - Teaching the lesson.

Step 4 - Evaluate the lesson and reflect on its effect.

Step 5 - Revise the lesson.

Step 6 - Teaching the revised lesson.

Step 7 - Evaluating and reflecting again; and,

Step 8 - Sharing the results.

Steps 6 to 8 are optional, meaning that the first cycle is more critical. Education systems that decided to follow the Japanese LS are doing so with some adaptations to suit their situations, and this has resulted in differing models from one country to the other. In the following sections, I briefly outline the LS of the following countries in United States of America (USA), United Kingdom (UK), Netherlands, Singapore, Malawi, and Zambia.

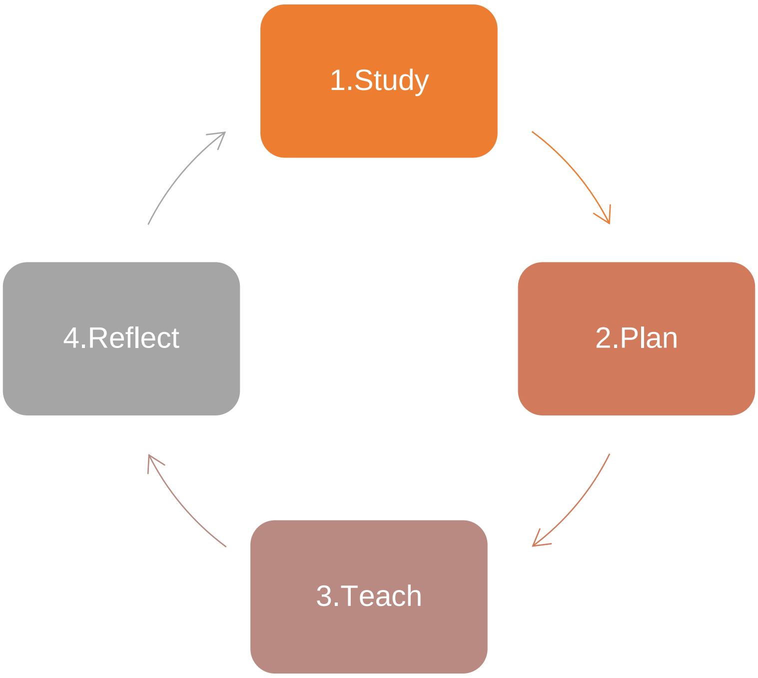
2.3.2 Lesson Study in the USA

The USA is one of the countries that are at the forefront of emulating the Japanese LS. Isoda et al. (2007) report that by September 2003 the USA had 140 LS groups active in 29 states, and more than 1100 teachers from 245 schools in the 80 school districts who were involved in LS. It is held that the USA considered LS in its education system in the late 90s, following the recommendations of Stigler and Hiebert (Lewis, 2015). Different states in that country seem to have adopted varying models. In this section, I discussed two models one practised in the state of California and the other in the Pacific Northwest region.

One of the most prevalent LS models in the USA is the one coined at Mills College by Lewis et al. (2022b). It has four stages which are: study, plan, teach, and reflect (see Figure 2.1).

Figure 2.1

Lesson Study Model Practised In California, Mills College



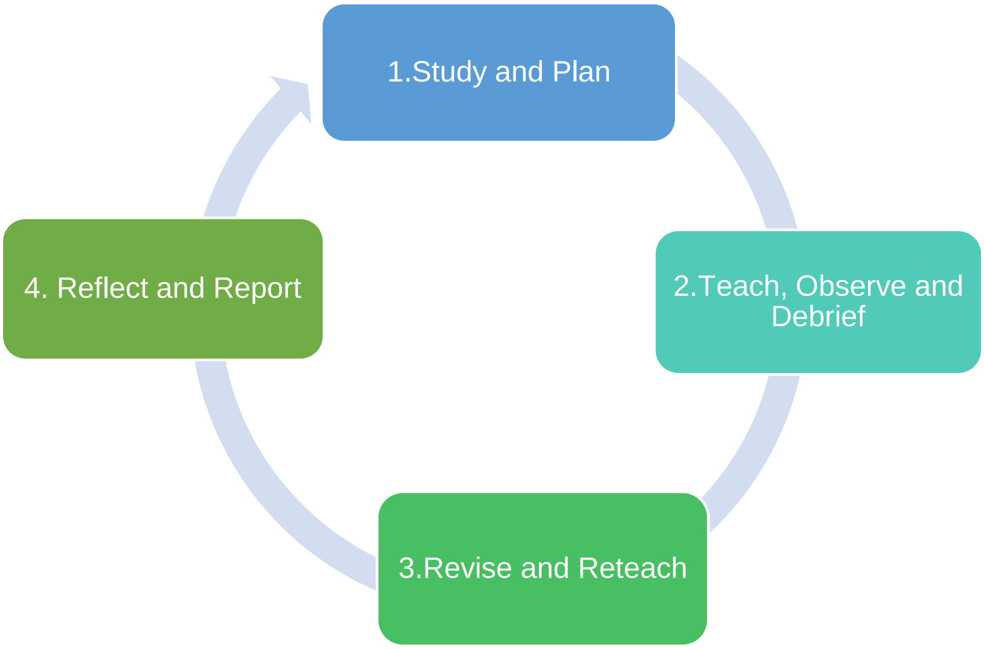
Note. Source: Lewis et al. (2022b)

During stage 1 (study), teachers collaboratively explore learners' goals according to the syllabus and lay standards in a chosen topic (Lewis et al., 2022b). This serves as a foundation for stage 2 (plan). Lewis et al. (2022b) report that in-depth lesson planning is done in stage 2 by formulating lesson objectives and choosing and solving class activities with learners' thinking in mind. While planning, teachers define the criteria for lesson observation to facilitate data collection. Stage 3, which is teach, entails the implementation of the plan where one teacher presents the lesson while others observe and collect data reflection. The data collected by observing teachers comprises learners' thinking and their reactions to subject matter and instruction. In stage 4 (reflect), teachers reflect on the lesson by focusing on the learners' thinking and learning. The reflection is the basis of lesson improvement. Lewis et al. (2022b) purport that teachers also discuss their professional gains in the whole LS cycle during the reflection stage.

The other model practised in the USA Pacific Northwest region that covers the states of Oregon, Washington, and Idaho is shown in Figure 2.2. The model has four stages namely: study and plan; teach, observe, and debrief; revise and reteach; and reflect and report. To complete the four stages of the cycle, participants have to pass through ten steps which are: develop collaboration norms; establish a research theme; identify and study the topic; plan the lesson; teach and observe the lesson; debrief and discuss observation data; revise the lesson; reteach, observe, and debrief; reflect and report; share and disseminate knowledge (Leong et al., 2021).

Figure 2.2

Lesson Study Model For The USA Pacific Northwest Region



Note: Source: Leong et al. (2021, p. 2)

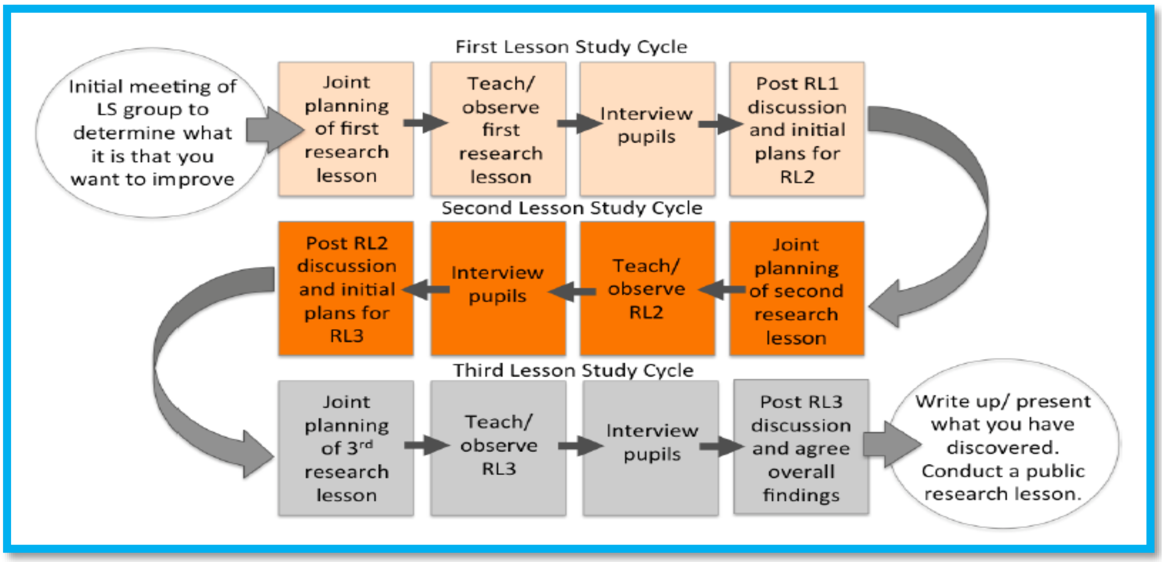
The Pacific Northwest region model is unique in that clusters processes that serve as separate stages from other models. For instance, stage 1 combines study and plan which exist as stages 1 and 2 respectively in the Mills College model. In some instances, debriefing exists as a separate stage. With most models revising and reteaching in stage 3 comes after reflection. It is therefore clear that states in the USA adapt LS to suit their contexts. Nevertheless, all the models that could be differing structures have the same goal of improving instruction and learners' understanding.

2.3.3 Lesson Study in the UK

In the UK, LS started in the 21st century (Dudley, 2014). The country adopted the Japanese LS and adapted it to its three-cycle process that has a total of twelve stages (see Figure 2.3).

Figure 2.3

The Lesson Study Process In The United Kingdom



Note: Source Dudley (2014, p. 6)

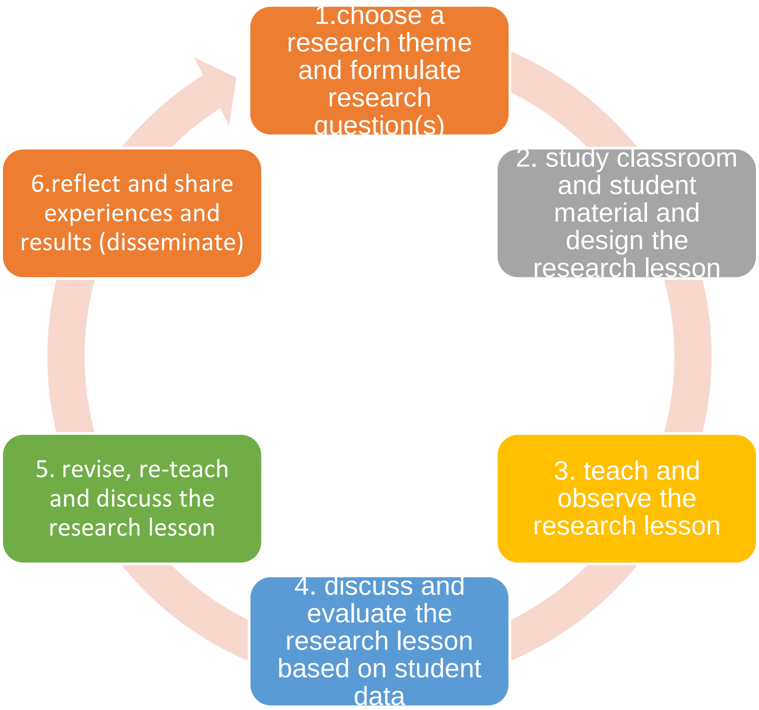
The cycles or process is distinct because it emphasises ‘case pupils’. At least three pupils are considered cases (Dudley, 2014). The group of educators focuses on these pupils during the process to check on the impact or effectiveness of their instruction. According to Dudley (2014), the pupils are interviewed after the research lesson to further ascertain the successes and challenges of the lesson. This model is strong since there is feedback from both learners and educators.

2.3.4 Lesson Study in the Netherlands

Lewis and Lee (2017) reports that Lesson Study was introduced to the Netherlands in 2009 by a doctoral student but gathered momentum in 2016. According to Uffen et al. (2022, p. 3), Netherlands LS model consists of six stages which are: “choose a research theme and formulate research question(s); study classroom and student material and design the research lesson; teach and observe the research lesson; discuss and evaluate the research lesson based on student data; revise, re-teach and discuss the research lesson; and reflect and share experiences and results (disseminate).” The model is shown in Figure 2.4.

Figure 2.4

Netherlands Lesson Study Cyle



Note. Source: Uffen et al. (2022)

I view Stages 5 and 6 as unique and important. Stage 5 sees the lesson being revised and re-taught thereby enrich data that informs improvement. In addition, stage 6 allows teachers to disseminate LS outcomes to other non-participant teachers. Such a practice could see good teaching skills pervading many classrooms. Uffen et al. (2022) report that the Netherlands LS model also incorporates case students like the UK model. This is advantageous in that feedback on the research lesson is enriched by perspectives from observing teachers as well as students.

2.3.5 Lesson Study in Singapore

In general, Singapore has a competent education system. I therefore found it important to explore its lesson study model. LS was introduced in 2004 in Singapore, and its cycle has three pronounced stages: planning session, research lesson, and post-discussion session. The adapted model for Singapore is shown in Figure 2.5. Lim et al. (2011) report that LS in Singapore exists in varied forms because of the adaptation of different schools to culture, needs, and priorities. LS in that country is being practised at the school and cluster level (Cheng & Yee, 2012).

Figure 2.5

Singapore Lesson Study Model

Note. Source: Lim et al. (2011)

The Singapore LS model is unique in that it has three stages. Again, it is not clear if there is re-teaching of the revised lesson after stage 3.

2.3.6 Lesson Study in Zambia

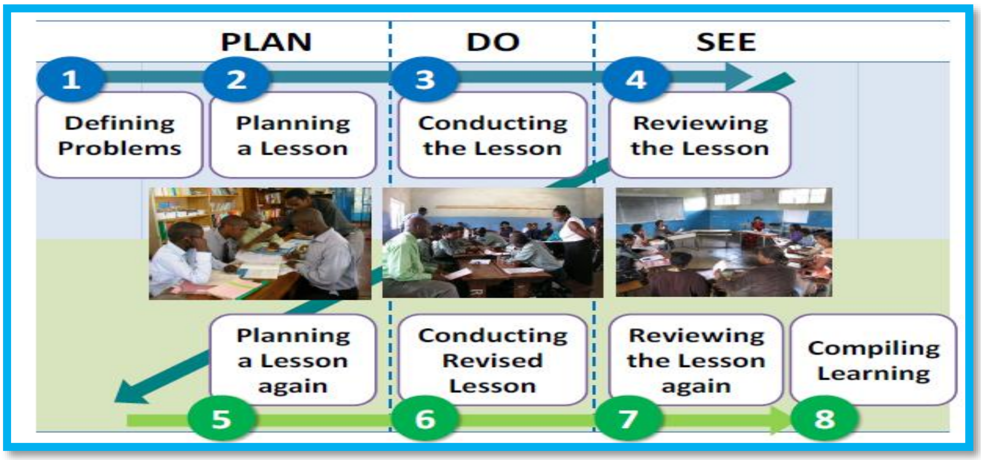
Zambia is one of the first countries in Africa to incept LS in their education system (Banda et al., 2014). In Zambia, the Lesson Study started in 2005 (Banda et al., 2014). It appears that the programme was launched nationally with the help of the Japan International Cooperation Agency (JICA). Like other countries, Zambia contextualised Lesson Study in its classrooms, starting with Mathematics and Sciences. The country's model has eight stages which are:

- Defining problems
- Planning a lesson
- Conducting the lesson
- Reviewing the lesson
- Planning a lesson again
- Conducting revised lesson
- Reviewing the lesson again
- Compiling learning

These eight stages are executed within the broad framework of Plan (stages 1, 2 and 5); Do (stages 3 and 6); and See (stages 4, 7 and 8) as shown in Figure 2.6.

Figure 2.6

The Zambian Lesson Study Model



Note: Source: Zambian Ministry of General Education & JICA, 2016

The categories of Plan, Do and See seem to respectively correspond to the Singaporean model consisting of the planning session, research lesson, and post-discussion session. Eventually, the eight stages concurred with those of the Japanese overarching process. However, in the Zambian model, it is not explicit if stages 6 to 8 are optional. In the South African LS model, however, re-teaching the improved or revised lesson is optional.

2.3.7 Lesson Study in Malawi

Lesson Study was introduced in Malawi in 2016 by the Japan International Cooperation Agency (JICA) through the University of Fukui (Kosaka et al., 2021). The LS practice is still at its infants according to (Kosaka et al., 2021). Malawi adopted the Zambian model with minor changes:(1) Defining Problems; (2) Planning a lesson(collaborative planning); (3) Conducting the lesson(first demo lesson); (4) Reviewing the lesson; (5) Planning a lesson again; (6) Conducting revised lesson(second demo lesson); (7) Reviewing the lesson again; and (8) Compiling learning (Kosaka et al., 2021, p. 238). The minor changes or adaptations by Malawi are indicated in brackets for stages (2), (3), and (6). However, it is not known how Malawi applies the Plan, Do, and See categories practised by Zambia. The Malawian model is interesting in that it demonstrates the importance and success of collaboration and adaptations amongst African countries (Malawi and Zambia) as well as overseas countries (Malawi and Japan). It would be a step ahead for African countries to form their Lesson Study association.

2.4 LESSON STUDY IN SOUTH AFRICA

Lesson Study (LS) in South Africa started in 1999 in the form of a project called Mpumalanga Secondary Science Initiative (MSSI) that was collaboratively manned by JICA, the Mpumalanga Department of Education (MDoE) and the University of Pretoria (Ono & Ferreira, 2010). The project was launched in Mpumalanga because of the poor quality of teaching, and in turn, the poor performance of Mathematics and Science learners. The project, which was school-based, targeted Mathematics and Science educators in secondary schools in a bid to improve the quality of teaching in these subjects. This sought to enhance teaching skills and subject knowledge.

The project started with training subject advisors, followed by lead teachers and Heads of Departments (HODs)/Departmental Heads (DHs). These officials, in turn, introduced LS to teachers through workshops. According to Ono and Ferreira (2010) evaluation, the project did not yield anticipated results since it clashed with educational policies and educators' teaching loads. A few schools in Mpumalanga practised LS only for a short time after its introduction after which it experienced a brief hiatus. Seemingly, the inception of the LS in South Africa was confined to one province. I hold the notion that trying it in several provinces could have seen it thrive better since each province has different insights, resources, demographics, and economic statuses.

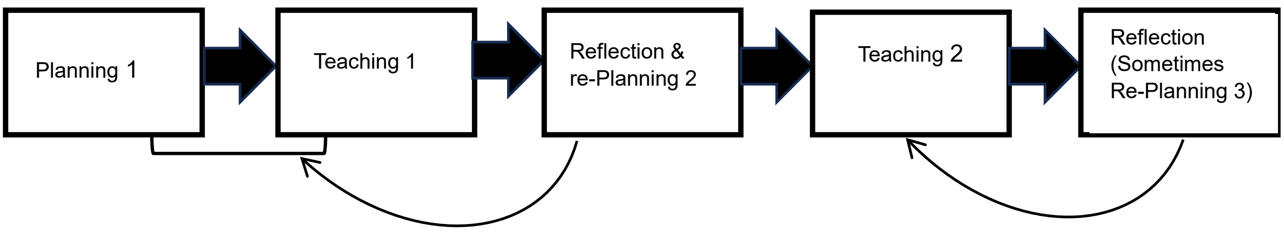
In South Africa, there are also various adaptations that have been reported by researchers and most of them seem to be at a fluid level. Bayaga (2013, p. 12) reports on one of the models adopted and adapted in South Africa that involves the following steps: "(a) formulating learning goals or objectives (b) designing the research lesson

(c) designing the study (d) teaching and observing the research lesson (e) analysing the evidence (f) repeating the process (g) documenting the lesson study."

Adler and Alshwaikh (2019) also researched in the South African context being guided by the model shown in Figure 2.5.

Figure 2.7

Lesson Study Model Used In Adler And Alshwaikh Project



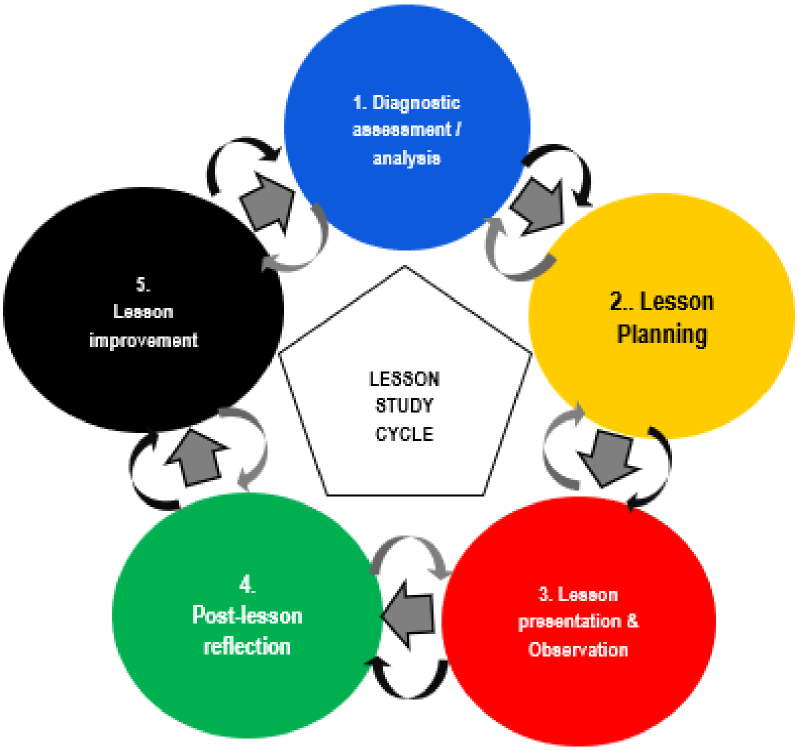
Note: Source - Adler and Alshwaikh (2019)

This model was confined to Mathematics, and it emphasised exemplification in teaching the subject. Two teaching (lesson presentation) stages exist in the cycle. Educators from a cluster in Gauteng province participated in the project that was conducted after school hours. The results of the project reveal that LS offered educators a better opportunity to learn the teaching of arithmetic and algebra through exemplifying.

There is another LS model that is presided over by the University of Pretoria which I presume to be the most popular South African model because most schools are implementing it (Sekao, 2023) (see Figure 2.6).

Figure 2.8

The South African Lesson Study Model



Note: Source - Sekao and Engelbrecht (2021)

The model has five stages which are: diagnostic analysis; collaborative lesson planning; lesson presentation and observation; post lesson reflection, and lesson improvement (Sekao & Engelbrecht, 2021). This model, to which my study is confined, is unique in that it starts with the diagnostic analysis stage. Furthermore, my exploration considered only three stages of this model which are: teachers' development of mathematical knowledge for teaching trigonometric functions in collaborative lesson planning, lesson presentation and observations, and post-lesson reflection. I did this as a way of delimiting the study and because of the fact that the three stages are the most relevant to my current study. Further elaborations on how these three stages link and relate to teachers' development of mathematical knowledge for teaching trigonometric functions through LS are found in the theoretical framework (see Chapter 3, Section 3.2). I shall go on to discuss the South African model stage by stage.

2.4.1 Stage 1 -Diagnostic Assessment/Analysis

Diagnostic assessment/analysis is stage 1 of the South African LS model, that uniquely identifies it (Sekao, 2023). Diagnostic assessment/analysis aims to identify learners' strengths and weaknesses (content gaps, common misconceptions and errors) in one or more topics, thereby informing intervention and or remediation measures that will result in conceptual understanding (Alfageh et al., 2024; Kanwal & Farooq, 2021; Ketterlin-Geller & Yovanoff, 2019; Mjenda et al., 2023; Sekao, 2023; Talosig, 2023). In the South African mathematics education system, there are formal and informal assessments, where diagnostic analysis could be applied (Ketterlin-Geller & Yovanoff, 2019). Formal assessment comprises end of term tests, mid-year, and end of year examinations. End of term tests are usually school-based assessments (SBAs) whereas mid-year and end of year examinations could be externally set by provincial and/or national education departments. The South African mathematics education system employs formal testing for end of term and end of year reporting. These formal assessments seem to be affiliated to summative assessment. Informal testing (which is aligned to formative assessment) takes the form of class tests. Sekao (2023) notes that teachers in a LS team/group can come up with a problematic topic through diagnostically analysing such formal and informal assessments. This is supported by Reinhold (2018) who argues that analysing learner by learner enables teachers to identify misconceptions and errors thereby developing knowledge of content and students (KCS), which seems to be incorporated in all LS stages. In support, Sekao (2023) explains that such a practice facilitates the formation of research lesson objectives during planning.

Trigonometric functions were considered for the current study as a problematic topic following revelations in the Department of Basic Education (DBE) National Senior Certificate (NSC) Diagnostic reports. These NSC diagnostic reports portray learners' persistent common misconceptions and errors (Ketterlin-Geller & Yovanoff, 2019) in trigonometric functions, where the average performance has dropped from 42% in 2015 to 34% in 2023. The main purpose of diagnostic assessment is to inform intervention and consequently planning. Intervention needs to be planned for, favourably in collaborative terms/environments synonymous with LS. The diagnostic assessment stage in the South African LS model is therefore unique and relevant in that proper intervention could only be informed by detailed diagnostic reports such as the DBE NSC diagnostic report(s). Teachers planning in LS do so with learners' misconceptions and errors in mind (Sekao, 2023). This concurs with the notion of Wongwatkit et al. (2020) that diagnostic assessments enable teachers to develop relevant learning materials during planning, a process that could promote learners' understanding of the subject matter.

2.4.2 Stage 2 - Collaborative Lesson Planning

Lesson planning or a lesson plan (known as *gakushushido-an* in Japan) is a pre-requisite for teaching and it is a key component of the LS processes (Sekao, 2023).

Lesson planning entails having: a topic; lesson objectives; instructional materials (*kyouzai-kenkyuu*) (such as textbooks, and curriculum policy documents, etc); teaching and learning activities (including assessment such as classwork and homework) to be executed in the classroom during lesson time (Cevikbas et al., 2023; Enama, 2021; Fujii, 2019; Lee & Takahashi, 2011; Sekao, 2023). In the South African LS model context, collaborative lesson planning is perceived as constituting Stage 2. Collaborative lesson planning is characterised by the continuous sharing of ideas and approaches by teachers to mould research lessons that promote conceptual understanding in learners (Caven et al., 2012; Sekao, 2023). According to Sekao (2023), studying curriculum materials (*kyouzai-kenkyuu* in Japanese) such as textbooks, manipulatives, and software is key to collaborative lesson planning and educative for teachers. From my synthesis of literature and experience in the teaching field, I find lesson planning serving as a spiral nexus between the teachers' CK/SMK and PCK. Indeed, it serves as a foundation for lesson presentation and observation; it informs and mirrors what is going to happen during the lesson execution session, and it serves as a template (observation criteria) for teachers observing the research lesson being taught (Isoda et al., 2007; Kuleshova, 2020; Sekao, 2023; Stigler & Hiebert, 2009; Sudejamnong et al., 2014). The teachers observing are guided by the objective(s) coined during collaborative lesson planning (Sekao, 2023). Again, the collaborative LS group(s) use planning as a basis/background for post-lesson reflection when they discuss the successes and challenges of what transpired in the classroom, considering what would have been planned (Eshchar-Netz & Vedder-Weiss, 2021). Furthermore, the discussion of lesson improvement (which is Stage 5 of the South African LS model) also refers to previous planning. It therefore follows that planning is threaded through all the LS stages of the South African model. This is in line with Clivaz and Ni Shuilleabhain (2019) who assert that LS stages overlap into one another. Similarly, Fujii (2019) posits that each stage in LS is closely linked to other stages in a cycle.

Eshchar-Netz and Vedder-Weiss (2021) report that collaborative lesson planning creates mentorship relationships between senior and novice teachers. Perhaps this mentorship is possible in planning trigonometric functions lessons in South African schools where teachers are a mixture of newly qualified and seasoned ones. Another finding is that collaborative lesson planning is characterised by pedagogical reasoning (Eshchar-Netz & Vedder-Weiss, 2021). Futter and Staub (2008) study revealed that collaboratively planned lessons were of better instructional quality than those planned individually. Bolívar and Ortiz (2017) study found that collaborative lesson planning provided environments where teachers improved their practices by discovering their strengths and weaknesses in content and pedagogy. The findings in Gutierez (2019) qualitative study were that there was content and pedagogical scaffolding during collaborative lesson planning that resulted in improved teaching skills and professional relationships amongst participating teachers. It posited that planning should consider learners' prior or assumed knowledge and connect content with real-world experiences for their understanding, rather than for the sake of covering the curriculum (Roussouw et al., 1998; Tatira, 2021). In a similar vein, Gholami (2022b) asserts that proper linking of prior knowledge and new topics helps the process of learning that promotes the understanding of mathematical concepts. This entails that planning should be learner-centred. Enama (2021) notes that lesson planning considers assumed/prior knowledge, class activities and their sequencing. This literature under discussion shows that collaborative lesson planning of trigonometric functions has the potential to offer teachers an opportunity to develop mathematical knowledge for teaching the topic. In fact, Sekao (2023) asserts that "the collaborative lesson planning process is a platform for in-depth teacher learning". Eshchar-Netz and Vedder-Weiss (2021) report that there is a dearth of studies that focus on the benefits of collaborative lesson planning. It, therefore, follows that my exploration of collaborative lesson planning could contribute towards closing that existential gap in the literature.

2.4.3 Stage 3 - Lesson Presentation and Observation

Lesson presentation and observation is the third stage of the South African LS model. Literature at this stage assisted me in gaining some insights into my second sub-question, namely, How does teachers' mathematical knowledge for teaching trigonometric functions evolve during the research lesson presentation? At this stage, one teacher from the LS team presents the lesson that was collaboratively planned by the group while other teachers observe. The observation involves moving around and taking down notes on learners' oral and written responses (Stigler & Hiebert, 2009). According to Fujii (2019), lesson presentation and observation is where the collaborative lesson planning input is manifested. This is a stage where teachers observe how learners respond to their planned objectives and/or activities, for instance, in trigonometric functions. Fujii (2019) and Sekao (2023) hold that other teachers observe the teacher presenting and collect data (or record findings) of what is happening during the session. These findings become the basis of post-lesson reflection. Upa et al. (2023) report that the observing teachers grow/learn/develop as they observe the presenting teacher.

2.4.4 Stage 4 – Post-lesson reflection

The fourth stage of the South African LS model is post-lesson reflection which coincides with my third sub-question: 'How do teachers' reflective practices hone their mathematical knowledge for teaching trigonometric functions?' Reflection is a process that allows teachers to discuss learners' learning, the presenting teacher to realise his strengths and weaknesses from the observers' comments, and the whole group to identify its lesson planning strengths and weaknesses, which, in turn, paves the way for improvement (Sekao, 2023; Upa et al., 2023). Kager et al. (2022) report that post-lesson reflection is fundamental in LS, where teachers need to act and think as a group while systematically questioning actions and situations that were observed during lesson presentation and observation. The discussions at the stage spontaneously generate ways of improving practices in the teaching of the topic (in terms of planning; sequencing of content, concepts, and objectives; assigning and sequencing of classwork and homework activities).

Moumou (2021) conducted a study that sought to establish the effects of reflection on teachers' MKT sub-domains. The researcher used a mixed-methods approach that involved 17 primary school mathematics teachers. Like in my current study, Moumou (2021) employed the interpretivist paradigm and analysed the qualitative data thematically. The findings were that reflection had a positive influence on the development of MKT sub-domains. From the continuing literature review, enhancing teachers' MKT sub-domains is directly proportional to learners' achievement. I anticipate post-lesson reflection by teachers participating in my study to hone their mathematical knowledge for teaching trigonometric functions. In line with this, critical and collaborative post-lesson reflection generates new knowledge based on practice (Perry & Lewis, 2011; Richit et al., 2022). Again, post-lesson reflection is an essential ingredient in teachers' professional growth, and it focuses on what transpired in the lesson and the reasons for it happening that way (Brant, 2006; Richit & Tomkelski, 2022; Upa et al., 2023). Furthermore, Brant (2006) and Upa et al. (2023) agree that there are very few studies that have focused on post-lesson reflection. This means that my current study serves to bolster and enrich literature in post-lesson reflection practices and research.

2.4.5 Stage 5 - Lesson Improvement

Lesson improvement is the last stage in the South African LS model. This stage is informed by or is based on collaborative post-lesson reflection discussions, lesson presentations and observations, and collaborative lesson planning (Sekao, 2023; Stigler & Hiebert, 2009). Improvements could involve realigning and/or altering objectives, sifting and/or enriching purposeful activities (classwork and homework), changing curriculum materials, and revisiting the sequencing of concepts (Stigler & Hiebert, 2009). As Sekao and Engelbrecht (2021) put it, the process also involves generating a comprehensive report that seeks to help practising teachers. My current research is not exploring this stage. However, the outcomes of lesson improvement could be unavoidably portrayed to the reader (the practising teacher) in earlier stages that were dealt with. This indicates that the LS stages are interlocking or overlapping.

2.5 TEACHING AND LEARNING OF TRIGONOMETRIC FUNCTIONS

2.5.1 Definition and Curricula of Trigonometric Functions

Trigonometric functions (also known as circular functions) are functions that give the relationship between the angles and sides of a triangle (Buyjus, 2022). Cuemath (2022, p. 1) defines trigonometric functions as “the basic functions (sine, cosine, tangent, cotangent, secant, and cosecant) that have a domain input value as an angle of a right triangle, and a numeric answer as the range”. For instance, the trigonometric function $f(x)=\sin\theta$ has a domain which is the angle θ given in degrees or radians, and a range of [-1;1] (Cuemath, 2022). Trigonometric functions are a branch of trigonometry which comprises the following: angle measures; trigonometric functions; trigonometric identities; inverse trigonometric functions; trigonometric equations; and solutions of triangles (Ancheta & Mark, 2022). Trigonometric functions are found in secondary schools’ and tertiary institutions’ curricula. In the South African context at the secondary school level, trigonometric functions are stipulated according to grades in the Curriculum and Assessment Policy Statement (CAPS) Grades 10-12 (DBE, 2011). The trigonometric functions are taught in Grades 10 and 11 and examined in the final Grade 12 examination. The terms ‘functions’ and ‘graphs’ are used interchangeably in the CAPS curriculum, and I am using them in the same manner in my study. At Grade 10, learners should firstly be able to, point by point, plot basic graphs defined by $y=\sin x$, $y=\cos x$, $y=\tan x$ for $x\in\left[0^\circ;360^\circ\right]$. Secondly, learners study the effects of parameters a and q in the graphs defined by $y=a\sin x+q$; $y=a\cos x+q$; and $y=a\tan x+q$ where a , $q\in Q$ for $x\in\left[0^\circ;360^\circ\right]$. Thirdly, Grade 10s are expected to draw and explicate these functions and find the equations of already drawn functions. Grade 11 learners are expected to plot basic graphs like Grade 10s, except that the domain changes to $x \in \left[-360^{\circ};360^{\circ}\right]$. Next, Grade 11s should be able to investigate the effects of parameters k (on $y = \sin kx$; $y = \cos kx$; and $y = \tan kx$) and p [on $y = \sin(x+p)$; $y = \cos(x+p)$; and $y = \tan(x+p)$]. Lastly, learners in Grade 11 should be able to sketch graphs defined by $y = a\sin k(x+p)$; $y = a\cos k(x+p)$; and $y = a\tan k(x+p)$ considering at most two parameters at a time.

Understanding trigonometric functions is considered important for further studies in higher-level topics such as calculus when it comes to limits, derivatives, and integrals (Gholami, 2022b; Maknun et al., 2019). Likewise, it is also acknowledged that an understanding and application of trigonometric functions is required in careers in biology, physics, astronomy, computer graphics, optics, technology, and many branches of engineering (Carlsen, 2018; Hertel & Cullen, 2011; Maula & Nu'man, 2021; Nanmumpuni & Retnawati, 2021; Nizeyimana et al., 2023). From my experience, the South African youth has a desire for such careers. Again, both departments of Basic Education and Higher Education in South Africa demonstrate bias towards these fields by investing more focus and funds on Mathematics and Sciences.

2.5.2 Effective Teaching of Trigonometric Functions

There is scarcity of research on the strategies of teaching trigonometric functions (Zeng, 2019). The synthesis of literature that I have done identified the following ways of teaching trigonometric functions: a combination of unit circle and technology; and approaches that include, amongst other things, knowledge of mathematics.

Ignacio (2022) conducted a study that used GeoGebra to simulate the unit circle as a strategy to teach the concept of trigonometric functions to 66 grade 10 learners (33 experimental group and 33 base group). Ignacio (2022) started by diagnosing learners' prior knowledge through a test. The experimental group was then engaged in hands-on simulation processes while the other group was taught using the traditional method. The traditional method could entail the lecture or the chalk-and-talk method. The findings showed that, the performance of learners in the base group was lower than of those in the experimental group; the use of GeoGebra afforded learners an understanding of trigonometric functions through the visualisation of concepts; learners learnt better through experience and reflecting on their use of GeoGebra, and that the teaching strategy fostered geometric and autonomous thinking amongst learners. Li et al. (2022) used a research and development method to investigate how Hawgent dynamic mathematics software could be used in the teaching and learning of trigonometric functions by teachers and learners. Hawgent dynamic mathematics software was developed by Guangzhou Hawgent Education Technology Co.Ltd. and has multiple and dynamic mathematical resources suited for use in the teaching of trigonometric functions and provision of interactive environments for learners (Li et al., 2022). One class of learners participated in the study that was a month long. The purpose of the study was to teach concepts, images, and properties of trigonometric functions. Like Ignacio (2022), Li et al. (2022) used the unit circle approach to pursue their objective. The study found that the software improved learners' understanding and interest in trigonometric functions. Another study by Loce (2021) also employed the unit circle method using Desmos software in probing learners' understanding of trigonometric functions. Loce (2021) found that Desmos software developed learners' understanding of concepts and transformations of trigonometric functions through manipulation and visualisation.

Magno (2022) perceives trigonometric functions as functions that originate from trigonometry and are important because of their use in the scientific industries. However, learners seem not to be aware of the modern use of trigonometric functions (Magno, 2022). This prompted Magno (2022) to come up with a way of teaching trigonometric functions that blended modelling of real-life situations and integration of GeoGebra. The researcher anticipated that the approach would help learners to visualise and understand trigonometric functions. Magno (2022) started by ascertaining learners' prior knowledge and possible content gaps using a questionnaire. Recordings of temperature for a local airport and heights/movements of water waves/tides for a nearby sea were used to model trigonometric functions in the study. The temperature variations for 2019 and 2020 depicted a cosine graph whose maximum, minimum, amplitude, period could be visualised by learners. The approximate values of the parameters were fed into GeoGebra, and learners used a slider to vary the parameters, observing the behaviour of the function. Weather information in terms of temperatures is important for both international and local travellers. The study also modelled recordings of tidal movements from the Tide Table for March 2019, which pictured the period and amplitude concepts associated with the cosine graph. Again, learners varied parameters (heights/amplitudes and periods of waves) using GeoGebra, noting the overall behaviour of the trigonometric function. Knowledge about tides is economically important for sailors and beach goers. Magno

(2022) concluded that modelling of real-life phenomenon and use of GeoGebra motivated learners and enhanced their understanding of trigonometric functions. In concurrence with Magno (2022) findings, Tatira (2021) notes that the use of real-world (or real-life) examples in the teaching and learning of trigonometric functions lessen the abstraction of the topic. The concepts (maximum, minimum, amplitude, and period) explored in Magno (2022) study also feature in the South African Grade 11 trigonometric functions syllabus (CAPS). However, in Magno (2022) study there has been no exploration of the concepts domain and range of trigonometric functions that my current study encompasses. Magno (2022) further reports that he developed new teaching skills (pedagogical knowledge) by conducting this study. Possibly, teachers participating in my study, and I could similarly benefit.

Two interesting studies separately examined teachers' pedagogical content knowledge (PCK) and content knowledge (CK) in trigonometric functions. Tatira (2021) investigated a case of three pre-service final year student teachers' PCK skills in the teaching of trigonometric functions. Tatira (2021) employed the Knowledge Quartet framework by Rowland et al. (2005) that helped him to sort the elements of PCK for each teacher. Four dimensions, (namely; foundation, transformation, connection and contingency) of the Knowledge Quartet framework with their corresponding codes were used in observations and analysis of video-recorded lessons (Tatira, 2021). The findings were that the teachers had basic PCK skills for trigonometric functions but displayed challenges in responding to prompt questions learners raised during the lessons. For instance, one teacher could not respond to a learner's question that sought clarification on why the tangent graph had a period of 180° (Tatira, 2021). While Tatira (2021) concentrated on PCK, Ogbonnaya and Mogari (2014) investigated the relationship between teachers' CK and Grade 11 learners' performance in trigonometric functions, which they found to be positively correlated. In essence, Ogbonnaya and Mogari (2014) found that learners who had been taught trigonometric functions by teachers with high CK in the topic performed better than those who were taught by teachers who struggled with the topic's CK. My current study attempts to close the gap in the literature by considering teachers' development of both CK and PCK in the teaching of trigonometric functions through LS.

From the above discussion in this section, it seems recent research on the teaching and learning of trigonometric functions thrives on the unit circle approach and /or use of technology. Zeng (2019) posits that the unit cycle method is advantageous in that it links the concepts of trigonometric functions with all quadrants. However, Zeng (2019) and Tatira (2021) agree that the method has limitations in the introduction of trigonometric functions since it does not facilitate the learners' conceptualisation of functions. Mosese and Ogbonnaya (2021) perceive teaching trigonometric functions using GeoGebra as effective in enhancing learners' understanding of the representation and interpretation of the functions.

Lesson Study, trigonometric functions, and technology

Stigler and Hiebert (1999) assert that collaboration is a natural element of lesson study. In the same vein, the US-based Partnership for 21st Century Learning cited in (Joynes et al., 2019) contends that the four Cs (Critical thinking, Communication, Collaboration and Creativity) are fundamental skills that should be cultivated in the teaching of core subjects (like mathematics in this current situation). The use of technology as a tool in teaching and learning is strongly associated with 21st Century Skills since it is attributed to collaboration, communication and sharing, creation of content and knowledge, evaluation and problem-solving (Voogt and Roblin, 2010 and Van Laar et al, 2017 both cited in Joynes et al. (2019)). From the ongoing discussion, collaboration is a thread that goes through LS, 21st-century skills, and the use of technology. Perhaps LS and technology are compatible.

Van Ryneveld (2016) insists that educators are required to be aware of the potential of various technologies and be able to integrate them into teaching and learning. I anticipate that teachers would learn to integrate technology in the LS context better than in a workshop set-up. Karavakou and Kynigos (2019) are recommending further research into the potentialities of using technology in the teaching and learning of trigonometric functions. I hold the notion that integrating technology in the context of LS would bring transformation in the teaching and learning of trigonometric functions in South Africa and the world over. It therefore follows that further studies in this area are recommended.

2.5.3 Learners' Conceptual Difficulties in Trigonometric Functions

In general, learning and teaching are two sides of the same coin. This section is driven by the assertion that learners' conceptual difficulties, misconceptions and /or errors in a topic imply challenges in teaching approaches (Sekao, 2023). In addition, Ogbonnaya and Mogari (2014) report that there is a strong relationship between learners' achievement and teachers' content knowledge in trigonometric functions. While trigonometric functions are important in the mathematics curriculum, many scholars and researchers agree that trigonometric functions are problematic to learners, and novice pre- and in-service teachers (Demir & Heck, 2013; Gholami, 2022b; Maknun et al., 2019; Martín-Fernández et al., 2019; Nizeyimana et al., 2023; Santiago, 2024; Sintema, 2020; Tatira, 2021; Weber, 2005; Zeng, 2019). Gholami (2022b) reports that misunderstanding and misconceptions are caused by teachers' use of traditional methods that promote the memorisation of trigonometric concepts by learners. These findings concur with the current situation in South African classrooms where teachers and learners are struggling with the topic (de Villiers & Jugmohan, 2012; Mosese & Ogbonnaya, 2021; Naidoo & Govender, 2014; Tatira, 2021). Mosese and Ogbonnaya (2021) found that learners had difficulties in representing and interpreting trigonometric functions. Zeng (2019) argues that the integration of concepts in algebra, geometry and number skills that characterise trigonometric functions poses understanding difficulties to learners. For instance, learners are confused whilst trying to relate $\cos\theta = \frac{adjacent}{hypothenuse}$ (that gives a numerical value introduced in earlier grades) to $y = \cos\theta$ (that is a trigonometric graph) (Zeng, 2019).

From my analysis and synthesis of the Department of Basic Education (DBE) Diagnostic reports from 2011 to 2023, the most common errors and misconceptions in trigonometric functions revolve around drawing (or sketching) and shape, amplitude, domain, range, period, maximum and minimum values, description of transformations, interpretation of graphs, and effects of parameters (DBE, 2011c, 2012, 2013, 2014, 2015a, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023). It is important to explore learners' misconceptions since this provides LS teams and teachers in general with a rich background for preparing and presenting suitable research lessons (Gholami, 2022b). From my experience in teaching graphs or functions, learners are introduced to drawing with a calculator to generate a table of values that enables point-by-point plotting. After mastering point-by-point plotting, learners are expected to be able to sketch graphs or functions showing the relevant shape and key points like the turning points, intercepts and or asymptotes. The 2017 DBE Diagnostic report states that some learners joined points with a ruler when drawing $y = 2\sin x$ (DBE, 2017). Learners should join these points with the features of the function in mind. In alignment with this, learners in the Munyaruhengeri et al. (2022) study viewed the point-by-point drawing of graphs as a long and laborious method. At the same time, most teachers in the study indulged in the rough sketching method, which is memory-based (Munyaruhengeri et al., 2022). Rough sketching is drawing a graph or function based on its features and not on accurate points or coordinates. From Munyaruhengeri et al. (2022) view, it seems the point-by-point method fosters the understanding of graphs better compared to rough sketching. Based on my experience in teaching graphs or functions, I concur with Munyaruhengeri et al. (2022) in that learners have to be introduced to the drawing of graphs or functions by plotting points from a table of values. It is after familiarising with the features of a graph or function (through plotting) that the learners may start to sketch.

It has also been noted that some learners confuse the period of the function or graph with the domain while others confuse the period with the amplitude of trigonometric graphs (DBE, 2013, 2020, 2022, 2023). Malambo (2021) study reports similar findings where one participant gave the range and the period of the sine function as ‘0,1,-1’ and 180 while the other stated them as ‘$0^\circ \le \theta \le 360^\circ$’ and ‘from -1 to 1’ respectively. Furthermore, Fahrudin and Pramudya (2019) reported that learners in their study had difficulties understanding the concept of periodicity when they gave the period of

$y = \sin(2x - 10^\circ)$ as 360°. The trigonometric function $\sin(2x - 10^\circ)$ depicts parameters k and p that are found in the South African curriculum. Kamber and Takaci (2018) found learners' challenges in periodicity being caused by the fact that trigonometric functions, unlike algebraic functions that learners knew as prior knowledge, were not one-to-one. Learners, therefore lacked the understanding of the relationship between the radian measure and the unit circle. The concept amplitude also appears to be problematic since learners gave it as $-\frac{1}{2}$ instead of $\frac{1}{2}$ in the 2020 final examination (DBE, 2020). Tuna (2013) reports that such misconceptions occur because teachers incorrectly explain definitions of concepts such as amplitude, to learners.

Learners often struggle to describe, interpret, and identify given functions and their transformations to the extent of being unable to differentiate between a sine and a cosine graph (DBE, 2012, 2013, 2015a, 2015b). Furthermore, Malambo (2021) reports that participants revealed patent misconceptions involving the properties of sine and cosine functions. The participants confused domains of trigonometric graphs to extreme values of the graphs and they evaluated inverses of trigonometric functions just like indices (Malambo, 2021). Examiners in South Africa also report that learners were unable to draw accurate graphs in terms of shape, turning points, x-intercepts, and given range (DBE, 2014, 2016, 2021). Another interesting diagnosis is that the tangent graph is problematic to learners since they have difficulty establishing the effects of both negative and positive parameters on the graph (DBE, 2018). Subsequently, a study Munyaruhengeri et al. (2022) conducted in Rwanda found that learners had difficulties in sketching the tangent graph compared to the sine and cosine graphs.

There is also a tendency to confuse maximum value with range where learners give the former as an interval instead of a single value. Gholami (2022b) conducted a study on the teaching of maximum and minimum values of trigonometric functions. The main finding was that discussing a variety of methods to solve problems in maximum and minimum values of trigonometry helped learners improve their problem-solving abilities. Furthermore, the study by Gholami (2022b) found that both learners and teachers developed a skill of generalising trigonometric function concepts. While still on maximum and minimum values of trigonometric functions, Malambo (2021) found that undergraduates gave π and 2π as the sine function's minimum and maximum values, respectively. Another participant in Malambo (2021) study took $+360^\circ$ as the maximum value for the cosine graph.

In a study by Ancheta and Mark (2022) that involved second year university students (10 electrical, 10 mechanical and 30 civil engineering), it was found that 64% had misconceptions about trigonometric functions that were attributed to students' attitudes toward listening and students' poor knowledge retention. These findings by Ancheta and Mark (2022) are more aligned to psychology. Perhaps, it would be important to conduct such a study in South Africa to enrich the body of knowledge in the teaching and learning of trigonometric functions at secondary schools. Mulyono and Hapizah (2021) conducted research where students' answers to questions on trigonometric functions were observed, focusing on symbols in mathematical expressions. The findings were that writing mathematical expressions without considering relevant symbols resulted in errors and misunderstanding of trigonometric functions. Mulyono and Hapizah (2021) research is interesting in that trigonometric functions in the South African Curriculum Assessment Policy Statement (CAPS) (Grades 10 to 12) are characterised by symbols (or notations) such as [ ]; ( ); <.>; $\leq, \geq$; $\in$. In the CAPS, these symbols are aligned to range, domain, intervals, and regions of interest. It is therefore important for teachers to properly address these symbols when teaching trigonometric functions.

The discussion of learners' conceptual difficulties in trigonometric functions in this section reveals that the difficulties emanate from misconceptions and errors associated with: teaching approaches and or methods; drawing and or sketching; concepts such as amplitude, domain, range, period, maximum and minimum values; description of transformations, interpretation of graphs, and effects of parameters; symbols or notation, and attitudes and poor knowledge retention. Researchers posit that most misconceptions about trigonometric functions could be addressed by learning using technology (de Villiers & Jugmohan, 2012; Hamzah et al., 2021; Kamber & Takaci, 2018; Magno, 2022; Makandidze, 2020; Mosese & Ogbonnaya, 2021; Mulyono & Hapizah, 2021). I hold the notion that LS processes could serve as effective intervention measures since LS is learner-centred and has the potential to sharpen teachers' content (CK) and teaching skills (PCK). In the next section, I discuss teachers' perspectives on the use of LS in the teaching of mathematics.

2.5.4 Observed attempt by South Africa to address the challenges in trigonometric functions

This section is based on my observations and experiences as a mathematics practitioner in the FET band. When scanning through the Grades 10 and 11 ATPs from 2018 to 2024, I observed varying teaching times allocated to trigonometric functions (DBE, 2018-2024) as shown in Table 2.1.

Table 2.1

Teaching Time Allocated To Trigonometric Functions From 2018 To 2024.

Year	2018	2019	2020	2021	2022	2023	2024
Days allocated to Grade 10	5	5	9	10	10	9	9
Days allocated to Grade 11	5	5	5	10	13	14	14

Note: Extracted from DBE-GDE ATPs from 2018 to 2024

The South African mathematics education's (or the curriculum developers') increase in teaching time allocated could have been an anticipated solution to the poor performance. It could be that the system thought increasing contact time would improve performance. However, from my observations before and after the time increase, teachers did not change their teaching approach. These teachers continue to cover the topic in two to three days where learners' written work would be characterised by point-by-point dot plots of the basic graphs of sine, cosine, and tangent. They could do this just to register in the learners' exercise books that they covered the topic. I suggest that these curriculum developers should be informed by thorough research concerning the contact time for trigonometric functions. Classroom-based research should investigate how the topic could be broken down and sequenced in terms of aspects and/or concepts. Teachers are attending JITT which covers trigonometric functions every year, but their teaching approach has not changed. It follows that both JITT and the increase in contact time are not making a positive impact on the teaching and learning of trigonometric functions. In my scrutiny of NSC examinations from 2011 to June 2024, I observed that the level of difficulty of trigonometric functions questions is taking a downward trend. The June 2024 is mostly made up of cognitive level 1 questions. Seemingly this strategy again by the examiners does not boost the average pass in trigonometric functions as portrayed by DBE NSC Diagnostic reports. These observations suggest that there is a challenge in the classroom in terms of teaching and learning of trigonometric functions. Hence the need for research like my current one.

2.6 TEACHERS' PERSPECTIVES ON THE USE OF LESSON STUDY IN THE TEACHING OF MATHEMATICS

Understanding the affordances of LS from participating teachers' perspectives has the potential to lure other teachers and convince stakeholders to support the practice (Druken, 2023; Fernandez, 2005). However, Druken (2023) reports that such studies on teachers' perspectives on LS are scarce. Druken (2023) asserts that instilling LS in a country requires resources and support from the education ministry at large. LS is still in its infancy in South Africa, hence there is a need to mobilise support from the Department of Basic Education (DBE) in the form of resources (to train teachers) and time. I hold that this could be achieved through spoken experiences by participating teachers.

In her qualitative study in the USA, Druken (2023) interviewed 33 teachers who participated in an LS involving the teaching of algebra to find out their perspectives on the purposes of mathematics LS. The teachers stated the following as purposes for engaging in mathematics LS: understanding syllabus requirements, appreciating learner-centredness, developing pedagogic skills, cultivating collaboration, developing curricular materials, and gaining content. Participants in the study said that they had learnt and gained confidence in teaching the new aspects of the syllabus through actively engaging with one another in LS rather than just passively reading them. Twenty-seven (27) out of thirty-three (33) teachers in Druken (2023) study viewed LS as learner-centred since it focused on anticipating, observing, analysing, and improving learner thinking. 70% of these teachers reported that they developed pedagogic skills through sharing instructional strategies in algebra. Teachers also reported that discussing challenges and sharing views within LS had cultivated collaboration and professional growth among them. It also emerged that teachers valued the production of curricular materials in LS in the form of refined lesson plans. The teachers went on to reveal that LS had enabled them to grow their subject matter of algebra through learning material and abstractions. It is possible that teachers participating in my current study could have similar perspectives on trigonometric functions.

Pre-service teachers in Fernandez and Robinson (2006) study viewed LS as useful in putting theory into practice, promoting collaboration and reflection. Richit and da Ponte

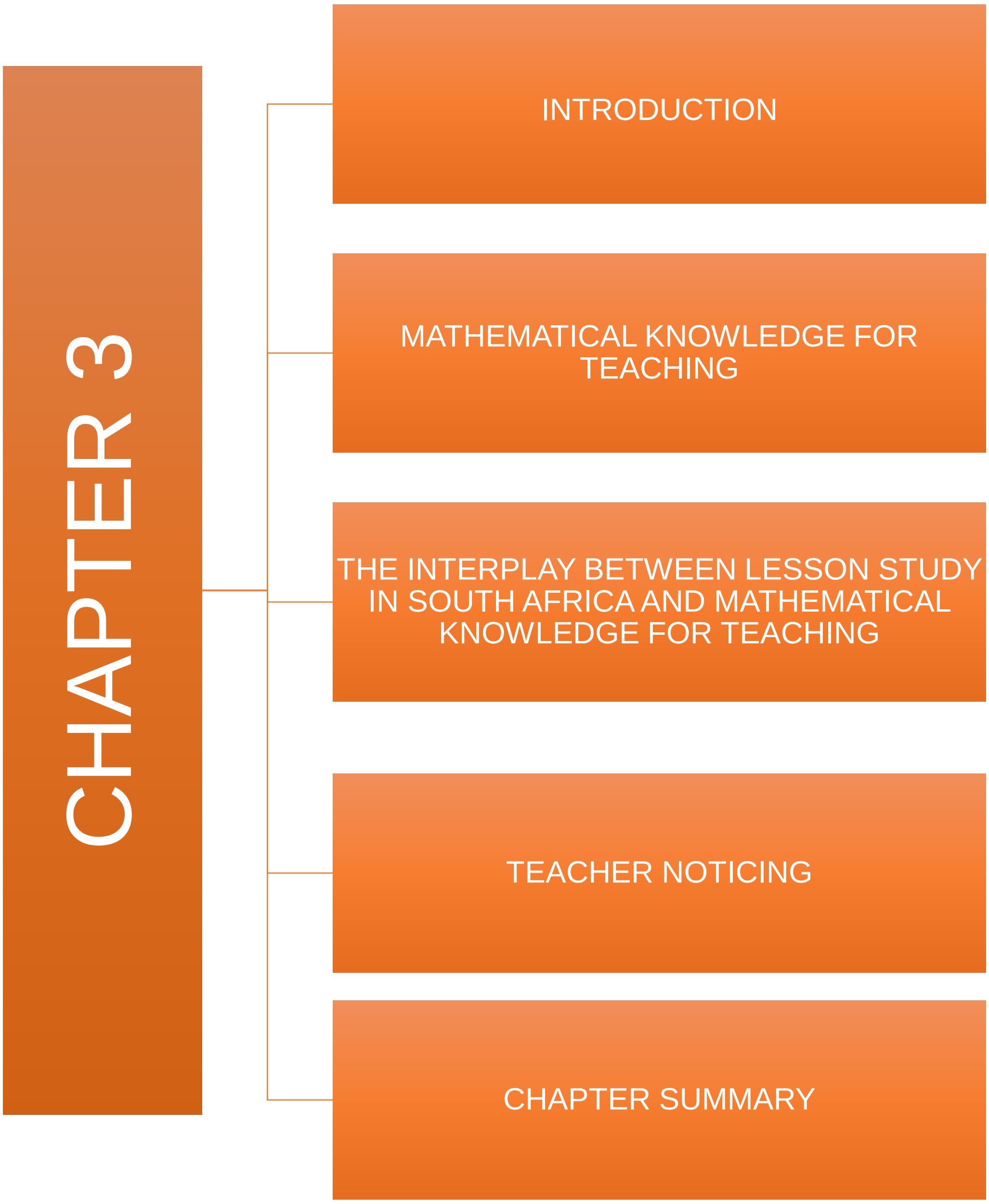
(2017) conducted another study involving in-service mathematics teachers where participants reported that LS afforded them the opportunity to develop mathematics content, learn new teaching strategies that promoted learner thinking, and improve their professional and personal relationships. Fauskanger et al. (2022) investigated primary school mathematics teachers' perspectives on the use of LS in the teaching and learning of mathematics. Teachers' reflections and lesson plans were the sources of data that were analysed using the qualitative content analysis approach. The study reported that teachers valued and practised traditional methods of teaching before participating in LS. After participating in LS, teachers had new perspectives that valued the importance of learner-centred teaching and discovery learning. Alamri (2020) probed 149 purposefully sampled primary school mathematics teachers on their perspectives on teaching mathematics through the LS strategy, using a questionnaire and qualitative interviews. Participating teachers reported that LS had cultivated the appreciation of LS in them and had helped them to improve their content knowledge, pedagogical knowledge, and knowledge for learner learning. From the ongoing discussion, studies were conducted with both pre- and in-service teachers in primary schools. However, the topics involved were not specified. My current study intends to shed more light on secondary school teachers' perspectives on the use of LS in teaching mathematics, particularly trigonometric functions in secondary schools. Since the study is informed by the interpretivist paradigm, I believe that the participating teachers' voices (during semi-structured interviews) will reach the South African mathematics education community.

Helmbold et al. (2021) reported that in South Africa, six grade one teachers who participated in a LS research that involved the teaching of mathematics perceived LS as enriching and fulfilling. The teachers' perspectives on LS were as follows: it improved their content knowledge; it improved their appreciation of the collaborative nature of LS, and they felt LS was better than previously known or experienced teacher professional development courses (Helmbold et al., 2021). Teachers in Helmbold et al. (2021)'s study further reported the fear of presenting a lesson in the presence of peers and self-expressing during discussions as the challenges they experienced. Venketsamy et al. (2022) conducted a study in South Africa where they workshopped three primary school mathematics teachers in LS. The participants purported that LS had fostered in them collaborative skills like planning together (Venketsamy et al., 2022). While Helmbold et al. (2021) and Venketsamy et al. (2022) acknowledge the scarcity of LS research in South Africa, I note the dearth of studies that report on teachers' perspectives on the use of LS in the teaching of Further Education and Training (FET) mathematics, in particular trigonometric functions.

2.7 CHAPTER SUMMARY

In this chapter, I reviewed literature bolstering my study under the following headings: Setting the scene: Landscape of teacher development in South Africa; Origins and definition of Lesson Study as a teacher development model; Lesson Study in South Africa; Teaching and learning of trigonometric functions; Teachers' perspectives in the use of Lesson Study in the teaching of mathematics. The literature presented revealed that LS is a teacher development model that has the potential to address problematic topics such as trigonometric functions. Literature from the second, third and fourth stages of the South African LS model (which are lesson planning, lesson presentation and observations, and post-lesson reflection respectively) served as a backing to the three sub-questions of my study. It was during the collaborative activities in these three stages that teachers developed their mathematical knowledge for teaching trigonometric functions. LS participants' perspectives were reported to have the potential to lure and convince other teachers and stakeholders to embrace and engage the LS model in teaching and learning. In the next chapter, I present the framework of my study.

3. CHAPTER 3: THEORETICAL FRAMEWORK



3.1 INTRODUCTION

In this chapter, I present the theoretical framework that enabled me to explore the teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study. A theoretical framework is a researcher's lens that enables an understanding of a phenomenon (Lempriere, 2019). The theoretical framework in my study served as a specialised lens through which I collected, examined and analysed data, understood, interpreted, explained and discussed the findings, and made recommendations and conclusions (Kivunja, 2018). I employed the Mathematical Knowledge for Teaching (MKT) as the framework for my study (Ball et al., 2008). My study's first three research questions emanated from and related to mathematical knowledge for teaching. These research questions spontaneously informed me the data, and on how I should collect it from the participating teachers. This resulted in me focusing on teachers' collaborative activities (during planning, teaching, and post-lesson reflections) that portrayed the development of mathematical knowledge for teaching. Sahidin et al. (2019) assert that the MKT framework supports mathematics education in research and practice in order to improve teaching and learning. Moreover, Mosvold (2022) reviewed literature on the studies involving mathematical knowledge for teaching in Africa from 2014 to 2021. Mosvold (2022) reported that there was an increase in the number of such studies, with most of them having been conducted in South Africa. In the second section of this chapter. and its related subsections, I detail the MKT framework and its relevance to my study.

3.2 MATHEMATICAL KNOWLEDGE FOR TEACHING

MKT was developed from Shulman's idea of pedagogical content knowledge (PCK) and empirically tested by Ball et al. (2008). Shulman's basic idea of PCK involved the following:

> general pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter; knowledge of learners and their characteristics; knowledge of educational contexts, ranging from workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures; knowledge of educational ends, purposes, and values, and their philosophical and historical grounds; content knowledge; curriculum knowledge, with a particular grasp of the materials and programs that serve as 'tools of the trade ' for teachers; pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their special form of professional understanding (Shulman, 1987, p. 8).

Ball et al. (2008) were inspired by Shulman's assertion that teaching required a special kind of content knowledge (CK) or subject matter knowledge (SMK). This inspiration resulted in Ball and her colleagues embarking on a project that cultivated MKT (Izsák et al., 2012). Furthermore, Chapman (2017) notes that Ball and colleagues were convinced by their series of studies that teachers needed MKT to teach mathematics well.

MKT is perceived by Hill et al. (2005) as the mathematical knowledge that is used to discharge the work of teaching mathematics. The work of teaching, according to Hill et al. (2005), entails clarifying terminology and abstractions to classes; casting light upon and/or explaining learners' responses to oral and written work; highlighting challenges and suggesting improvements on mathematics content literature; articulating visual, symbolic, physical, contextual and verbal mathematics to learners; and exemplifying to learners mathematical abstractions, multi-stepped solutions or proving. Chapman (2017) also argues that MKT is a specialised type of knowledge that involves decision-making within the context of mathematics teaching, and it thrives on practice-based (hands-on or classroom-based) professional learning. Possessing skills like examining and interpreting thinking associated with learners' errors and misconceptions, establishing gaps in learners' understanding of mathematical content, and determining the best way to articulate mathematical content for learners' conceptual understanding is characteristic of MKT (Chapman, 2017). From what is embodied in the work of teaching as asserted by Hill et al. (2005) and in the skills and abilities outlined by Chapman (2017), it is clear that MKT pervades 3the LS stages such as diagnostic analysis, lesson planning, lesson presentation and observation, and post-lesson reflection. I generally understand that for Ball et al. (2008), MKT is a refinery of Shulman's work aimed at sharpening the art of teaching mathematics. It also follows that MKT is mostly acquired in a classroom context and is learner-centred.

The MKT by Ball et al. (2008) comprises two knowledge bases, namely; SMK/CK (mastering the material/matter) and PCK (delivery of material to learners or pedagogic mastery) (Sahidin et al., 2019). Subject matter knowledge is also known as (CK).

These two knowledge bases are characterised by six domains, three for each as shown in Figure 3.1.

Figure 3.1

Mathematical Knowledge For Teaching By Ball Et Al. (2008)

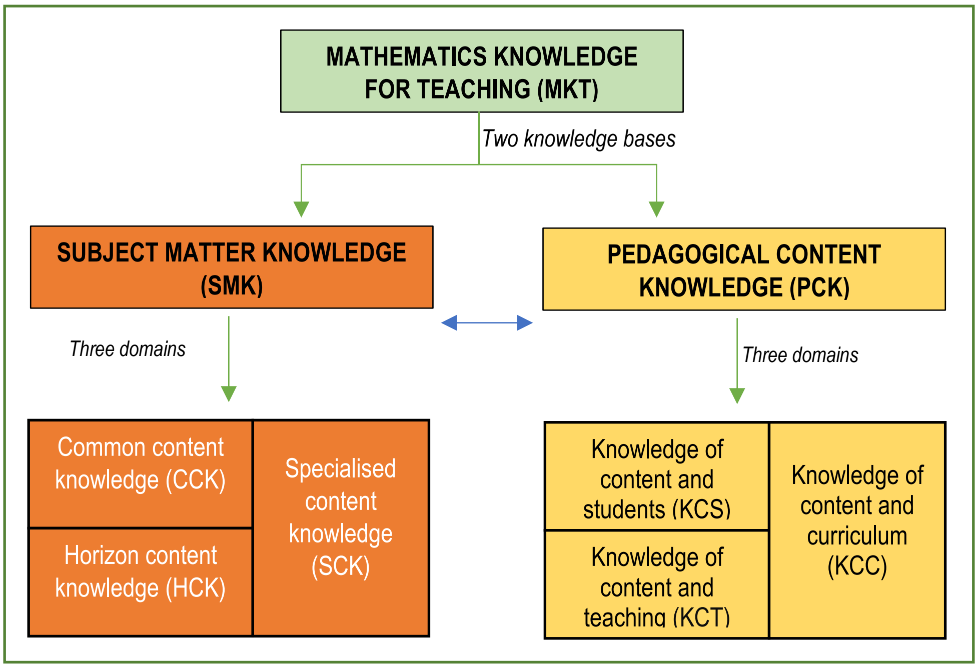


Figure 3.1 shows that MKT is split into two knowledge bases, which are SMK or CK and PCK. SMK or CK is divided into three domains namely Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialised Content Knowledge (SCK). The other knowledge base, PCK, breaks down into three domains Knowledge of Content and Curriculum (KCC), Knowledge of Content and Students (KCS), and Knowledge of Content and Teaching (KCT).

Depaepe et al. (2015) hold that PCK relies on subject matter knowledge while SMK is independent of PCK. However, Baki and Arslan (2016) and Proctor (2019) hold a differing and intriguing view, which I agree with, that effective teaching cannot thrive on mastery of subject matter only without pedagogical content knowledge. In agreement, Sahidin et al. (2019) posit that the ability of subject matter and pedagogic mastery is like two sides of a coin and is one unit that is inseparable from the other. These notions render SMK and PCK necessary constituencies of teachers' MKT. Accordingly, teachers who lack in terms of matter and pedagogic mastery in trigonometric functions may not be able to effectively teach the learners to understand a topic.

From Figure 3.1: SMK houses Common Content Knowledge (CCK), Specialised Content Knowledge (SCK) and Horizon Content Knowledge (HCK) while the PCK base embodies Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC) (Ball et al., 2008). In the ongoing discussion, I show how the MKT framework's domains are applicable, utilised, and incorporated in the LS stages as well as the way it guided my study. The domains overlap on five different stages of the South African LS cycle like in the study by Ní Shúilleabháin and Clivaz (2017) (see Appendix A). This is the reason why I probed all the domains on the three stages (lesson planning; lesson presentation and observation, and post-lesson reflection) that are aligned with my study. The indicators/elements of domains enabled me to collect data during observations and interviews. This, in turn, facilitated the analysis.

In the next two sections, I discuss the two knowledge bases (SMK and PCK). Within each knowledge base, I detail and elaborate on each domain.

3.2.1 Subject Matter Knowledge

Common Content Knowledge (CCK)

CCK refers to mathematical knowledge that can be easily understood by people who are not subject specialists. Chinnappan and Cheah (2012) perceive CCK as general knowledge in mathematics that can be executed by a literate individual. As Moumou (2021) puts it, CCK is mathematical knowledge possessed by any person who has undergone school mathematics and/or any mathematician. This could be perceived as assumed/prior knowledge of a particular topic, and this helps teachers during the planning stage and the introduction of a lesson. It helps in that assumed/prior knowledge connects what is already known with new knowledge of the topic being introduced. For example, when drawing graphs, the method of the table of values that is already known from algebraic graphs could facilitate the introduction of drawing trigonometric graphs. CCK is rated vital since it helps teachers comprehend and follow the content that they are instructing to learners (Kelcey et al., 2019).

Specialised Content Knowledge (SCK)

SCK constitutes the knowledge and skills that enable a teacher to trace the thinking of learners, explain the underlying algorithm, and explain the correctness of solutions to mathematical problems (Chua, 2019). This knowledge aligns itself with identifying misconceptions about a topic, and it helps teachers in conducting diagnostic analysis of learners' work. The development of SCK could be observed when teachers identify purposeful instructional activities in trigonometric functions during the planning stage. Proctor (2019) posits that SCK emphasises teachers' establishment of the errors learners commit and an analysis of the presentation of the solutions. In support, Moumou (2021) perceives SCK as knowledge special to teachers (and not just to any mathematician) that enables them to respond to learners' clarity-seeking questions (like 'how' and 'why'). Considering Proctor's and Moumou's assertions, I view SCK in trigonometric functions as knowledge that will enable a teacher to explain the dialectics in the effects of parameters to learners. The teacher presenting a research lesson on trigonometric functions and the observers could apply and or execute SCK in assessing learners' responses to oral questions and class activities. This is also extendable to the reflection phase.

Horizon Content Knowledge (HCK)

According to Proctor (2019), HCK is useful in helping teachers understand learners' foundational knowledge of a topic to come up with pedagogical approaches that will help learners grasp the new concept(s). It is held that HCK concerns itself with teachers' abilities to see the relationship or connections among mathematics topics in each curriculum (Moumou, 2021). For instance, in CAPS the vertical translation of trigonometric functions associated with parameter q is comparable to algebraic functions. Teachers' discussions during collaboration lesson planning could build on this aspect, thereby facilitating teachers' reaching across to learners during lessons. There are also questions like: For which values of x is $f(x) < g(x)$? that feature in both algebraic and trigonometric functions. This feature could be used as leverage by teachers during both planning and lesson time. From my understanding, HCK helps teachers in ascertaining and employing assumed knowledge.

3.2.2 Pedagogical Content Knowledge

Knowledge of Content and Students (KCS)

Chua (2019) views KCS as a combination of knowing about students and knowing about mathematics, which informs the teacher's ability to design and plan a lesson that stimulates learners. KCS is the knowledge that teachers articulate when they consider learners' thinking approaches about a concept and anticipate how learners would react to their presentation of the concept (Ball et al., 2008; Hill et al., 2005; Moumou, 2021). This entails choosing purposeful activities during the planning phase, activities that will match the learners' level of abilities and provoke their thinking. At this instance, teachers can edit activities from textbooks and/or create their own that they see as capable of promoting conceptual understanding. Such a practice is bolstered by Proctor (2019) who notes that KCS transpires when teachers engage in sifting tasks/activities that will interest learners with possible misconceptions, difficulties and common errors in mind.

Knowledge of Content and Teaching (KCT)

The second domain under the PCK is KCT, which marries knowing about teaching and mathematics (Ball et al., 2008). Proctor (2019) argues that KCT is the knowledge that teachers employ to prepare/plan lessons to promote learners' understanding. The teacher can select and sequence purposeful activities and tasks that will help learners understand the topic at hand. Optimum KCT occurs when teachers successfully combine/interweave/integrate content knowledge, learning and teaching materials, and instructional strategies (teaching techniques) to promote conceptual understanding in learners (Moumou, 2021; Shulman, 1987). In my study, this coincided mostly with the planning and the teaching stages. I could observe these KCT tenets during teachers' collaborative planning and teaching stages.

Knowledge of Content and Curriculum (KCC)

KCC refers to the correct interpretation of the content of the curriculum scope of the grade (Chua, 2019). Trigonometric functions are taught from grade 10 to 11 in South Africa. It, therefore, follows that KCC enables teachers to select and teach the correct level of trigonometric functions for the grade concerned. For instance, according to CAPS, each grade has its specifications in terms of domains, parameters, and ways of drawing graphs to be taught in each grade. In the South African context, I see KCC being guided by the ATP and CAPS. Moumou (2021) contends that KCC is important because it enables teachers to familiarise themselves with the relevant mathematics curriculum at a particular time.

From the numerous elaborations on MKT knowledge bases and their respective domains in this section, it follows that the incorporation of domains in the classroom could enhance teaching and learning trigonometric functions and other topics in mathematics. Having discussed the MKT theoretical framework by Ball et al. (2008), I proceed to deliberate on the compatibility of the South African Lesson Study model and the framework.

3.3 THE INTERPLAY BETWEEN LESSON STUDY IN SOUTH AFRICA AND MATHEMATICAL KNOWLEDGE FOR TEACHING

MKT leads to improved instruction and has been linked to positive learner achievement (Almadi, 2022; Fischman & Riggs, 2021; KOÇAK et al., 2021; Kuleshova, 2020; Moumou, 2021; Pournara et al., 2015; Proctor, 2019). Equally important, vast research reports have revealed that LS contexts nurture and enhance MKT development (Almadi, 2022; Clivaz & Ni Shuilleabhain, 2019; Helmbold et al., 2021; Huang & Shimizu, 2016; Jita & Ige, 2019; Kang, 2016; KOÇAK et al., 2021; Ni Shuilleabhain, 2016; Ní Shúilleabháin & Clivaz, 2017; Perry & Lewis, 2011; Willems & Van den Bossche, 2019). It is held that collaborative activities in LS allow teachers to scaffold one another concerning SMK and PCK. Furthermore, exploring how knowledge for teaching mathematics develops in pre- and in-service teachers is key to understanding the complex set of knowledge and skills required to teach mathematics (Lewis et al., 2009; Ní Shúilleabháin & Clivaz, 2017). This renders exploring how teachers develop mathematical knowledge for teaching trigonometric functions through LS important in the South African mathematics education system. In the ongoing discussion, I intend to synthesise the literature that buttresses the MKT-LS relation.

Helmbold et al. (2021) conducted a case study that explored the impact of LS on six primary school mathematics teachers' knowledge. Two LS cycles were held outside schooling hours. The study found that LS had a positive influence on the development of teachers' MKT. However, Helmbold et al. (2021)'s study deviated from the LS practices in that the teaching was not done according to the school's timetable in order to adhere to the organic nature of LS. Teaching according to the school's timetable enhances the natural setting in the classroom. This implies that the findings from Helmbold et al. (2021) may not be portraying genuine LS processes. Almadi (2022) also studied how LS enhanced primary school mathematics, science, and Arabic teachers' CK and PCK. Questionnaires and interviews were used to solicit data from 50 teachers. In addition, four school leaders were interviewed to tap their perceptions on the impact of LS on CK and PCK and school learning communities. The findings were that participants believed that participating in LS enhanced teachers' CK and PCK as well as foster viable school-based learning communities. Furthermore, the study reported improved teacher performances and corresponding learner outcomes. Scheduling of collaborative meeting times, lack of CK and PCK in respective subjects by some teachers, and lack of experience in LS were some of the challenges in the study. Findings from Almadi's study serve as a background to answering research sub-question 4 of my study, namely: 'What are the teachers' perspectives on the use of Lesson Study in the development of mathematical knowledge for teaching trigonometric functions?'

In my study, I incorporated the South African LS model whose cycle has five stages (see Figure 2.2 in Chapter 2 Section 2.4). Each stage has been detailed in Chapter 2 (see Section 2.4). Like Sudejamnong et al. (2014), my study explored teachers' development of mathematical knowledge for teaching trigonometric functions in three stages of the LS cycle (lesson planning; lesson presentation and observation, and post lesson reflection). Sudejamnong et al. (2014)'s study had its three corresponding stages, that is, plan; act and observe; and reflection. This study is interesting in that it used MKT as one of its frameworks. Again, Sudejamnong et al. (2014) used video recordings, interviews and field notes as data sources, and the outcome was that teachers developed knowledge during the planning stage by, discussing and sharing knowledge from the textbooks; discussing lesson objectives and constructing open-ended questions, and anticipating learners' thinking. During the act and observe stage (that corresponds with the lesson presentation and observation stage of the South African LS model), Sudejamnong et al. (2014) report that teachers developed mathematical knowledge for teaching, when they presented open-ended problems; by observing learners' responses to problem-solving questions and related thinking, and by extrapolating learners' thinking. Furthermore, the Sudejamnong et al. (2014) study revealed that knowledge development amongst the participants occurred when they reflected on the learners' goal accomplishment, the learners' learning processes and thinking approaches, and the use of instructional materials. In addition, teachers are reported to have also developed knowledge from the comments and recommendations provided by knowledgeable others (*koshi* in Japanese). A knowledgeable other (in the context of LS) is someone who has profound knowledge and experience in LS and whose function is to guide and support LS participants (Bilge & Dede, 2023). From my synthesis of previous research, Sudejamnong et al. (2014) study is one of the most powerful studies that reveal that LS as a teacher development model provides settings where MKT thrives.

Another study of relevance is that conducted by Clivaz and Ni Shuilleabhain (2019) in Switzerland. The qualitative study used MKT as one of its frameworks and examined the mathematical knowledge embodied by eight primary school mathematics teachers in each LS stage. The researchers used an LS cycle model based on Lewis et al. (2006), a model that comprised four stages: study curriculum; plan lesson; conduct and observe lesson, and reflect on the lesson. Video recordings, like in my current study, were used to collect data during the LS processes. Clivaz and Ni Shuilleabhain (2019) used the MKT and the Levels of Teacher Activity frameworks to analyse the knowledge embodied in each LS stage where MKT domains were used as codes. The appealing finding in their study was that MKT development occurs during each stage of LS, meaning that all six domains are embodied and developed at any one stage. In my study, I anticipated the embodiment/incorporation of domains to vary in occurrence in each of the South African LS stages and this is portrayed in Figure 3.2.

Figure 3.2

MKT Domains In Relation To The South African Lesson Study Model.

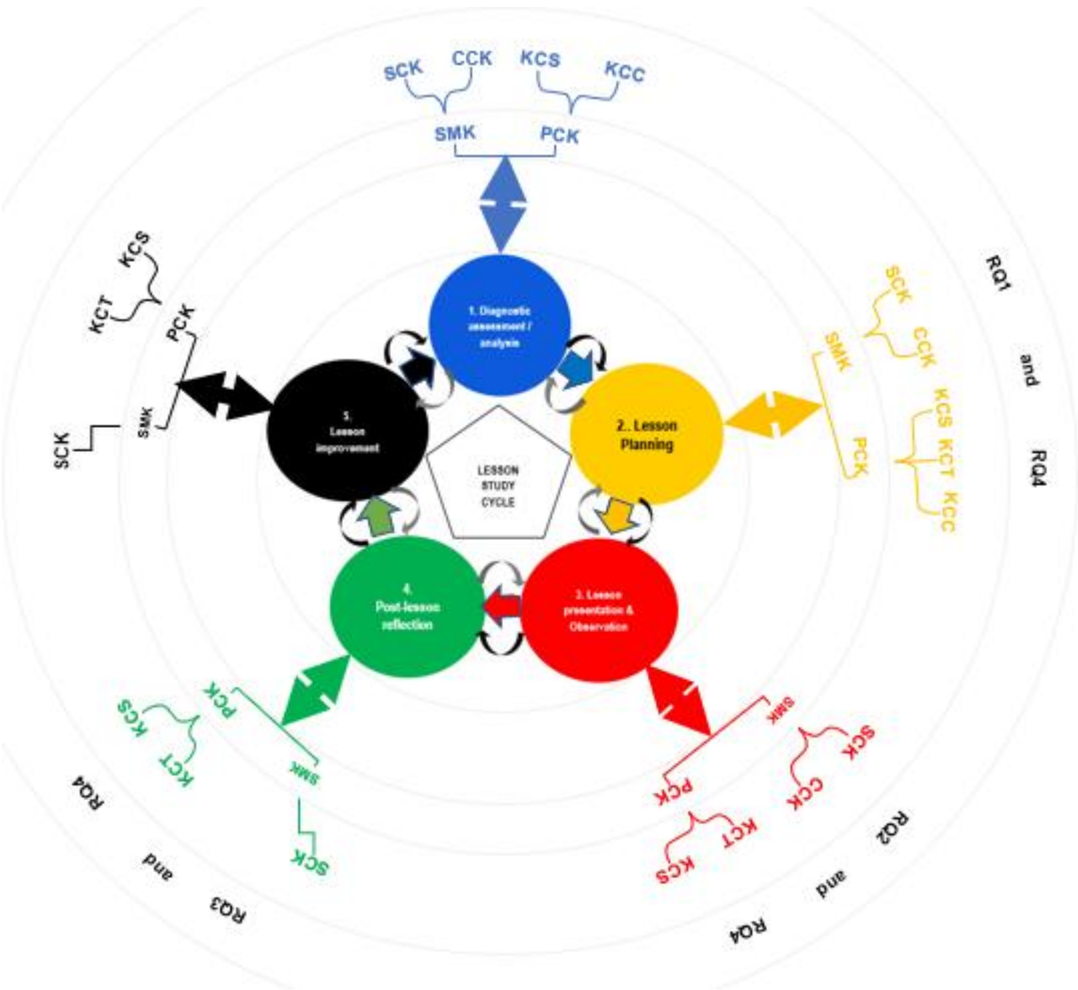


Figure 3.2 shows the purported relationship between the MKT theoretical framework (see Figure 3.1) and the South African LS model (see Figure 2.2 in Chapter 2 section 2.4). The inner structure portrays the five stages of the South African LS model: Stage 1 (blue circle) – diagnostic assessment or analysis; Stage 2 (yellow circle) – lesson planning; Stage 3 (red circle) – lesson presentation and observation; Stage 4 (green circle) – post-lesson reflection, and Stage 5 (black circle) – lesson improvement. The second layer of the structure comprises arrows entailing the link(s) between each LS

stage and MKT framework knowledge bases (SMK and PCK). The knowledge bases that split into selected domains form the third layer. The outer layer contains research questions (RQ) 1 to 4 (that is, RQ1, RQ2, RQ3, and RQ4) that are coinciding with relevant LS stages. For instance, it follows that collaborative activities in stage 1 help to answer RQ1 and RQ4.

Each stage of the South African LS model nurtures PCK and SMK (in equal or different weightings) while embodying varying/selected domains and providing them with an environment to develop. For instance, Stage 1 has SMK and PCK equally weighted (as indicated by the same font size) with their related domains (SCK and CCK) and (KCS and KCC) respectively. Stage 2, like Stage 1, has PCK and SMK weighing the same. However, stages 3 to 5 have PCK dominating SMK (with a smaller font). It means that PCK in those stages in embodied/incorporated more than SMK. The diagram also shows that what is occurring in stages 2 to 4 helps me to answer research questions 1 and 4; 2 and 4; 3 and 4 respectively. In line with Clivaz and Ni Shuilleabhain (2019) way of analysis, I used MKT domains as codes. The other finding reported by Clivaz and Ni Shuilleabhain (2019) is that LS stages are not discrete and separated from one another, resulting in a tenet of any one stage pervading the whole cycle. This also concurs with the South African LS model in which the back-and-forth arrows between the stages were indicated by Sekao and Engelbrecht (2022) (see Figure 2.2 in Chapter 2). Due to the interplay between MKT domains and the cyclic process of LS in South Africa, I see the development of mathematical knowledge for teaching trigonometric functions being nurtured.

Silverman and Thompson (2008), who are proponents of Ball et al. (2008) MKT framework, posit that MKT is developed when teachers search for understanding ways that support learner learning by deliberating on content and pedagogical knowledge. As Proctor (2019) puts it, developing MKT occurs when teachers blend their understanding of subject matter with that of teaching and learners. Silverman and Thompson (2008) argue that their perspective on MKT applies to both pre- and in-service teachers. In essence, Silverman and Thompson (2008) see teachers developing MKT in any one topic when they:

> (1) have developed a key developmental understanding within which that topic exists; (2) have constructed models of the variety of ways learners may understand the content (decentring); (3) have an image of how someone else might come to think of the mathematical idea in a similar way; (4) have an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea, and (5) have an image of how learners who have come to think about the mathematical idea in the specific way are empowered to learn other, related mathematical ideas (Silverman & Thompson, 2008, pp. 17-18)

In my study context, this perspective extrapolates, reinforces, enriches, and augments Ball et al. (2008) MKT framework. Furthermore, the perspective facilitated and refined my exploration and understanding of teachers' development of mathematical knowledge for teaching trigonometric functions in the South African LS model stages by addressing the "how" of MKT development. From my understanding, the word "image" that characterises Silverman and Thompson (2008) perspective refers to mental pictures that teachers have during LS collaborative activities. I view this as the thread weaving through all LS stages where teachers always do everything with learners in mind and/or seeing things through the eyes of learners. In line with this, Sekao (2023) insists that teachers in LS always consider how learners are likely to respond to their planned lessons. It therefore follows that teachers would have developed new knowledge of teaching mathematics after engaging in all these learner-centred collaborative activities (Proctor, 2019).

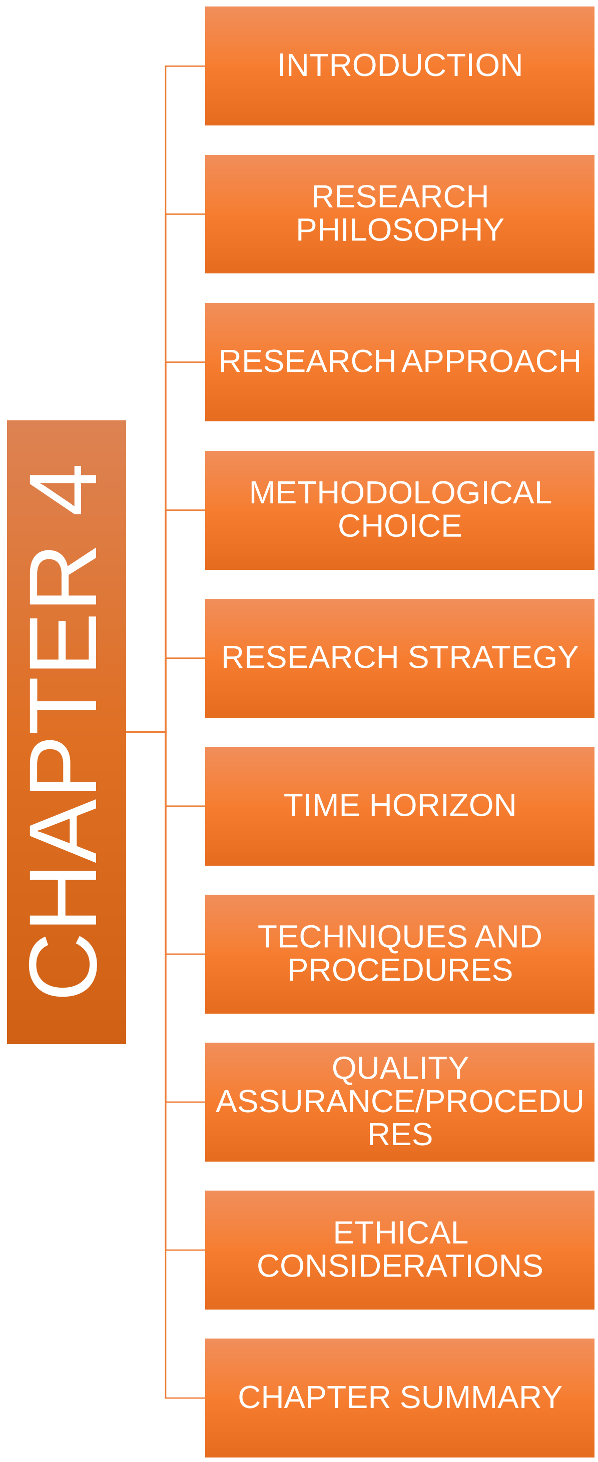
3.4 TEACHER NOTICING

In the ongoing discussion, I find it worthwhile to infuse teacher noticing or professional noticing which appears to have a seamless existence between MKT and LS. Teacher/professional noticing is when teachers observe, respond, discuss, interpret, and or make sense of what is happening (in terms of teachers' activities and learners' reactions, responses, and thinking) during LS processes (Biccard, 2020; Choy, 2016; Dindyal et al., 2021; Jacobs et al., 2010; Weyers et al., 2023; Williams, 2022; Yang et al., 2021). In essence, teacher/professional noticing is a means of teacher knowledge/professional development that thrives in LS (Biccard, 2020; Dick, 2017; Dindyal et al., 2021). Research reports that there is a positive correlation between MKT and teacher/professional noticing (Copur-Gencturk & Tolar, 2022; Dick, 2017; Hoth et al., 2022; Jong et al., 2021; Larrain & Kaiser, 2022; Weyers et al., 2023). Interestingly, Choy (2016) purports that teachers take mental snapshots while seeing what is happening in any one LS stage and use them as a means of knowledge development. In my study, it therefore follows that teacher/professional noticing (and hence, knowledge development) could occur during collaborative lesson planning, lesson presentation and observation, and post-lesson reflection.

3.5 CHAPTER SUMMARY

In this chapter, I deliberated on Ball et al. (2008) MKT theoretical framework that served as the lens for my study. LS provides environments in which mathematical knowledge for teaching flourishes. The LS stage-MKT domain aspect(s) interweave, which is buttressed by Silverman and Thompson (2008) perspective ,together with previous research, have provided a magnified lens for my study. This has greatly facilitated my observations and influenced my methodology and analysis/understanding or making sense of data. In so doing, I have been able to address my four research questions.

4. CHAPTER 4: RESEARCH METHODOLOGY



4.1 INTRODUCTION

The focus of this qualitative study was to explore teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study. The current methodology chapter explained how and what I did to pursue this exploration. The structure of the current chapter was guided by the research onion (Figure 4.1) espoused by Saunders et al. (2023).

Figure 4.1

Saunders' Research Onion

Note. Adapted from Saunders, et al. (2023). *Research methods for business students*. Pearson education.

The research onion comprises six layers, that is, research philosophy; research approach, methodological choice; research strategy; time horizon; techniques and procedures (Saunders et al., 2023).

4.2 RESEARCH PHILOSOPHY

Research philosophy is a system of beliefs and assumptions about developing new knowledge in a particular field (Saunders et al., 2023). A research philosophy is also known as a research paradigm and I will use these two words interchangeably. Ling and Ling (2016, p. 2) perceive a research paradigm as referring “to a world view or logic that underpins all aspects of a research undertaking from the intent or motivation for the research to the final design, conduct, and outcomes of the research.” Employing a research philosophy ensures that all the elements of research are consistent, congruent, and compatible with one another (Ling & Ling, 2016). It follows that the research philosophy in my study threaded together and aligned all the chapters and their respective sections and subsections. Furthermore, the research philosophy afforded my study process rigour, enabling me to understand the problem (teachers’ development of mathematical knowledge for teaching trigonometric functions through Lesson Study) and develop recommendations (Ling & Ling, 2016).

Positivist, neo-positivist, interpretivist, transformative, pragmatic, and super complexity are the six research paradigms that Ling and Ling (2016) have identified as the most common in educational research. In the current study, I adopted the interpretive paradigm which emphasises that humans (in this case, participating teachers) are different from physical phenomena because they create meanings that are studied by researchers (Saunders et al., 2023). As Ling and Ling (2016) put it, the interpretive research philosophy insists on gathering and analysis of data by the researcher and on the social construction of understandings.

The ontological assumption of interpretivist philosophy is concerned with the subjective nature of reality, and it holds that reality exists in multiple perspectives presented by different individuals (Creswell & Poth, 2018; Sławecki, 2018; Thanh & Thanh, 2015). Six teachers participated in my study. I held the notion that whatever information I got from the teachers would represent the realities that answer my research questions. During the planning stage, I accepted teachers' input and varying ideas on structuring, designing, and writing lesson plans. Again, in the lesson presentation and observation stage, I embraced the participating teachers' differing ways of teaching trigonometric functions that were also different from mine in concurrence with the ontological assumption in qualitative research (Creswell & Poth, 2018). Multiple realities also emanated from the post-lesson reflection stage(s) and during interviews when teachers gave numerous differing views verbally. This idea of multiple realities informed my use of multiple data collection methods: observations, document analysis, and interviews (Willis et al., 2007).

Ling and Ling (2016) posit that epistemology is the subjectivist understanding of the nature of knowledge and how it may be known. In an interpretive qualitative study context, according to Creswell and Poth (2018), the epistemological assumption entails that a researcher tries to get as close as possible to the participants being studied to solicit their subjective experiences. In line with this, I immersed myself in a school environment where the teachers worked during data collection. I observed and video-recorded these teachers during their lesson planning, lesson presentation, and post-lesson reflection to know and understand their development of mathematical knowledge for teaching trigonometric functions in the context of LS. Interviewing participating teachers also allowed me to gather their subjective experiences on the use of LS in the development of mathematical knowledge for teaching trigonometric functions.

4.3 RESEARCH APPROACH

Although the term *research approach* is used by other researchers as a collective term to refer to either qualitative approach, quantitative or mixed method approach (Bager-Charleson & McBeath, 2020; ÇALIK, 2022; Chali et al., 2022; McBeath & Bager-Charleson, 2020; Mulisa, 2022), Saunders et al. (2023) view research approach as concerned with theory testing or theory building. For Saunders et al. (2023), therefore, research approach can assume a form of inductive approach, deductive approach or abductive approach. Given this view, and the fact that I have adopted the research onion to structure my methodology chapter, I align myself with Saunders et al. (2023) regarding the use of the term *research approach*.

Qualitative studies are compatible with both the inductive approach and deductive approach (Creswell & Poth, 2018; Tracy, 2020). In my study, I applied INDUCTIVE-Deductive reasoning because it enabled me to come up with conclusions based on evidence and reasoning (Miessler, 2020). My study is more inductive than deductive (hence INDUCTIVE-Deductive) because it relied more on data collection to understand how teachers develop mathematical knowledge for teaching trigonometric functions in a LS context.

The inductive approach involves starting with observation and data collection, moving to description and analysis to form a theory (Melnikovas, 2018). In the same vein, Tracy (2020) views the inductive approach as a bottom-up approach where meanings emerge from participants and their environment. My approach was inductive in that I collected data through an analysis of documents such as lesson plans, and by observing and interviewing participating teachers in the school they taught. I went on to analyse the data and came up with findings and conclusions that are new in the teaching of trigonometric functions.

Melnikovas (2018) maintains that deduction is when a study starts with an existing theory followed by formulating a question or hypothesis and data collection to confirm or reject the hypothesis. The current study is deductive, firstly because it is informed by the Mathematical Knowledge Teaching (MKT) framework, and I collected data in an LS context. MKT and LS are already established theories and models respectively. During the structuring of my study and data collection, I had some expectations based on teachers' development of subject matter and pedagogical knowledge during planning, lesson presentation and observation, and post-lesson reflection. The MKT framework and the LS model assisted me in understanding and interpreting the collected data.

4.4 METHODOLOGICAL CHOICE

Melnikovas (2018) argues that methodological choice seeks to determine the use of either quantitative or qualitative methods, or various mixtures of both. The method for my study is qualitative (as I mentioned in earlier sections) since it thrives on multiple realities (unlike quantitative which relies only on numbers) in the form of multiple forms of evidence like spoken words (Creswell & Poth, 2018), which is in line with the assumptions of the interpretive philosophy. I am of the view that a qualitative study is concerned with the pursuance of a phenomenon (teaching of trigonometric functions in an LS context) in participants (teachers) in their natural setting (school) and understanding the meanings that they give through observation, document analysis, and interviews (Klem et al., 2021; Merriam & Grenier, 2019; Willig, 2022). According to the concurrence of Creswell and Poth (2018), Merriam and Grenier (2019), and Tracy (2020), qualitative studies have the following characteristics: a natural setting, researcher as a key instrument; multiple methods of data collection, reasoning inductively and deductively, participants' multiple perspectives and meanings, context-dependent, emergent design, reflexivity, and holistic account. These characteristics appear to encompass all of Saunders et al. (2023) onion layers, thereby rendering congruency and compatibility to the elements of my entire study. Again, the stated characteristics strengthen my choice of a qualitative method (Merriam & Grenier, 2019).

I opted for a qualitative research method because I needed to explore and deeply understand how teachers develop mathematical knowledge for teaching trigonometric functions through LS (Creswell & Poth, 2018; Willis et al., 2007). The excerpts from interviews and video recordings of lesson planning, lesson presentation and post-lesson reflection sessions as well as selected texts from lesson plans strengthened the findings of my study. As the primary researcher, I managed to analyse data whilst I was still collecting; and this resulted in robust analysis and interpretation of data (Merriam & Grenier, 2019).

4.5 RESEARCH STRATEGY

What Saunders et al. (2023) refer to as research strategy is commonly known as research design, whilst Creswell and Poth (2018) and Willig (2022) view it as an approach. Yin (2018) notes that case studies are common in social sciences (like mathematics education). A case study can either be made up of a single case or multiple cases (Yin, 2018). My study is a qualitative single case study. Kumar (2014) contends that the case selected in a case study becomes the basis of a thorough, holistic, and in-depth exploration of the aspect(s) that a researcher wants to find out about. The case of my study is FET mathematics teachers from a single school whose development of mathematical knowledge for teaching trigonometric functions through LS was explored. I chose my case of FET teachers at the school because I had sufficient access to data through observations, documents, and interviews. Trigonometric functions are taught at the FET level. It follows that my research questions were tentatively defining the case of FET teachers and this is supported by Yin (2018) who posits that a case could derive from research questions.

I settled for a qualitative case study design because my primary research question was a "how" question that explored a contemporary phenomenon (the teaching of trigonometric functions in Term Three 2023) (Yin, 2018). The case study strategy was of immense relevance to my study since it focused on exploring and gaining insight into FET mathematics teachers' development of mathematical knowledge rather than confirming and quantifying it (Kumar, 2014). This is supported by Willig (2022, p. 41) who observes that a case study nurtures "in-depth, intensive and sharply focused explorations". In the same vein, Yin (2018) argues that a case study is suitable for focusing in-depth on a small group like the six teachers who participated in my study. The case of FET mathematics teachers generated new knowledge (through induction) in the teaching of trigonometric functions and enriched (through deduction) the MKT framework and the LS model (Faulve-Montojo, 2020; Willig, 2022). Again, Kivunja and Kuyini (2017), Towers et al. (2020) and Willis et al. (2007) assert, almost similarly, that qualitative case study designs are favourable to interpretivists. In concurrence with the interpretivist research philosophy, my case study throve from data collected through observations, documents and interviews (multiple methods of data collection) in the teachers' school environment (natural setting) (Creswell & Poth, 2018; Willig, 2022). The use of multiple methods of data collection that are compatible with case studies promotes triangulation as asserted by Yin (2018). The other benefit of the case study is that it renders more detail to the research than when a large sample is used (Kumar, 2014). Furthermore, the single case of FET mathematics teachers entailed deep analysis compared to diluting multiple cases (Creswell & Poth, 2018). In addition, Yin (2018) contends that a case study is compatible with tenets of chosen theoretical frameworks like the MKT in my case. Again, a case study offers high-quality flexibility and open-mindedness when it comes to techniques of data collection and analysis (Grinnell, 1981 cited in Kumar (2014)).

The most common limitation is that qualitative case study findings may not be generalised. In the current study, the case of FET teachers was typical of mathematics teachers in other South African schools and hence the findings from this case study might provide insight into the events and situations prevalent in those schools (Kumar, 2014). Again, such a case study could be replicated in similar educational environments. The potential for replication was facilitated by my deep description of the school setting, the time-frame and the participating teachers (Creswell & Poth, 2018). Thick description helps practitioners and/or professionals such as teachers and researchers to replicate a study thereby building new knowledge (Bassey, 2001). The description of the whole of my case study enable readers to understand the phenomenon (development of teachers' mathematical knowledge for teaching trigonometric functions through LS). This in turn could guide the reader/researcher to replicate my study in their own situations /settings. My case study could be replicated in the same or different topic, subject, location, syllabuses offering trigonometric functions. More important is the fact that educational (or social sciences) qualitative case studies like mine are more useful to professionals (i.e. teachers and researchers) than being concerned about their generalisability (Bassey 1981, cited in Bassey (2001).

4.6 TIME HORIZON

The duration of a study could be cross-sectional, implying that data is collected over a short time or longitudinal, where collection of data is done repeatedly over a long period to compare data (Melnikovas, 2018). My study was cross-sectional since data was collected during the time of teaching trigonometric functions according to the Annual Teaching Plan (ATP). Again, data collection was clustered around the LS cycle which was a relatively short period.

4.7 TECHNIQUES AND PROCEDURES

Techniques and procedures are the final and innermost layer of Saunders et al. (2023) research onion which is concerned with sampling, data collection, and data analysis procedures. These procedures are influenced by the choices made in the outer layers of the research onion (Melnikovas, 2018).

4.7.1 Sampling/selection of participants

Sampling is when a researcher selects participants or items for a study using probability and/or non-probability means (Mweshi & Sakyi, 2020). On one hand, probability sampling entails participants being randomly selected to be representative of a population and is commonly used in quantitative studies (Gill, 2020; Mweshi & Sakyi, 2020; Tutz, 2023). On the other hand, non-probability is where the selection of a few participants with experiences that answer the study's research questions is non-random (Gill, 2020; Mweshi & Sakyi, 2020; Tutz, 2023). Non-probability sampling is characterised by smaller numbers compared to probability, and is associated with qualitative research (Gill, 2020). In my study, I non-probabilistically (that is, purposefully and conveniently) sampled six teachers.

Tracy (2013) defines purposeful sampling as choosing a meaningful sample that fits the parameters of the project's research questions and goals. In the same vein, Kumar (2014) sees purposeful or judgemental sampling as common in qualitative research and is considered when a researcher has identified people who are likely to have the required information and are willing to share it with the researcher. Teachers who participated in my study volunteered to participate when I approached them.

Convenience sampling is done based on the accessibility and availability of participants (Mweshi & Sakyi, 2020). Despite the disadvantage of non-generalisability associated with purposeful and convenience sampling, their advantages weigh more than this disadvantage (Creswell & Poth, 2018). Creswell (2016), B. Farrugia (2019), and Mweshi and Sakyi (2020) agree that convenience sampling uses less time, money, and effort. As for purposeful sampling, Campbell et al. (2020) and Andrade (2021) concur that the nature of the sampling promotes the rigour and trustworthiness of the data and results of a study. Although the findings of my study are non-generalisable to all schools in South Africa, they can be replicated to other schools of a similar setting.

The participants in my study were teachers who were professionally qualified to teach FET mathematics (trigonometric functions) from a school in one education district in Gauteng province in South Africa that was conveniently located for me. The school was a public ordinary and situated in a township. It was a section 21, quintile 4 no fee-paying school. Quintile is a system in the Department of Education that classifies schools according to their economic status where quantile 1 is the poorest and quintile 5 the richest (DoE, 2006). Quintiles 4 and 5 are fee-paying. However, schools could request to be exempted from paying school fees when the economic status of the community declines, like what happened to the school where I collected data (DoE, 2006). Section 21 schools are schools that are allocated funding and resources by the government according to their quintile classification (DoE, 2006). The demographics of the teachers are shown in Table 4.1.

Table 4.1

Demographics Of Teachers

Demographic characteristic	Grade(s) taught	Years of experience in teaching mathematics	Qualifications
Teacher A	8 and 9	1.7	Bachelor of Education [Mathematics and Language]
Teacher B	8 and 9	1	Bachelor of Education [Natural Sciences]
Teacher C	8, 10 and 11	0.7	Bachelor of Education [Mathematics, Science and Technology]
Teacher D	9, 10 and 12	10	Bachelor of Science [Mathematics and Chemistry] and Post Graduate Certificate in Education
Teacher E	8 to 12	24	Bachelor of Education Honours [Mathematics Education]
Teacher F	10 to 12	11	Bachelor of Education [Natural Sciences] and Bachelor of Education Honours [Leadership and Management]

All six mathematics teachers in the high school where the study was undertaken constituted the participants. I formed a LS group with these teachers with the help of a knowledgeable other, who is a lecturer in charge of LS at university, more than a year ahead of the data collection phase. This gave the teachers time to acquaint themselves with the LS processes and procedures. Moumou (2021) also formed a LS team well ahead of data collection. One of the experienced teachers facilitated LS processes by coordinating the collaborative activities. Table 4.1 shows that the LS study group was composed of teachers with experiences that ranged from less than a year to 24 years. Also, the teachers taught mathematics in various grades. This case, comprising a small number (six teachers) enabled me to delve into the teachers' development of mathematical knowledge in teaching trigonometric functions through LS in depth (L. Farrugia, 2019; Gill, 2020). The attendance of the teachers was not uniform throughout the research process due to other commitments and or duties (see Table 4.2).

Table 4.2

Teachers' Attendance Schedule

DATE	DURATION	ACTIVITY BY TEACHERS	COMMENTS ON ATTENDANCE
Day 1 25 July 2023	15:10 to 16:34	Lesson planning for lessons 1 and 2	Teacher E was occupied with assessment activities.
Day 2 26 July 2023	15:13 to 16:06	Lesson planning – refining lessons 1 and 2	Teacher E was occupied with assessment activities.
	16:06 to 16:59	Lesson planning for lesson 3	Teacher E was occupied with assessment activities.
Day 3 27 July 2023	08:45 to 09:45	Lesson presentation and observation – Lesson 1	All present
	15:00 to 15:41	Post-lesson reflection of Lesson 1	Teacher A had gone to attend a Natural Sciences workshop
	15:41 to 16:15	Lesson planning – refining lesson 2	Teacher A had gone to attend a Natural Sciences workshop
Day 4 28 July 2023	08:45 to 09:45	Lesson presentation and observation – Lesson 2	Teacher E was occupied with assessment activities.
	14:44 to 15:02	Post-lesson reflection of Lesson 2	Teacher E was occupied with assessment activities.
	15:29	Lesson planning – refining lesson 3	Teacher E was occupied with assessment activities.
Day 5 31 July 2023	13:30 to 14:30	Lesson presentation and observation – Lesson 3	All present
	14:54 to 15:35	Post-lesson reflection of Lesson 3	Teacher D had an afternoon lesson (extra classes) with Grade 12s and Teacher E was attending a School Management Team meeting.
Day 6 02 August 2023	15:10 to 16:25	Responding to the interview	All present

Teacher E is the school's assessment team (SAT) coordinator. The teacher was overseeing the whole school's assessment programme in terms of planning dates and times of writing SBAs and recording the marks. These duties resulted in Teacher E missing most of the afternoon LS activities.

4.7.2 Data collection

Data collection is an essential process in a study. Mazhar et al. (2021) assert that a study cannot be completed without data gathering. According to Creswell and Poth (2018), data collection entails interwoven activities that seek to accumulate information that will be analysed to answer research questions. The data that I collected assisted me in finding possible solutions and backing up my arguments in the teachers' development of mathematical knowledge for teaching trigonometric functions through LS (Rizzolo, 2023). Data collection is linked to other components of the study as shown in Table 4.3. Table 4.3, shows how research questions, the lesson study stage(s), data collection method(s), participants and the theoretical framework were linked towards answering my study's main research question 'How do teachers develop mathematical knowledge for teaching trigonometric functions through Lesson Study?'

Table 4.3

Linking Key Components Of The Study.

Research Question	Lesson Study stage	Data Collection Method	Participants	Framework
1. How do teachers develop mathematical knowledge for teaching trigonometric functions in the collaborative lesson planning stage?	2. Lesson planning	Observation (video recorded) Document analysis of lesson plan(s)	All observed.	CCK SCK KCS KCT HCK KCC
2. How does teachers' mathematics knowledge for teaching trigonometric functions evolve during the research lesson presentation?	3. Lesson presentation and observation	Observation (video recorded) Observation notes	All observed.	CCK SCK KCS KCT HCK KCC
3. How do teachers' reflective practice hone their mathematical knowledge for teaching trigonometric functions?	4. Post-lesson reflection	Observation (video recorded)	All observed.	CCK SCK KCS KCT HCK KCC
4. What are the teachers' perspectives on using Lesson Study in developing mathematical knowledge for teaching trigonometric functions?	All	Semi-structured interviews	All were interviewed as a group	CCK SCK KCS KCT HCK KCC

According to Sławecki (2018), the ontological assumption of interpretivism advocates a subjectivist attitude toward reality, thereby accepting the existence of many locally constructed and reconstructed realities. It is against this background that I employed multiple methods of data collection in this study, namely; observation, documents, and interviews. My employment of these three methods is also supported by Creswell and Poth (2018) and Golafshani (2003) who argue that a more valid, reliable, and diverse construction of realities is achieved by engaging multiple data collection methods such as observations, interviews, and recordings. I collected data according to the schedule shown in Table 4.4.

Table 4.4

Data Collection Schedule

DATE	DURATION	ACTIVITY BY TEACHERS	ACTIVITY BY RESEARCHER	RESEARCH QUESTION
Day 1 25/07/2023	15:10 to 16:34	Lesson planning for lessons 1 and 2	Observation-Video recording	1
Day 2 26/07/2023	15:13 to 16:06	Lesson planning – refining lessons 1 and 2	Observation – Video recording	1
	16:06 to 16:59	Lesson planning for lesson 3	Observation-Video recording	1
Day 3 27/07/2023	08:45 to 09:45	Lesson presentation and observation – Lesson 1 Teacher F teaching	Observation-Video recording	2
	15:00 to 15:41	Post-lesson reflection of Lesson 1	Observation – Video recording	3
	15:41 to 16:15	Lesson planning – refining lesson 2	Observation – Video recording	1
Day 4 28/07/2023	08:45 to 09:45	Lesson presentation and observation – Lesson 2 Teacher F teaching	Observation-Video recording lesson	2
	14:44 to 15:02	Post-lesson reflection of Lesson 2	Observation-Video recording	3
	15:29	Lesson planning – refining lesson 3	Observation-Video recording	1
Day 5 31/07/2023	13:30 to 14:30	Lesson presentation and observation – Lesson 3 Teacher F teaching	Observation – Video recording	2
	14:54 to 15:35	Post-lesson reflection of Lesson 3	Observation – Video recording	3
Day 6 02/08/2023	15:10 to 16:25	Responding to the interview	Interviewing and video recording	4

I collected data (to answer research questions 1 to 3) from three stages of LS (lesson planning, lesson presentation and observations, and post-lesson reflection) through document analysis (lesson plans) and observations. Three LS cycles were conducted on the teaching of trigonometric functions in Grade 11. Teachers planned the three lessons on Days 1 and 2 (a time frame of three hours and ten minutes) after school, which ended at 14:30. Three lesson presentations and observations were done on Days 3 to 5 during school hours as per the timetable and each period was an hour long. The corresponding post-lesson reflections for the three lessons were done after school since teachers had to attend their classes. The fourth research question was answered using data collected through interviews. Two days after the LS processes had been completed, I interviewed teachers to solicit their perspectives on using LS in developing mathematical knowledge for teaching trigonometric functions.

4.7.2.1 Observation

Observation in qualitative research refers to the collection of data through seeing and hearing participants in their natural settings (Billups, 2021). Qualitative observation is either participant or non-participant (Billups, 2021; Busetto et al., 2020; Creswell & Poth, 2018). A participant observer engages in activities being done by those being studied while a non-participant observer is watching and detached from the participants' activities and behaviours (Billups, 2021; Busetto et al., 2020). In my study, I remained a non-participant observer to minimise obstructing the natural setting of the phenomenon. Meyer and Wilkerson (2011) hold that during the lesson presentation and observation stage observers should float around the room without interfering with the research lesson. In addition, Chinnappan and Cheah (2012) insist that LS ought to be dominated by participating teachers with minimum interference from the researcher.

I observed participating teachers during the lesson planning, lesson presentation and observation, and post-lesson reflection stages of LS during days 1 to 5 as shown in Table 4.3. These observations afforded me a first-hand or live experience of teachers' development of mathematical knowledge for teaching trigonometric functions through LS (Merriam & Grenier, 2019). Such a stance is informed by the interpretivist epistemology that prescribes a researcher to be immersed in the participants' teaching lives and their environment. This is supported by Sławecki (2018) who contends that the interpretive researcher accepts subjectivism when observing. The challenge in observation is that it is impossible to capture everything that is happening with pen and paper (Creswell & Poth, 2018). To curb this, I video-recorded all the three LS stages and this is recommended by Borg et al. (2005) for observation sessions. Again, the origins of LS are characterised by video recordings of sessions as reported by (Stigler & Hiebert, 2009).

Video recording the observations is advantageous in capturing both participants' verbal and non-verbal behaviours (Billups, 2021). This afforded me a deep insight into teachers' development of mathematical knowledge for teaching trigonometric functions through LS. Video recordings of the presentation of the lessons supported the analysis of post-lesson reflection sessions where a scenario under discussion could be replayed. Furthermore, replaying the video recordings enabled me to relate teachers' behaviours with the components of the MKT framework during data analysis. The aspects/indicators of MKT domains/components, like those outlined by Ní Shúilleabháin and Clivaz (2017) as portrayed in the tool/instrument in Appendix A, facilitated my observations. Each instrument was used during the lesson planning, lesson presentation and observation, and post-lesson reflection stages of LS, respectively.

4.7.2.2 Document analysis

Document study or document analysis is when a researcher examines textual materials produced by participants (in this case lesson plans) to gain an understanding of the phenomenon being studied (Billups, 2021; Bowen, 2009; Busetto et al., 2020). Bowen (2009) asserts that documents are compatible with qualitative case studies. Furthermore, Bowen (2009) reports that documents can also be used as the only data collection method for a particular study. This signifies the strength of document analysis as a method of data collection. Punch and Oancea (2014) hold that documents are a rich source of data that can either be historic or contemporary. In my study, lesson plans produced by the participating teachers were contemporary and they served as pedagogical documents.

Ndihokubwayo et al. (2020) perceive a lesson plan as a crucial road map guiding the teacher before the implementation of the lesson. In my study, lesson plan production (lesson planning) by teachers preceded and informed lesson presentation and observation, as well as post lesson reflection. During the planning, teachers discussed and agreed upon the sequencing of content, selection of examples, and class activities. I used the lesson plans (documents) for augmentation and triangulation with observations and interviews (Billups, 2021; Bowen, 2009; Creswell & Poth, 2018). The study of teachers’ lesson plans facilitated my tracking and understanding of how the teachers used LS to develop their mathematical knowledge for teaching trigonometric functions. The difference between rough and final drafts of lesson plans signified the development of teachers' mathematical knowledge for teaching trigonometric functions. In the same vein, Ndihokubwayo et al. (2022) and Zainil and Fauzan (2020) corroborate that the production of good-quality lesson plans by teachers portrays their level of pedagogical competence. I examined the lesson plans in relation to the discussions done during planning, the lesson presentation and observation conducted by teachers, and the post-lesson discussions held by teachers. This enabled me to trace the teachers' development of mathematical knowledge for teaching trigonometric functions from one LS stage to the other. Excerpts from the lesson plans helped me to bolster my findings. Again, selected texts in the lesson plans informed some interview questions to answer research question 4 – 'What are the teachers' perspectives on the use of Lesson Study in the development of mathematical knowledge for teaching trigonometric functions?'

The use of documents to collect data is advantageous in that it needs less time and, hence, is cost-effective because data is already available in the document (Bowen, 2009). In addition, the lesson plans generated by teachers in my study can be reviewed, analysed, and studied repeatedly. One of the disadvantages of documents, as Bowen (2009) puts it, is that documents may contain insufficient detail required for research questions. In my study, this flaw was mitigated by the fact that the lesson plans had been specifically designed for the teaching of trigonometric functions and were meant to inform lesson presentation and observation. Also, documents are sometimes difficult to retrieve (Bowen, 2009). However, the lesson plans in my study were not prone to this limitation since they were produced by participating teachers. From the ongoing discussion, the advantages of using documents outweigh the flaws.

4.7.2.3 Semi-structured interviews

An interview is a conversation characterised by questions and responses between the participant and researcher where their interaction produces data needed to answer a research topic (Puspitarini & Hanif, 2019; Willig, 2022). Interviews can be conducted either on-line or face-to-face (Silverman, 2020). Furthermore, Silverman (2020) posits that there are three types of interviews, namely; structured interviews, semi-structured interviews and open-ended interviews. In my study, I used face-to-face semi-structured interviews where I interviewed teachers in their natural school environment in line with the interpretive paradigm. Semi-structured interviews are characterised by pre-determined questions that are informed by the study topic and/or the researcher's experience in the participants' work environment (King et al., 2019; Silverman, 2020). Unlike structured interviews, semi-structured interviews allow the researcher to seek further clarification on a participant's response.

Participating teachers were interviewed as a group for one and a quarter hour on day 6, two days after the LS cycle sessions had been completed as shown in Table 4.3. The semi-structured interview schedule in Appendix B was used and I video-recorded the interviews. The interview aimed to solicit teachers' perspectives on the use of LS in developing mathematical knowledge for teaching trigonometric functions. This helped in answering research question 4: 'What are the teachers' perspectives on the use of Lesson Study in the development of mathematical knowledge for teaching trigonometric functions?'. Employing semi-structured interviews has the strength of 'squeezing' perspectives and experiences on a topic out of participants (Billups, 2021). Semi-structured interviews eliminate the use of unnecessary and ambiguous questions by the researcher. Again, by using semi-structured interviews, I had the advantage of discovering new aspects of the problem by exploring in detail the explanations supplied by the teachers through probing or follow-up questions (Bless et al., 2006; Silverman, 2020).

The main disadvantage of semi-structured interviews is that they bring in bias (through asking leading questions) if the interviewer is not competent (Bless et al., 2006). I alleviated this by asking questions that were aligned to the research question of the study. This was supported by my interview schedule or protocol which guided, customised, and facilitated my interviewing of the teachers (Billups, 2021; Creswell & Poth, 2018; King et al., 2019). Furthermore, my supervisor gave guidance and recommendations for the finalisation of the interview protocol.

4.7.3 Data Analysis and Interpretation

Data were analysed qualitatively to determine emerging categories and gain deeper insights into the teachers’ use of LS to develop mathematical knowledge for teaching trigonometric functions. Pascua (2014) defines qualitative data analysis as a process that seeks to reduce and make sense of vast amounts of information from different sources (interviews, transcripts, lesson plans, videos) so that impressions that shed light on research questions can emerge. In my study, the analysis was done according to the domains in the theoretical framework and through thematic analysis which are the two approaches that qualitative analysis thrives on (Pascua, 2014). Data in my study consisted of document analysis (lesson plans and teacher notes), observations, and interviews. Initial MKT domain codes (and indicators) emanating deductively from the MKT framework as adapted from Ní Shúilleabháin and Clivaz (2017) and Clivaz and Ni Shuilleabhain (2019) served as the basis of data analysis in this study (see Appendix A). Nowell et al. (2017) posit that such a pre-existing coding framework provides a detailed analysis of aspects and indicators of the data that a researcher is interested in exploring. My data analysis was mainly based on and guided by a conglomerate of (Braun & Clarke, 2006) six phases of thematic analysis and the embedded technique postulated by Adu (2023).

Braun and Clarke (2006) perceive thematic analysis as a method for identifying, analysing, and reporting patterns and themes within data. The process involves searching across a data set (in this case transcripts from interviews and observations) to find repeated patterns of meaning (Braun & Clarke, 2006). The nature of the process instilled rigour in my study through triangulation. This assertion is supported by Kiger and Varpio (2020) who report thematic analysis as a powerful and flexible method that is appropriate for seeking to understand experiences, thoughts, or behaviours across a data set.

Thematic analysis, according to Braun and Clarke (2006), is achieved through six phases which are: (1) familiarising oneself with collected data, (2) generating initial codes, (3) searching for themes, (4) reviewing themes, (5) defining and naming themes, and (6) producing the report (see Figure 4.2).

Figure 4.2

Braun And Clarke's Six Phases Of Thematic Analysis.

To familiarise myself with data collected through two collaborative lesson planning sessions; three lesson presentation and observation sessions; three post-lesson reflection sessions (including refining of lesson plans 2 and 3 during sessions 1 and 2, respectively); and one interview session, I replayed and watched each video recording per session several times. From this practice, I started to identify areas of the recordings that were of relevance to the research questions of my study.

Adu (2023) defines the embedded technique as the data analysis where a researcher integrates or incorporates the framework into research questions or vice versa, and this is deductive. I hold that the embedded technique facilitates the structuring and presentation of data analysis. In essence, this informed my formulation of themes from the four research questions of my study during data analysis. Phases (2), (3), (4) and (5) approach got clustered together spontaneously for the following recordings: collaborative lesson planning; lesson presentation and observation; and post-lesson reflection. This followed the fact that I used the MKT domains as codes and research questions as themes for the data in these recordings.

My thematic analysis was enhanced by using Atlas.ti® software which is suited for transcribed qualitative data as well as videos. It is contended that the analysis involves a constant moving back and forward between the entire data set, the coded extracts of data that one is analysing, and the analysis of the data that one is producing. I used Atlas.ti® to code and comment on areas that portrayed tenets of MK domains. As for the interviews, I started with many codes (emanating from the teachers' utterances) that gradually reduced to six categories. I did the coding in the interviews with LS and MKT in mind.

There is flexibility and non-linearity in the analysis of qualitative data and this enabled me to use varied ways of analysing data in addition to Braun and Clarke (2006) approach (Karga Giritli, 2023; Lester et al., 2020). In actuality, it is argued that thematic analysis in general [including that of Braun and Clarke (2006)] is flexible, thereby allowing analysis of data from different sources (Amartey et al., 2024; Barns et al., 2021; Cassell & Bishop, 2019; Lester et al., 2020; Wood et al., 2020; Woodward et al., 2020; Zeimet, 2012). It follows that the Braun and Clarke (2006) approach is not rigid.

4.7.3.1 Analysis and Interpretation of Lesson Plans

Data from lesson plans and teacher notes produced by teachers during the collaborative lesson planning; lesson presentation and observation; and post-lesson reflection stages were analysed through document analysis. It is held that document analysis is compatible with the interpretive paradigm that informed the current study (Bowen, 2009; Braun & Clarke, 2006). Morgan (2022) argues that document analysis is both a data collection and analysis method. The analytic procedure, as Bowen

(2009) puts it, encompasses finding, selecting, appraising (making of), and synthesising data contained in documents. I analysed lesson plans (from the production of rough drafts to the final) and teacher notes by identifying indicators of teachers' development of mathematical knowledge in trigonometric functions. This was done in line with the tenets of LS and the theoretical framework. In addition, I analysed the texts in the lesson plans in relation to the teachers' discussions during collaborative planning, the lesson presentations and observations, the teachers' discussions during post-lesson reflections, and what teachers said during interviews. In essence, I interwove document analysis with observations.

4.7.3.2 Analysis and Interpretation of Observations

The observations done during the collaborative lesson planning stage, the lesson presentation and observation stage, and the post-lesson reflection stage were all captured in video. I started analysing the observations by transcribing the video recordings and commenting on each transcription. I re-played the transcriptions and read the comments several times for familiarisation purposes followed by coding (Clarke et al., 2015). The six domains of the MKT framework assisted me in coming up with codes, categories, and themes and defining and refining them. Atlas.ti® software facilitated my coding and categorisation. I further achieved intense analysis by replaying the videos whilst completing the tools in Appendices B1 to B3. These tools are made up of possible MKT indicators (in the form of tenets of each of the six domains: CCK, SCK, KCS, KCT, HCK, and KCC) per LS stage that I adapted from Ní Shúilleabháin and Clivaz (2017). Within Atlas.ti, I assigned a colour for each domain/code, and these are the same colours that I used in the bar graphs associated with each LS stage (see Chapter 5 Figures 5.1, 5.15 and 5.30). Bavdekar (2015) posits that colouring of bars is important since it facilitates visualisation by the reader. Each bar represented the approximate number of occurrences of a domain (in the form of participating teachers' utterances and all other collaborative activities) within a LS stage. My use of bar graphs in presenting findings gleaned from the manifestation of MKT domains is informed by Clivaz and Ni Shuilleabhain (2019). Bar graphs are compatible with qualitative data analysis (Bavdekar, 2015; Brewer et al., 2012; Çatman Aksoy & Işıksal Bostan, 2021; Martinez, 2015; Ng, 2016; Rolfes et al., 2018).

4.7.3.3 Analysis and Interpretation of Interviews

The analysis of interviews started by transcribing verbatim and familiarising myself with the video recordings. I then used Atlas.ti® software to code the transcriptions, which, in turn, facilitated the generation of themes which I named to produce a report (Braun & Clarke, 2006). In the process, I also analysed the concurrences and possible contradictions of the teachers' perspectives on the development of teachers' mathematical knowledge for teaching trigonometric functions through LS. My analysis of the interviews was also done in relation to what was observed during the collaborative planning, the lesson presentation and observation, and the post-lesson reflection stages. The interviews triangulated observations and document analysis.

4.8 QUALITY ASSURANCE/CRITERIA

The quality assurance/criteria of my study were achieved through trustworthiness. Qualitative researchers insist that trustworthiness, which was invented by Lincoln and Guba in 1985, is the appropriate means of evaluating or quality-assuring qualitative studies (Billups, 2021; Creswell & Poth, 2018; King et al., 2019; Kumar, 2014; Stahl &

King, 2020). In my study, trustworthiness was accomplished through four indicators which are credibility, transferability, dependability, and confirmability.

The credibility of a qualitative study entails the truthfulness of findings on the phenomenon being explored (Billups, 2021; Stahl & King, 2020). The credibility of teachers' uses of LS to develop mathematical knowledge for teaching trigonometric functions was established through triangulation. Triangulation refers to the use of multiple methods of data collection, data sources, analysts, and/or theories on a single phenomenon for corroborating the findings (Billups, 2021). In my study, I triangulated by using document analysis, observations, and interviews as methods of data collection. The use of these three methods of data collection is a tenet of the interpretive paradigm that informed my study.

Transferability is viewed by Trochim and Donnelly (2007) as the degree to which the results of a qualitative study can be transferred to other contexts or settings. I attained transferability through thick description where I extensively and thoroughly described the process that I employed for the study so that other researchers may follow and replicate it (Billups, 2021; Stahl & King, 2020). This involved detailing the participating teachers' characteristics/demographics and their school location and environment (see Section 4.7.1). I further thoroughly described how I collected data through document analysis, observations, and interviews (see Section 4.7.2). I also stated the time taken to collect data and the duration of the whole study (see Table 4.3).

The third indicator, which is dependability, is concerned with whether the same results would be obtained if the same thing could be observed twice (Trochim & Donnelly, 2007). In addition, Abidin et al. (2024) assert that dependability could be achieved by thoroughly narrating, describing and documenting methodology, data collection, and analysis processes. In this study, I attained dependability by keeping an extensive and detailed record of the study process (see Sections 4.1 to 4.7), for others to replicate to ascertain the level of dependability. In fact, the greater part of the study was captured in video.

According to Trochim and Donnelly (2007), confirmability is the degree to which the results of a study are confirmed or corroborated by others. In the same vein, Billups (2021) perceives confirmability as the element of trustworthiness that is concerned with the accuracy of the findings of a phenomenon. In my study, I achieved confirmability through audit trails and reflexivity. Audit trails are reflected in the way that I clarified the processes and procedures followed to complete my study. This enables readers of my study and other researchers to replicate it and compare the results (Billups, 2021; Kumar, 2014). I attained reflexivity by stating my experiences in conducting the study as well as disclosing possible biases (Billups, 2021; Creswell & Poth, 2018).

4.9 ETHICAL CONSIDERATIONS

Researchers need to address ethical issues throughout various stages of the study process by ensuring that the participants' rights are protected (Creswell & Poth, 2018; L. Farrugia, 2019). In my study, I protected teachers' and learners' rights by not harming them, avoiding deception, getting informed consent from them (see letters to teachers and learners in Appendix D), and establishing privacy and confidentiality (Tracy, 2013).

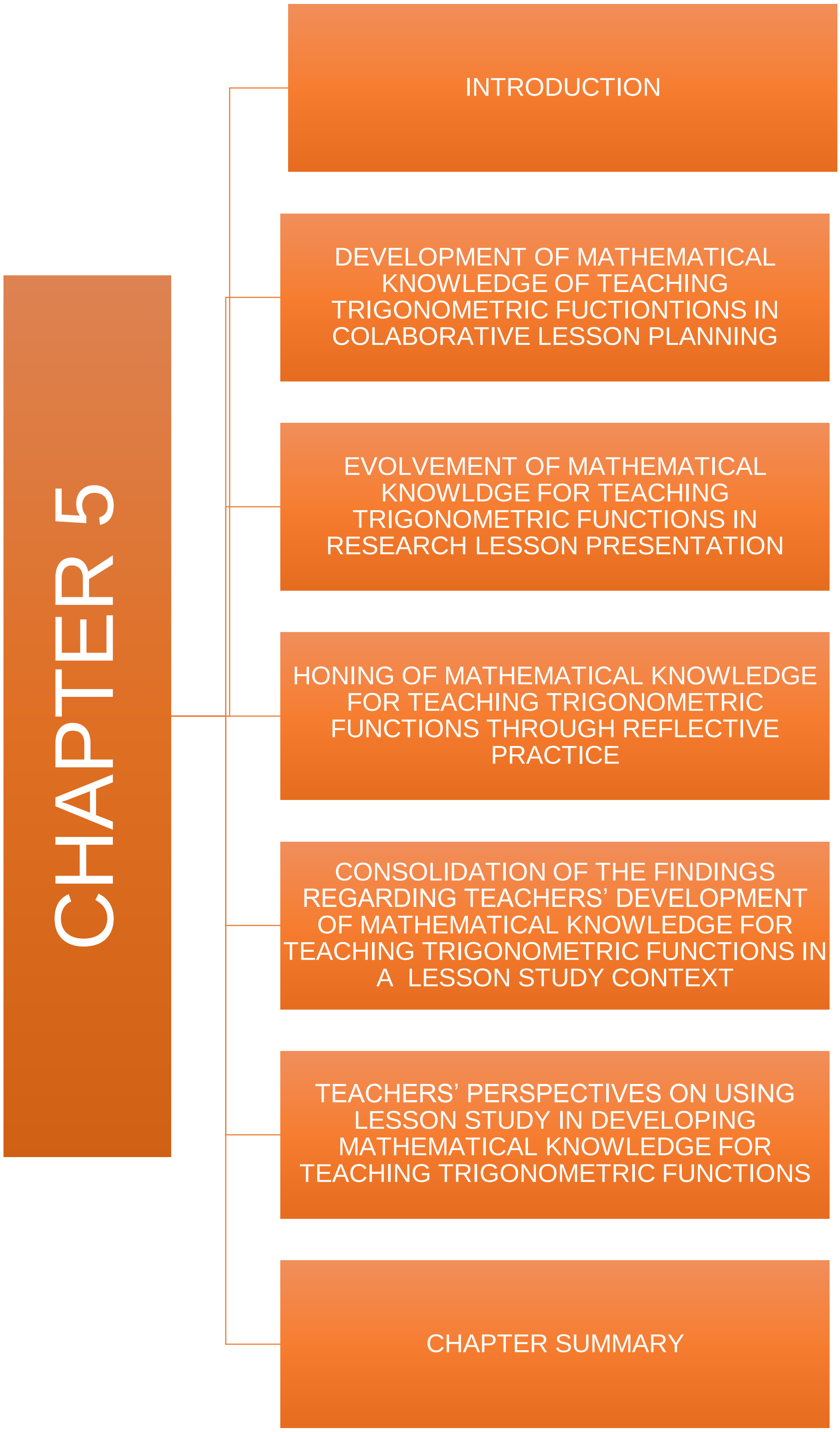
My journey to address ethical issues started with my seeking ethical approval from the Faculty of Education's Ethics Committee at the University of Pretoria, and permission from the Gauteng Department of Education to conduct this study. The approval and permission were a certification that participants were not going to be harmed psychologically and or physically (Lincoln & Guba, 1989; Suri, 2020). Further permission was sought from the principal of the school (see letter to the principal in Appendix D). Informed consent requires that the researcher informs participants about the purpose of the study and requests their participation (Creswell & Poth, 2018; L. Farrugia, 2019; Lincoln & Guba, 1989). I secured informed written consent from participants, particularly teachers. As for learners, I sought written consent from their parents. In both cases I clearly stated that their participation was voluntary and that they were free to opt out of the research any time without facing consequences (see letter to parents in Appendix D). After I collected data according to the approval by the Ethics Committee, I applied for the Ethics Clearance certificate from the Ethics Committee.

Confidentiality and anonymity/privacy encompass the need to conceal participants' identities and safeguard data obtained from them (Billups, 2021; Creswell & Poth, 2018). In my study, I used pseudonyms for the participating teachers by naming them Teacher A, Teacher B, Teacher C, Teacher D, Teacher E, and Teacher F. The faces of learners who were in class during the lesson presentation and observation stages were blurred to hide their identities. Again, data collected from the participants in the form of lesson plans, video-recorded observations and interviews were used for research purposes only. I secured the video recordings with a password on my computer.

4.10 CHAPTER SUMMARY

In this chapter, I gave details of the research methodology processes that sought to answer the research questions of my study. The methodology followed Saunders et al. (2023) research onion and was informed by the interpretivist paradigm. My study was an exploratory qualitative case study in which six teachers participated. I collected data by observing the teachers during the three LS stages, namely; lesson planning, lesson presentation and observation, and post-lesson reflection. Again, the lesson plans (documents) generated by the teachers were a data source. In addition, I interviewed the teachers after they had completed the LS processes to solicit their perspectives on the use of LS to develop teachers' mathematical knowledge for teaching trigonometric functions. Data from observations and interviews were analysed using the thematic analysis stages coined by Braun and Clarke (2006). Atlas.ti software was employed in this thematic analysis. I analysed lesson plans using document analysis. The MKT framework and LS tenets assisted me to understand and interpret data. My methodology chapter ended by detailing quality assurance and ethical considerations. I presented the findings of my study in Chapter 5, which is next.

5. CHAPTER 5: FINDINGS



5.1 INTRODUCTION

In the current study I explored how teachers developed mathematical knowledge for teaching trigonometric functions through Lesson Study (LS). In this chapter, I presented the findings from the data collected during the three stages of the South African LS model (i.e. collaborative lesson planning, lesson presentation and observation, post-lesson reflection) and interviews. The findings presented in the current chapter were used to answer the main research question: How do teachers develop mathematical knowledge for teaching trigonometric functions through Lesson Study? I accomplished this by pursuing the following sub-questions: (i) How do teachers develop mathematical knowledge for teaching trigonometric functions in the collaborative lesson planning stage? (ii) How does teachers' mathematical knowledge for teaching trigonometric functions evolve during research lesson presentation? (iii) How do teachers' reflective practices hone their mathematical knowledge for teaching trigonometric functions? (iv) What are the teachers' perspectives on the use of Lesson Study in developing mathematical knowledge for teaching trigonometric functions? I observed six high school teachers as they participated in three cycles of the in-school South African LS model. I then interviewed the teachers after they had completed the LS activities. All LS activities and interviews were video recorded. Document analysis comprised of lesson plans and teacher notes.

My data analysis was a blend of Braun and Clarke (2006) six-step thematic analysis and the embedded technique proposed by Adu (2023) (see Chapter 4 Section 4.7.3). I searched for themes and patterns within and across observations, interviews, lesson plans, and teacher notes. This involved playing and replaying video recordings and scanning through lesson plans and teacher notes many times. My analysis was enhanced by the use of Atlas.ti in the transcription of videos. Adu (2023) embedded technique helped me to structure and present my data analysis by incorporating and integrating the framework into the research questions. The Mathematical Knowledge for Teaching framework (MKT) helped me to understand and interpret data, that is, analyse data (Grant & Osanloo, 2014; Luft et al., 2022). I used the MKT domains (namely; Common Content Knowledge (CCK), Specialised Content Knowledge (SCK), Horizon Content Knowledge (HCK), Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC)) as codes and research questions as themes. The synthesis of my observations indicated that the MKT domains (CCK, SCK, HCK, KCS, KCT, and KCC) thrived on collaborative lesson planning, lesson presentation and observation, and post-lesson observation stages (Clivaz & Ni Shuilleabhain, 2019; Ní Shúilleabháin & Clivaz, 2017). The first three research questions directly corresponded with the LS stages thereby rendering the embedded technique of incorporating the framework into research questions relevant to my study. From the data collected during each stage of the LS, I could identify/observe MKT domains from the participating teachers' collaborative activities. I, in turn, coded and quoted the tenets of the domains in the videos using Atlas. ti software. The domains overlapped frequently leaving limited room for a clear-cut tenet to a single domain. This notion concurred with Thanheiser et al. (2010) and Moumou (2021) who hold that MKT domains have blurred and/ or overlapping boundaries among themselves.

The presentation of findings was structured according to the four sub-questions which I used as themes (see Table 5.1).

Table 5.1

Themes And Categories

Themes	CATEGORIES
Development of mathematical knowledge for teaching trigonometric functions in collaborative lesson planning	- collaborative planning of research lesson 1 - collaborative planning of research lesson 2 - collaborative planning of research lesson 3
Evolvement of mathematical knowledge for teaching trigonometric functions in research lesson presentation	- presentation and observation of research lesson 1 - presentation and observation of research lesson 2 - presentation and observation of research lesson 3
Honing of mathematical knowledge for teaching trigonometric functions	- post-lesson reflection of research lesson 1 - post-lesson reflection of research lesson 2 - post-lesson reflection of research lesson 3
Teachers' perspectives on using LS in developing mathematical knowledge for teaching trigonometric functions	- teachers' experiences in collaborative lesson planning - affordances of implementing and observing the collaboratively planned lesson - teachers' experiences in post-lesson reflection dialogical discussions - trigonometric functions content acquired by teachers from participating in LS. - pedagogic/teaching strategies learnt from the LS processes. - teachers' closing remarks on the use of Lesson Study in developing mathematical knowledge for teaching trigonometric functions.

Under the guidance of Braun and Clarke (2006) six phases of thematic analysis, I coded the data according to the existence/incorporation of any MKT domain(s) under each of the categories. Each MKT domain was identified according to its tenets observed in the teachers' collaborative activities (see column 3 of Table 5.2 in Section 5.5). I wove the analysis of lesson plans and teacher notes across the LS stages. My analysis of interviews also followed Braun and Clarke (2006) approach while I understood the data through the MKT lens. There were instances where teachers used other languages other than English during their collaborative discussions. I indicated the languages in italics, followed by their translation in square brackets.

5.2 DEVELOPMENT OF MATHEMATICAL KNOWLEDGE FOR TEACHING TRIGONOMETRIC FUNCTIONS IN COLLABORATIVE LESSON PLANNING

In this section, I presented the findings of the first secondary research question of my study. The research question sought to explore how teachers developed mathematical knowledge for teaching trigonometric functions in the collaborative lesson-planning stage. To achieve the exploration, I observed the manifestations of Mathematical Knowledge for Teaching (MKT) domains in participating teachers' activities and discussions during their collaborative lesson planning sessions. The characteristics (tenets or indicators) of domains found in column three of Table 5.1 helped in my observations and analysis. In particular, I identified the tenets of CCK, SCK, KCS, KCT, HCK, and KCC portrayed by teachers' collaborative lesson-planning activities and discussions. The teachers planned three lessons over two days. I video-recorded all the planning sessions to facilitate the analysis of findings.

5.2.1 Context of the Planning Sessions

According to the prescripts of LS, collaborative lesson planning involves certain key elements like stating objectives, studying teaching materials (*kyouzai kenkyuu*) (curriculum and instructional materials) and selecting purposeful activities (Sekao, 2023). A total of six teachers participated in the school-based Lesson Study involving the teaching of trigonometric functions. The collaborative lesson planning sessions were done after school in the computer laboratory from 15:10 to 16:34 on the first day and from 15:13 to 16:59 on the second day. The teachers sat in an arrangement as shown in Figure 5.1. The teachers wore name tags where their pseudonyms were written. Teacher E could not attend both collaborative lesson planning sessions due to administration duties in the school assessment team. The teacher managed to attend the refining of lesson 2 which was done after the post-lesson reflection of lesson 1 (see Table 4.2 on attendance schedule in Chapter 4 sub-section 4.7.1).

Figure 5.1

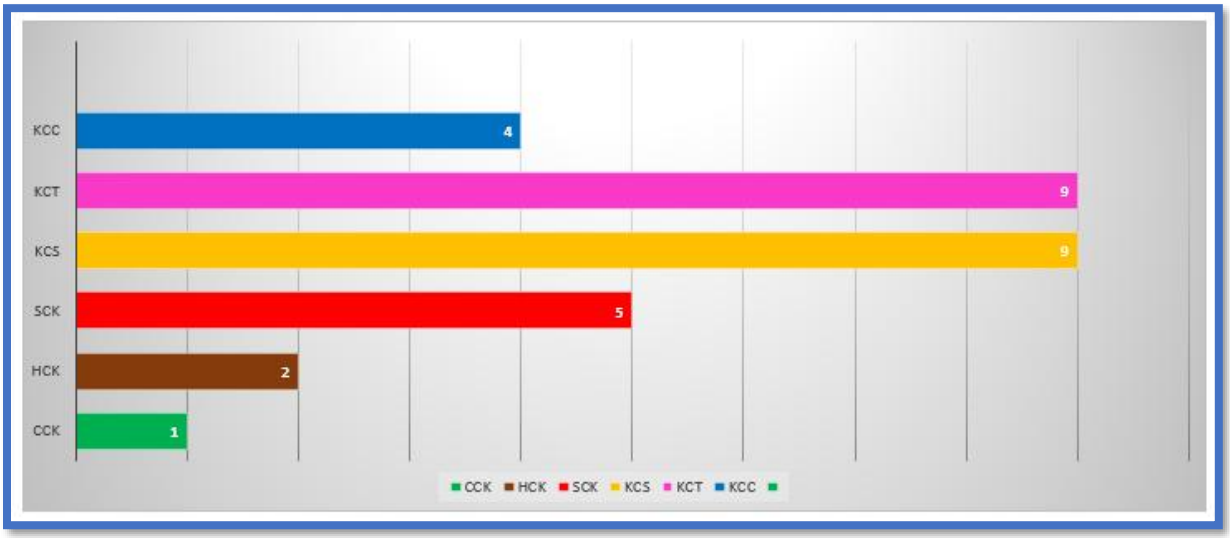
Teachers Planning In The Computer Laboratory.

The teachers brought curriculum materials (Sekao, 2023) in the form of different mathematics textbooks, printed ATP, as well as instructional materials such as calculators, laptops, cell phones and notebooks (see Figure 5.1). The CAPS document was in electronic format since I observed teachers viewing it from their cell phones or laptops. All the teachers scribbled some notes in their notebooks during collaborative planning, while Teacher F was tasked to compile, type, and print the final lesson plans. I interwove the analysis of the teacher's notes and lesson plans with observations.

Although the tenets of different domains tended to overlap, the approximate manifestation (occurrences or indicators) of MKT domains during teachers' collaborative lesson-planning sessions are shown in the bar graph in Figure 5.2. The graph is similar to that of Clivaz and Ni Shuilleabhain (2019) (see Chapter 4 section 4.7.3.2).

Figure 5.2

Occurrences Of Domains During Collaborative Lesson Planning.



The graph (Figure 5.2) shows that teachers incorporated and engaged in KCT, KCS and SCK more frequently than in other domains.

5.2.2 Collaborative planning of Research Lesson 1

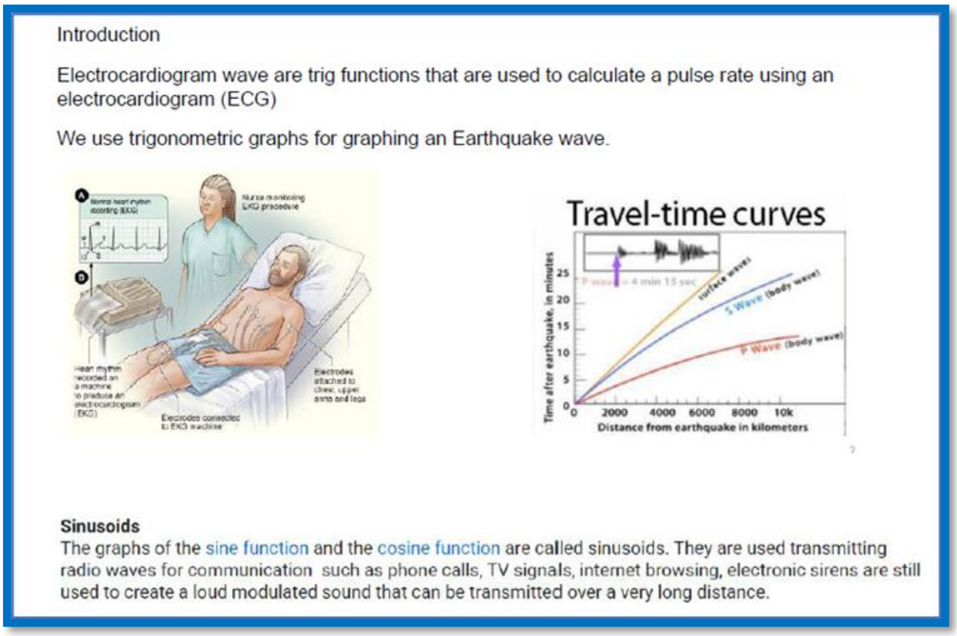
Research lesson 1 (see Appendix C) was aimed at firstly enabling the learners to, point-by-point, plot the basic graphs/functions $y = \sin x$, $y = \cos x$, and $y = \tan x$. The second objective of lesson 1 required learners to investigate the effects of parameter *k* on the functions $y = \sin kx$, $y = \cos kx$, and $y = \tan kx$. According to my observations, teachers' dialogical interactions during their collaborative planning of research lesson 1 bore all the six domains of the MKT. I elaborated on the domains in the six sub-sections below. The tenets of the domains are contained in Table 5.2.

5.2.2.1 Common Content Knowledge (CCK)

The collaborative school-based LS team agreed upon the importance of introducing trigonometric functions by identifying where they were used in real-life situations. The introductory part from lesson plan 1 (see Appendix C) is shown in Figure 5.3.

Figure 5.3

Applications Of Trigonometric Graphs.



Notes: Assembled from various internet sources by teachers

The pulse screen in the hospital displays changes in amplitude (parameter a introduced in Grade 10) and in period (parameter k to be introduced in Grade 11). This introduction portrays CCK. The textbooks that the teachers used during the planning did not cover anything on the real-life applications of trigonometric functions. The development of such CCK by participating teachers for the introduction of trigonometric functions shows that teachers were collaboratively thinking about and agreeing on strategies for introducing the topic so that their learners would understand better.

5.2.2.2 Horizon Content Knowledge (HCK)

The discussion on using trigonometric functions in real-life applications also qualifies as HCK. This is because the application (see Figure 5.3) is not found in the Grade 11 ATP but later (in the far horizon) in the life of careers. Again, assumed knowledge in the form of point-by-point plotting of basic graphs ($y = \sin kx$, $y = \cos kx$, and $y = \tan kx$) and effects of parameters a and q served as HCK during teachers' planning of lesson 1.

5.2.2.3 Knowledge of Content and Curriculum (KCC)

My observations of teachers' collaborative lesson-planning sessions on day 1 revealed four examples, indicators, tenets, or characteristics of KCC. Firstly, the teachers consulted curriculum materials or pedagogical documents (ATP and CAPS) from hard copies and downloads on their phones, respectively. Teacher D (paraphrasing from the ATP) noted that: "…in Grade 11 learners should be able to investigate the effects of parameter k …These learners are from Grade 10, I think we need to link up with what we are going to teach them". In agreement, Teacher F said: "Prior knowledge first. Prior knowledge first." In addition, Teacher D pointed out: "In Grade 11, we are introducing the negative sign because in Grade 10 we started from 0° to 360°. In Grade 11 we start to work from -360° ." The intervals mentioned by Teacher D referred to prescribed domains of trigonometric graphs in each grade.

Secondly, Teacher C asked other participants this question: *"Kanti ama* effects of a and q don't they know them from Grade 10?" [Is it not that the learners learnt the effects of a and q in Grade 10?]. In response, Teacher F said, “They draw them, however according to this (lifting the ATP document) we are supposed to do point-by-point plotting we must do it …ko Grade 11…*akere* [isn’t it] it’s built up from Grade 10…” [They draw them, however, according to this (lifting the ATP document) we are supposed to do point-by-point plotting, we must do it …in Grade 11…I mean it’s a build-up from Grade 10…]. This dialogue signalled teachers’ awareness and acknowledgement of the Grade 10 curriculum that linked to the Grade 11 trigonometric functions content.

Thirdly, the teachers debated that q had been learnt in Grade 10 and that p should be introduced in Grade 11. Teacher C said, “As you are saying q is their prior knowledge…” Teacher B interjected, saying: “*Sesiya introducer u* p” [We are going to introduce p].

Fourthly, on day 2 while compiling a class activity for lesson 1, Teacher D reminded other teachers about the domain of trigonometric functions that should be used in Grade 11. Teacher D’s spoke thus: “Let’s change it. In Grade 11 they should get used to this negative story. The equation you can leave it. The interval.” Teacher F agreed and added, “Oh, yaa yaa.” [Oh, yes, yes]. Teacher D continued and said, “We just want them to get used to this negative interval.” The group changed the interval for $y = \tan\left(\frac{1}{2}x\right)$ from 0° to 360° (according to what was in the textbook) to $x \in \left[-180^\circ;180^\circ\right]$. This suggests that the teachers had tailored and or altered the activities in the textbook to suitable purposeful activities.

From their discussions, it became evident that the teachers were engaging one another on aspects of trigonometric functions done in each grade. The teachers were deliberating on how the content in the two grades (Grades 10 and 11) could be sequenced and aligned. This made these interactions KCC according to the definitions and tenets of the domain.

5.2.2.4 Knowledge of Content and Students (KCS)

Whilst planning, teachers also delved into collaborative discussions and/or activities that suggested KCS four times. Teacher B raised the notion that Lesson 1 should start by plotting the basic graphs followed by investigating the effects of p. This became evident where Teacher B said:

> *Manje besisa bheka the effects of p. Iyasi guider iCAPS. Basitshelilile bathi after plotting the basic graphs we have to investigate the effects of parameter p on the graph of functions defined by* $y = \sin x$*. Asibajahi sizalabo kuhle. Kusukela kule positive abayaziyo sihamba labo. Nethi sesi introducer ubani u* p*.* [We were still considering the effects of parameter p according to the CAPS. The CAPS prescribes the plotting of basic graphs of functions such as $y = \sin x$ followed by investigating the effects of parameter p. Let us not rush the learners, but smoothly introduce them to parameter p.]

Teacher F and Teacher D said simultaneously, "*Sifundile"* [We have learnt something from what you are saying]. All the teachers laughed with excitement. Teacher B continued, "…*Vele ngeke siqhubeke ngoba abanye isabahlula leyo basic. Because* *once singaba right lapho ku foundation singahamba kahle vele."* [Truly, we cannot continue without addressing basic graphs first because some learners are still struggling with them. Everything will flow smoothly once the foundation has been addressed]. All the teachers agreed that the first lesson should focus on basic graphs and parameter p . This would also be a way of sequencing and aligning content that is characteristic of KCT, SCK and KCC.

The extract from Teacher C's utterances brought awareness of KCS to other teachers during the discussion:

> We have to test the learners' prior knowledge. *Phela* [By the way] we know our kids if you ask them 'did you do this and this' they will say 'yes'. Let us see with a written exercise if they can remember the point-by-point plotting.

Teacher C emphasised the fact that learners would always agree to have done a topic, though without necessarily understanding it. Teacher F agreed that the learners should be given work on point-by-point plotting at the beginning of the lesson: "These ones they are going to do them, these ones."

The teachers grappled with the effects of parameter k until Teacher F admitted that they had misconceptions as teachers. This was only after Teacher D had explained. Teacher D explained in the following manner: "Once you increase the value of k the period reduces but the frequency increases. If you can look. Let's say from here to there the period is 360°, but in that 360° this graph will make two graphs".

With excitement, all the teachers said, “We are answered!”. Teacher F continued, “We are answered. Yeah misconceptions.” The parameter k affects the period. So, the teachers took a lot of time struggling to phrase the effects of k and come up with elaborations that would enable them to teach with confidence. In the end, the outcome appeared in the lesson plan (see lesson 1 in Appendix C) as follows:

> There is this thing that I want to ask. A learner asked me ‘Why is it tan graph the period is 180°?’ *Askere ku* [Isn’t it in] Grade 11 when I started I said the graph of sine starts from 0° to 360°; $y = \sin x$, it starts from 0° until 360°. That is the completion of one wave, that’s what I said. And the graph of cos again one wavelength from 0° to 360°. However, tan 0° to 180°. Why? A learner asked me ‘Why *Meneer* [sir] is it like this? What is the reason?’.

Teacher D responded (drawing the tangent graph on paper while all teachers watched): “For tan, the complete wavelength is from 90° to 270°.” All the other teachers asked at the same time “Why?” Teacher D continued, “You can see this part (pointing at the tangent graph component found at 90°) is similar to this part (pointing at the tangent graph component found at 270°).” (see Figure 5.4).

Figure 5.4

Teacher D Explaining The Periodicity Of The Tangent Graph To Other Teachers.

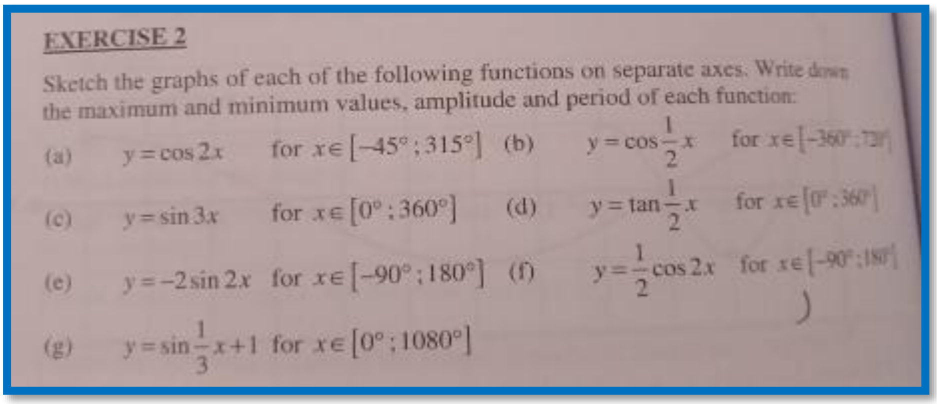
Teacher F responded: “Ooh, oh a complete wavelength is at between 90° and 270°, which is a 180° movement?” Teacher D said, “Yes.” All the teachers laughed to signal their understanding of Teacher D’s explanation and Teacher C said, (with excitement): *“Siyabangena!”* [“We are understanding it very well”]. Teacher D went on to explain the notion thus: “This approach will help us when we introduce from the negative side.” From this discussion, it is clear that teachers previously defined the period of trigonometric graphs in relation (with reference) to the origin of the cartesian plane. This discussion was also typical of the development of SCK.

5.2.2.6 Specialised Content Knowledge (SCK)

Whilst planning, teachers managed to evaluate and choose an exercise suitable for their lesson(s). I heard Teacher B saying, “Yaa *liyabona lama* questions *la*. *Iyangichaza* *le* textbook because *ngala ma* questions for *ama* basic functions *akhona la. Sibanike iquestion. Sengiyasho ku* Exercise 2" [Colleagues do you see these questions? The questions from this textbook are interesting. Let us give them questions from Exercise 2]. In response, Teacher F agreed: "Ok. They say, 'Sketch the graph of each of the following…' and they start from the negative sign. *Yaaa* [Yes], it's fine, it's fine, it's fine. So, let me just pick a few graphs." Exercise 2 discovered and evaluated by the group is shown in Figure 5.5. Teacher F referred to the work in Exercise 2 as graphs, yet they were graphical equations or graphs of functions. They would be graphs after sketching.

Figure 5.5

Chosen Exercise For Parameters a, q *And* k



EXERCISE 2

Sketch the graphs of each of the following functions on separate axes. Write down the maximum and minimum values, amplitude and period of each function.

(a) $y=\cos 2x$ for $x\in[-45°;315°]$ (b) $y=\cos\frac{1}{2}x$ for $x\in[-360°;720°]$

(c) $y=\sin 3x$ for $x\in[0°;360°]$ (d) $y=\tan\frac{1}{2}x$ for $x\in[0°;360°]$

(e) $y=-2\sin 2x$ for $x\in[-90°;180°]$ (f) $y=\frac{1}{2}\cos 2x$ for $x\in[-90°;180°]$

(g) $y=\sin\frac{1}{3}x+1$ for $x\in[0°;1080°]$

Notes: *Mind Action Series* Grade 11 Phillips et al. (2012, p. 192)

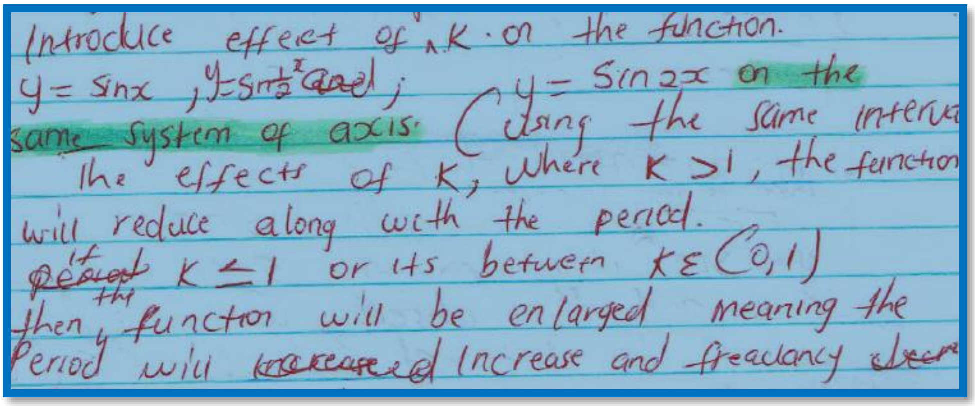
The questions in the exercise interested the teachers. This demonstrated the identification of purposeful instructional activities and is typical of SCK.

Exercise 2 (Figure 5.5) required learners to sketch the graphs defined by the given functions and this prompted Teacher A to seek clarification: "Sketching and plotting, they are different?" To explain Teacher D said, "Sketching, it's like you are not concerned about points…" There was an interjection from Teacher F, who asked, "So, Teacher D what you are saying is when we are doing point-by-point plotting we should have a graph *papernyana* [paper] for accuracy?" Teacher D answered, "You see those books that the learners are using have lines and it's easy for them to use them as a graph." Learners in South African schools use lined exercise books like the one in Figure 5.6.

The group also agreed to build the concept behind parameter k by assigning a class activity in lesson 1 that required learners to plot on the same set of axes: $y = \sin x$, $y = \sin 2x$, and $y = \sin\left(\frac{1}{2}x\right)$. During planning, I observed that teachers were also writing notes of what they were discussing. This is evidenced by Teacher C's notes as shown in Figure 5.6.

Figure 5.6

Notes On Parameter K By Teacher C


Introduce effect of K on the function.
y = sinx, y = sin ½x and; y = sin 2x on the same system of axis. (Using the same interval
The effects of K, where K > 1, the function will reduce along with the period.
if K ≤ 1 or its between K ε (0, 1)
then, the function will be enlarged meaning the Period will Increase and freaquancy decre

The notes by Teacher C summarised how the graph of a trigonometric function changes when the value of k assumes different values. Although this finding was an illustration of SCK, it also demonstrated teachers' KCT of trigonometric functions.

5.2.3 Collaborative Planning of Research Lesson 2

In research lesson 2, learners were expected to investigate the effects of the parameter p on functions defined by $y=\sin(x+p)$, $y=\cos(x+p)$,and $y=\tan(x+p)$. The planning for lesson 2 was done on days 1 and 2, and it was refined during the post-lesson reflection of lesson 1. The teachers' activities portrayed KCS, KCT and SCK.

5.2.3.1 Knowledge of Content and Students (KCS)

As teachers worked around the introduction of the parameter p to learners, Teacher B brought the following to the group: "If *uyibeka kanje* as *uminiyera etsho* it will be easy for learners to use a calculator for $y=\sin x+q$ but *bayomfaka kanjani leyo* value *ka* p *ku* calculator for $y=\sin(x+p)$?" [If you do it the way sir said, it will be easy for learners to use a calculator for $y=\sin x+q$ but how are they going to punch the value of p into the calculator for $y=\sin(x+p)$?"].

There was a deafening silence. Teacher C (shaking her right-hand palm) broke the silence by saying *"Iyangi confuser angiyi* understand *kahle."* [It confuses me, I don't understand it well]. In response, Teacher B said: "If iconfuser *lawe* obvious *iyo confuser labantwana*." [If it confuses you, it is obvious that it will confuse learners as well].` In turn, Teacher F suggested, “We must do something straightforward.” All the other teachers agreed with Teacher F and paved a way that would introduce p without confusing learners.

The teachers’ dialogical interactions revealed sharing of KCS in the tangent function compared to sine and cosine functions.Teacher C pointed out that: “*Considering ukuthi utan uyasogolisa* what can we do to make it understandable for the learners*? Ku sine laku* cosine it’s better but *sokufika ku tan iyasogolisa*.” [Considering that the tangent function is a challenge to learners, what can we do to make it understandable for the learners? Learners can handle the sine and the cosine function better, but it is always a challenge when it comes to the tangent function].

Teacher B responded by saying “*But abantwana laba* if they can plot, it shouldn’t be an issue.” [But if these learners can plot it, it should not be an issue]. These utterances indicate that teachers perceived plotting as being key to mastering the effects of parameters like p.

When the teachers were sifting through activities in the textbooks, they realised that most of the graphical equations involved the positive a parameter only. They then decided to mix positive and negative equations for learners to be able to handle both. Teacher F said, “*Are bazame* [Let us try them]. *Are bazame* [Let us try them] $y=-\sin\left(x+30^{\circ}\right)$. *Are bazame* [Let us try them]. I do not want a point whereby our kids are now used to positive.” Teacher C agreed: “And *iyamocha ma sebebona abo* negative *bayasaba*.” [I agree, it misleads learners to an extent that they get scared when they see negative graphical equations.]. The teachers were thinking about the impact of chosen examples and/or class activities on the learners.

5.2.3.2 Knowledge of Content and Teaching (KCT)

Synthesising the teachers' notes, I found that teachers had improved the class activity on introducing the parameter p from what was shown in Figure 5.7 to what appeared in Figure 5.8 and Figure 5.9.

Figure 5.7

Initially Proposed Activity From Teacher F's Notes.

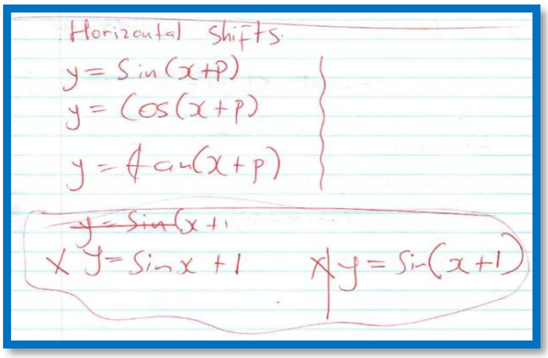


Figure 5.8

Improved Activity From Teacher F's Notes.

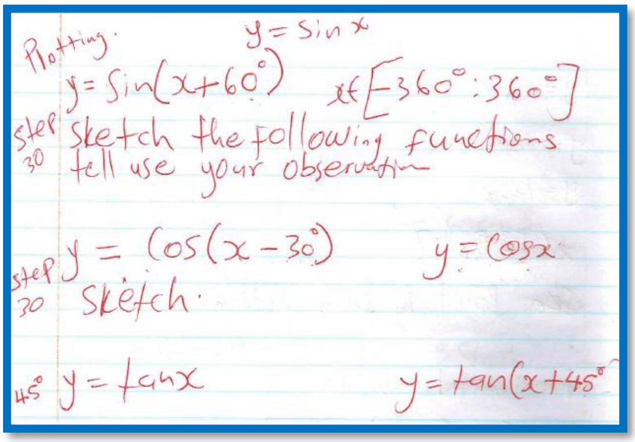
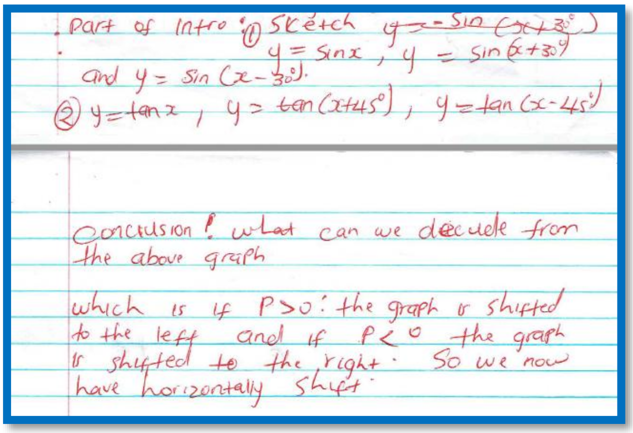


Figure 5.9

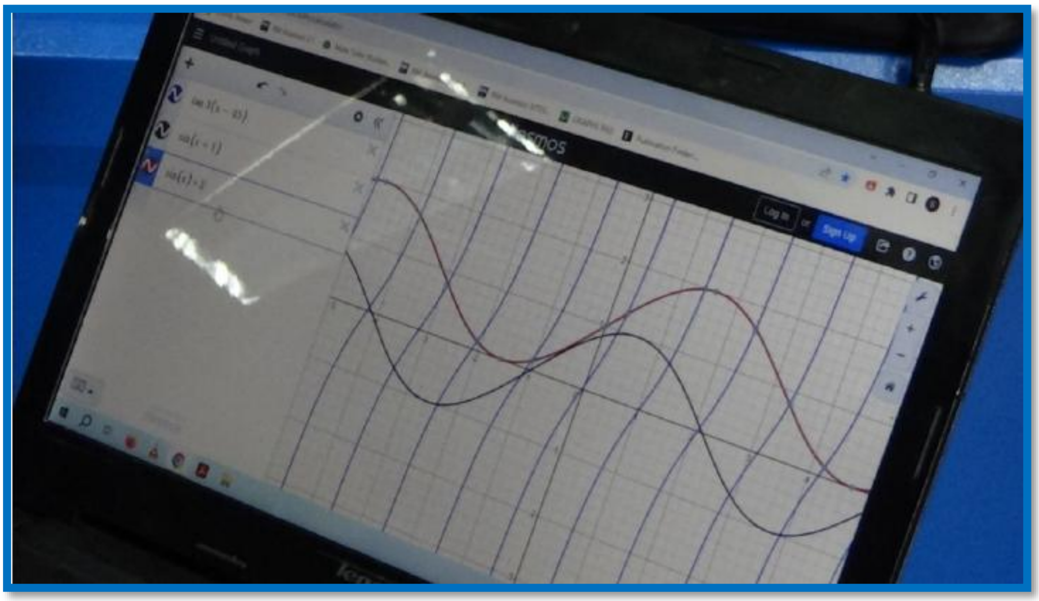
Improved Activity From Teacher C's Notes.



Teacher C bolstered the discussion by saying: "I think *iso ingaba betternyana, yaa ma iso ibetter*." [I think it is fine if we put it like that]. Everyone was heard agreeing. Opening the software Desmos, Teacher F said: "At least here you can draw more than one graph at the same time…This is an App." The other teachers laughed with excitement as Teacher F started to project the graphs generated through Desmos graphing calculator software (see Figure 5.10).

Figure 5.10

Projection Of Y=Tan3(X-45), Y=Sin(X+1) And Y=Sin X+1 On Desmos Software.



Teacher F assured other teachers that he would work on the domains of the graphs at a later stage. The educators found the software helpful in guiding learners towards the effects of parameters on trigonometric functions.

Whilst planning, the group had overlooked or skipped the parameter k. Teacher A suggested that the parameter k be addressed in Lesson 2 because learners were going to be required to use the knowledge thereof to interpret graphs in Lesson 3. Teacher A observed, “In Lesson 2 we never included the parameter k that will be used in Lesson 3 to find the equation of the given graphs.” Teacher F and the rest agreed that it had been a mistake to exclude it during the previous day's planning. Teacher F noted, “I think yesterday we made a mistake by not including k. We also need to include the issue *ya* k *ko* lesson plan the way you were explaining.” [We also need to include the parameter k in lesson 2 the way you were explaining.]

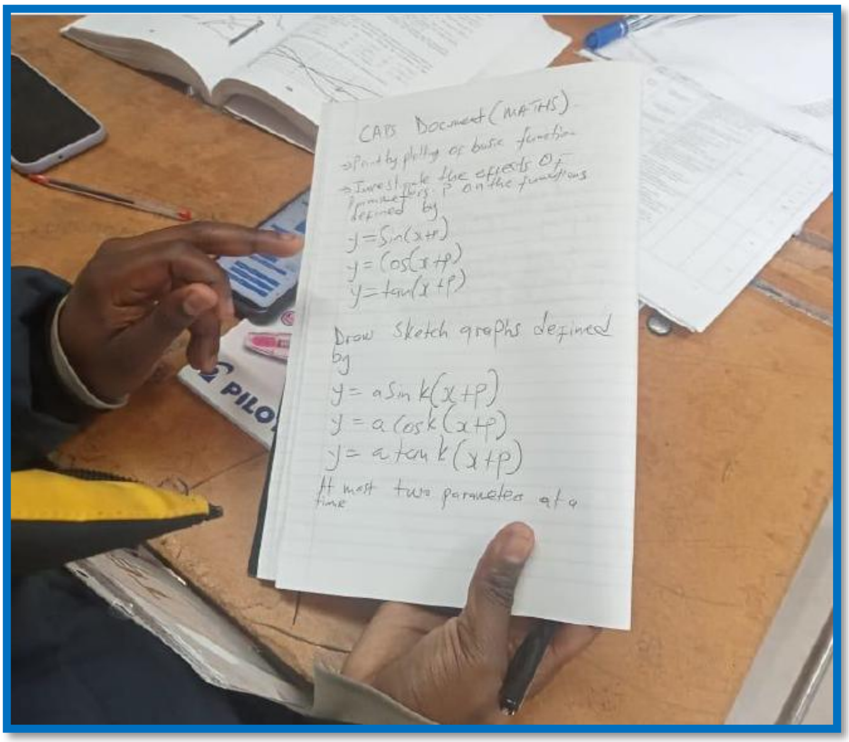
Teacher C suggested that the presenter should use Desmos software to project the three graphs (in the same set of axes) for the learners and ask them to identify the transformations. Teacher D pointed out that using 1 degree for p in $y = \sin\left(x + 1^{\circ}\right)$ would not portray a recognisable horizontal shift for the sine graph (see Figures 5.6 and 5.9). Teacher D put it as follows: “Now see here our angles, 1? If you take an angle of 1 it will look like that graph didn’t shift.” The rest of the teachers sighed, “Ooooooh”. Teacher D continued and said: “That’s why I was looking at it and thinking 1? It will look like they will get the same graph and learners say, ‘the graphs look the same’. They won’t see that shift” (moving his right hand to demonstrate a horizontal shift). The teachers' discussions came up with what would work better for learners' understanding. Development of mathematical knowledge for teaching trigonometric functions thrives in such engagements.

5.2.3.3 Specialised Content Knowledge (SCK)

Teachers’ outlining of the aspects pertaining to parameter p during their deliberations portrayed SCK. This was seen in the teachers’ discussion notes as shown in Figure 5.11.

Figure 5.11

Aspects In Grade 11 Trigonometric Functions Compiled By Teachers.



Again, Teacher B cautioned the group saying that: "*Njengoba vele ipolicy itshilo vele*. *Siyaku* $y = \sin(x+q)$. *Singamfaki u* q *singamfaki u* a. *Siye* from the basic *abayaziyo* $y = \sin x$ *uaffectwa ngubani ngu* p *kuphela*." [According to CAPS, we should deal with the parameter p alone without parameters q and a. We will have to start from the basic graph $y = \sin x$ and see how it is affected by parameter p.]. All teachers agreed to the input. Teacher B anticipated difficulty in handling p and other parameters simultaneously during the introduction.

5.2.4 Collaborative Planning of Research Lesson 3

In research lesson 3 learners were expected to determine the equations of given/drawn graphs/functions. Lesson 3 was planned/prepared in two phases like lesson 2. The initial planning/preparation was done on days 1 and 2. The lesson was then revisited, re-aligned, and refined during post-lesson reflections on lessons 1 and 2. KCS, KCT and SCK are the domains that were evident during my observation of the planning of lesson 3.

5.2.4.1 Knowledge of Content and Students (KCS)

Teacher D raised the issue of language when teaching the aspect at hand, especially, ‘drawing the original graph’ versus ‘drawing the basic graph.’ Teacher D tabled her concern to other teachers thus, “*Abantu base khaya ilanguage. Ma uthi ‘original’ umntwana uzothini.”* [Colleagues, let us mind our language. How will the learner take/understand it when we say original?]. Teacher B came in and said: “Draw original or basic? *Asithi* [Let us say] draw the basic?” The teachers ended up using ‘basic’ after agreeing that the word was more appropriate. KCT and SCK also featured in this discussion. On one hand, the agreement that the teachers reached on the use of the word ‘basic’ was pedagogical, hence KCT. On the other hand, the skill and capability of contextualising the word basic in trigonometric functions was aligned with SCK.

As the process of sifting examples/class activities continued, teachers came across an exercise that required learners to find the values of x. Teacher F argued that teachers should use roots when referring to x values early so that learners get used to the language. Teacher F observed:

> Now after finding the equations, they have to stress the issue of roots. There is an issue of roots. This question reads (reading from the book) 'Determine graphically the values of x for which $f(x)$ is equal to…' *Ama roots ngama roots. Ama roots ngama* x *-values. So, reko* grade 8 and 9 *besika berekisa* x , x , x consistently we must also include the word roots so that they get used to the idea of saying no when we talk about x *kahle kahle* we are talking about the roots of the equation. [Roots of an equation refer to x -values. In grades 8 and 9 we used the term x -values. Now we should use the term roots so that our learners get accustomed to it. You understand? It's like in the June exam, question 1 was not straightforward. They gave them a tough question 1 that needed the understanding of roots *kanjalo kanjalo* [and so on].

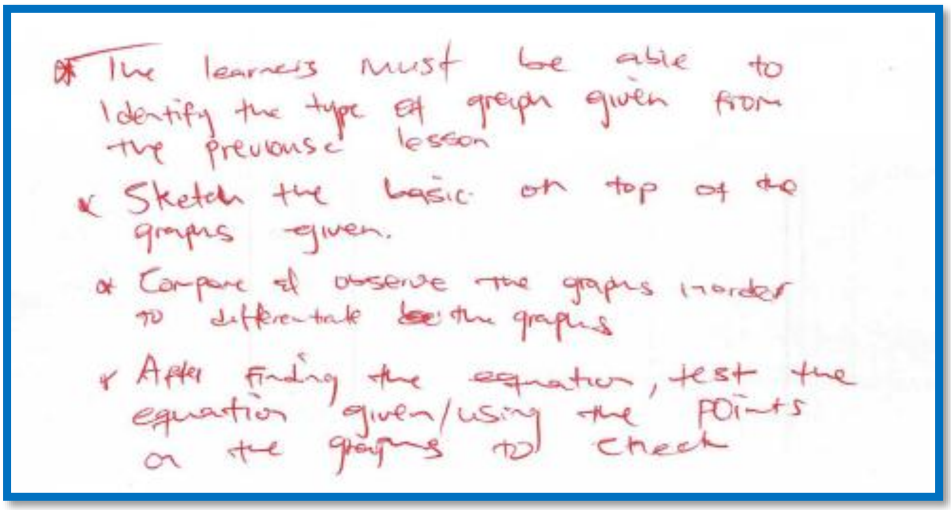
Teacher D again raised the notion that the tangent graph was difficult to handle for both learners and teachers. Teacher D observed: "Because these kids in tan. *Eee itan iyaba sokolisa*." [Because these learners struggle with the tangent graph]. Teachers B and C uttered simultaneously, in agreement with Teacher D. In the end the teachers agreed that it was important for them to include a reasonable number of activities that included the tangent graph. Based on the scenarios sponsored by Teachers D and F, it became evident in the conversation that teachers were mindful of learners' thinking, therefore they viewed the content to be taught through the learners' eyes.

5.2.4.2 Knowledge of Content and Teaching (KCT)

Teachers discussed the steps involved in interpreting graphs (determining or coming up with equations of given/drawn graphs) in preparation for lesson 3. Teacher B managed to capture the steps in her notes as shown in Figure 5.12.

Figure 5.12

Steps For Determining Equations Of Given Graphs From Teacher B's Notes.



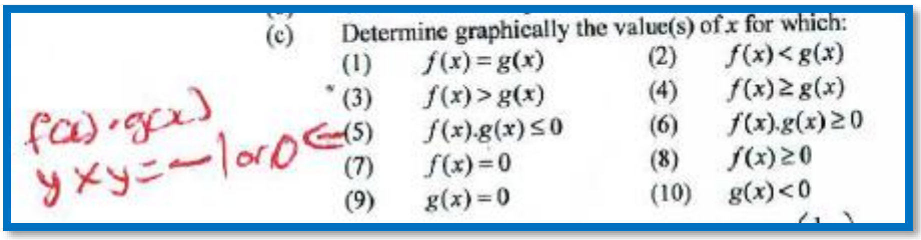
* The learners must be able to identify the type of graph given from the previous lesson
* Sketch the basic on top of the graphs given.
* Compare & observe the graphs inorder to differentiate the graphs
* After finding the equation, test the equation given/using the points on the graphs to check

Notes: The steps read [The learners must be able to identify the type of graph given from the previous lesson; Sketch the basic on top of the graphs given; Compare and observe the graphs to differentiate the graphs; After finding the equation, test the equation using the points on the graphs to check.]

The language (notation) of presenting intervals when answering questions like 'Determine graphically the value(s) of x for which… ' was discussed by the teachers, that is, the square brackets [ ] (that means inclusion of the upper and/or lower values) versus the round bracket ( ) (that signal exclusion of lower and/or upper values). One of the most interesting problems that they discussed was question c(5) in Figure 5.13.

Figure 5.13

Activity On Finding The Roots In Trigonometric Functions.



(c) Determine graphically the value(s) of x for which:

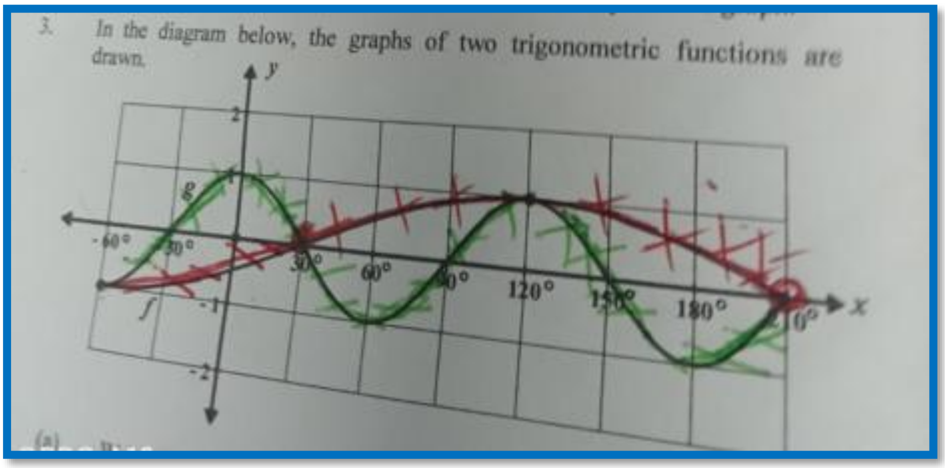
(1)	$f(x)=g(x)$	(2)	$f(x)<g(x)$
(3)	$f(x)>g(x)$	(4)	$f(x)\geq g(x)$
(5)	$f(x).g(x)\leq 0$	(6)	$f(x).g(x)\geq 0$
(7)	$f(x)=0$	(8)	$f(x)\geq 0$
(9)	$g(x)=0$	(10)	$g(x)<0$

Notes: *Mind Action Series* Grade 11 Phillips et al. (2012, p. 195)

Teacher B said, "Because *kulo* equal sign *thina bebetshonjalo.* Once *wabona u-equal ibracket lakho liba yi* square." [Our teachers taught us to use [ ] once we saw ≤.] Teachers also took turns to share how they could teach learners solve the problems that they had chosen for a class activity on roots. The idea of labelling parts of a graph above the x-axis as positive and those below as negative was adopted by the group after being raised by teachers B, D and F (see Figure 5.14).\

Figure 5.14

Marking Of Parts Of Graphs Above And Below The x -Axis With + And – Respectively.



Notes: Mind Action Series Grade 11 Phillips et al. (2012, p. 195)

Teacher D explained the solution of the problem to other teachers thus:

> I tell them when dealing with y , this is y (pointing at the graph)…If you multiply two things, because this is less than zero, we want a negative. Now y that is positive is at the top, and y that is negative is at the bottom. So, meaning that if one graph is at the bottom and the other one is at the top, then it's positive negative, then I know I am going to get a negative. Look here, just check if one graph is at the top and the other one at the bottom of the y -axis.

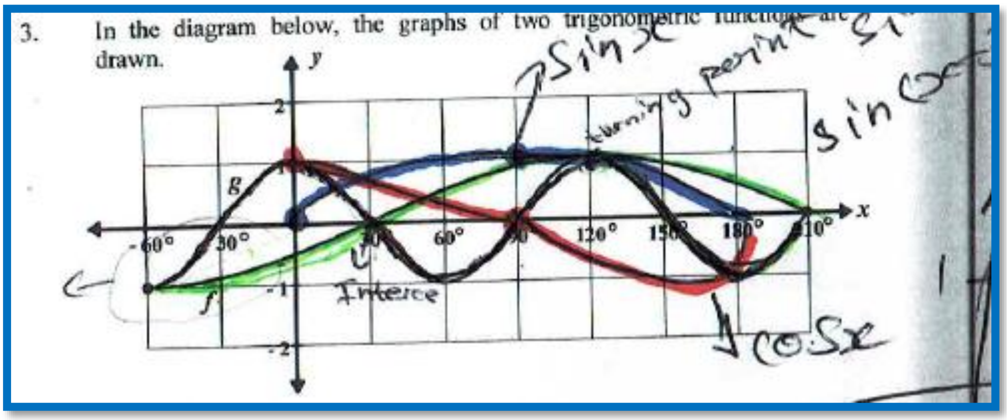
What Teacher D was saying coincided with what was scribbled in red and green ink on Figure 5.14. The teachers had to solve these activities together before assigning them to learners. Such a practice prevented the teacher from getting stuck in front of learners. More importantly, the colouring would help learners to visualise the concept being explained by teachers.

5.2.4.3 Specialised Content Knowledge (SCK)

The teachers engaged and learnt from one another in the area of determining equations of given/drawn graphs. Teacher F went on to suggest that different colours be used to trace the graphs since that promoted visualisation of the transformations. This resulted in other teachers coming up with multicoloured scribblings of basic and transformed graphs like the one shown in Figure 5.15. These collaborative activities by teachers affiliate with SCK because visualisation is an element of mathematical representations (Ní Shúilleabháin & Clivaz, 2017). The idea of deliberating on the incorporation of a visual presentation to learners suggests KCS as well since teachers planned with learners in mind. Furthermore, these activities depict the development of KCT because teachers delved into content and how to teach as they solved the questions.

Figure 5.15

Colouring By Teacher A: Different Trigonometric Functions Drawn On The Same Set Of Axes.



Notes: Mind Action Series Grade 11 Phillips et al. (2012, p. 195)

This aspect of interpreting transformation (transformed graphs) seems to be challenging to participating teachers. They took more than 40 minutes to engage on subject matter knowledge and on how to present it to learners.

5.3 EVOLVEMENT OF MATHEMATICAL KNOWLEDGE FOR TEACHING TRIGONOMETRIC FUNCTIONS IN RESEARCH LESSON PRESENTATION

The second secondary research question delved into how teachers' mathematical knowledge for teaching trigonometric functions evolved during research lesson presentation and observation. In other words, I explored how teachers' knowledge of teaching trigonometric functions unfolded or gradually developed during research lesson presentation and observation.

5.3.1 Context of the lesson presentation and observation session/site

Lesson presentation and observation is when the presenting teacher articulates the collaboratively planned objectives to the learners while other teachers observe and record findings without disrupting the lesson (Sekao, 2023). As Sekao (2023) puts it, the main focus of the lesson is to ascertain learners' thinking by observing how the learners respond to oral and written questions. The three lessons were delivered in a Grade 11 classroom that had 39 learners, and its setting was as shown in Figure 5.16.

Figure 5.16

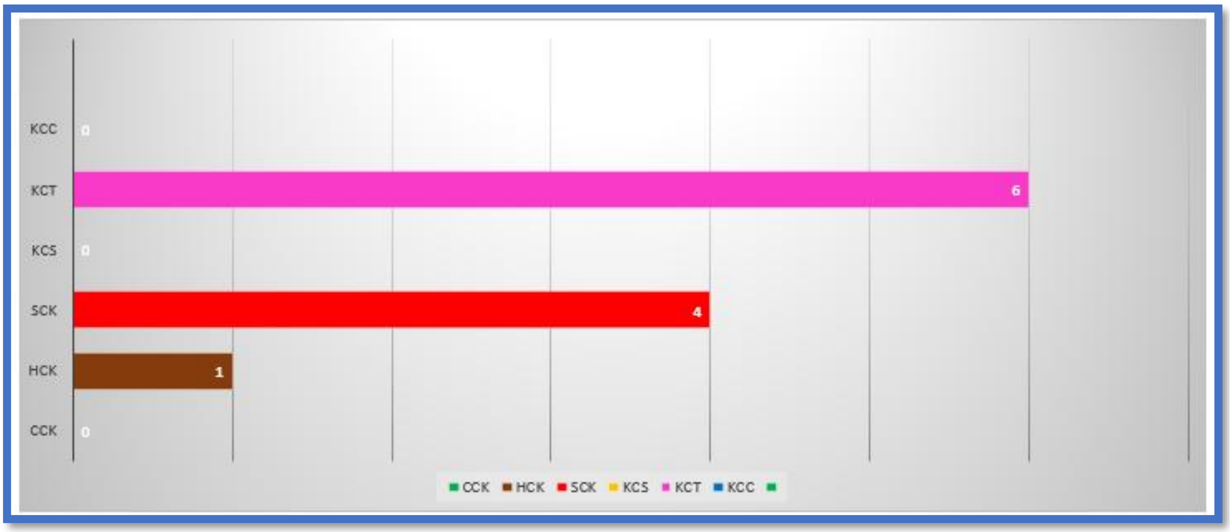
Grade 11 Classroom Set Up.

The front set-up had a smart-board and a whiteboard side by side. This facilitated the teacher's operations between demonstrations using the Desmos software and writing on the whiteboard. On the front desk was a laptop that Teacher F (the presenter for all three lessons) used to project the graphs on the smart-board. All six teachers were present during the presentation of research lessons 1 and 3. Teacher E did not observe lesson 2 because he was engaged in the school's assessment activities. However, the absence of one teacher did not have a bearing on lesson presentation.

While lessons 1 to 3 were being presented, I employed and anticipated the six MKT domains during my observations of the lessons, but only three featured as shown in Figure 5.17.

Figure 5.17

Occurrences Of Domains During Lesson Presentation And Observation.



From the graph, KCT was the dominant domain followed by SCK and HCK, respectively.

5.3.2 Presentation and observation of research lesson 1

The mathematical knowledge for teaching trigonometric functions evolved through three domains; HCK, KCT and SCK during the presentation and observation of research lesson 1.

5.3.2.1 Horizon Content Knowledge (HCK)

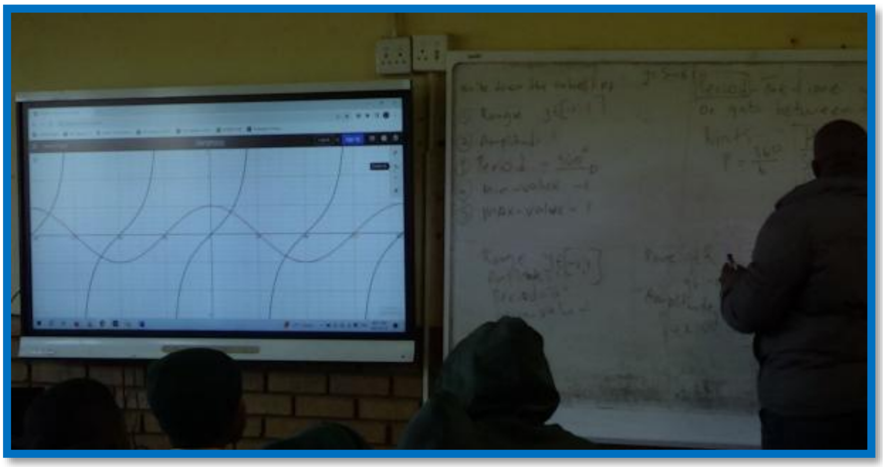
The picture of the electrocardiogram in the hospital projected by the teacher (see Figure 5.3) brought learners closer to trigonometric functions due to its sinusoidal nature. I observed this when the learners described its sinusoidal nature during the whole class discussion. This was also confirmed by Teacher C during post-lesson reflection on lesson 1 (see section 5.4.2.1). The electrocardiograms used in hospitals are not part of the Grade 11 CAPS/ATP, but they serve as the horizon/future (application) of trigonometric functions (Wasserman & Stockton, 2013). From my observation, the use of trigonometric functions in real-life that was discussed in the research lesson seems to have stimulated learners' interest. At this instant, the learners' reactions (thinking) could have made teachers notice, realise and appreciate that HCK (which is "considering other uses of a mathematical knowledge" according to Ní Shúilleabháin and Clivaz (2017, p. 124)) was important in introducing and motivating learners in trigonometric functions. However, I noticed and observed that the real-life examples of the use of trigonometric functions discussed with learners were only related to the sine and cosine graphs and not the tangent graphs. The teachers mentioned several times that the tangent graph was difficult for both learners and teachers. Being unable to identify the tangent graph's real-life application(s) could be exacerbating these difficulties.

5.3.2.2 Knowledge of Content and Teaching (KCT)

During the presentation of lesson 1, the presenter (Teacher F) sequenced the trigonometric functions content (a tenet of KCT according to Ni Shuilleabhain and Clivaz, 2017) by starting with the plotting of basic graphs followed by identifying their properties (see Figure 5.18).

Figure 5.18

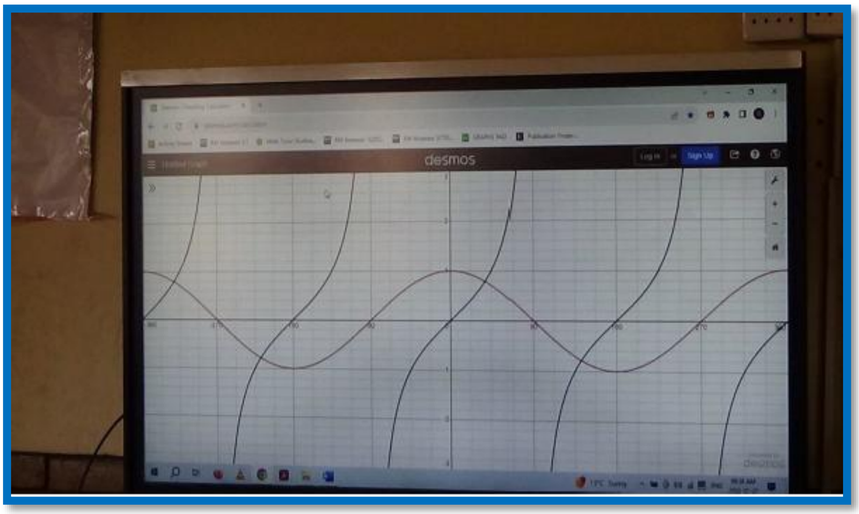
Basic Graphs Of Cosine And Tangent Displayed On The Smart-Board Using Desmos Software



The teacher used Desmos software to project, demonstrate, and illustrate properties of the basic graphs and this promoted and supported learners' understanding of trigonometric functions. Furthermore, the presenting teacher displayed KCT when he used the software to explain the concept of amplitude, maximum and minimum values of the tangent graph (compared to) the cosine graph (see Figure 5.19 and Figure 5.20).

Figure 5.19

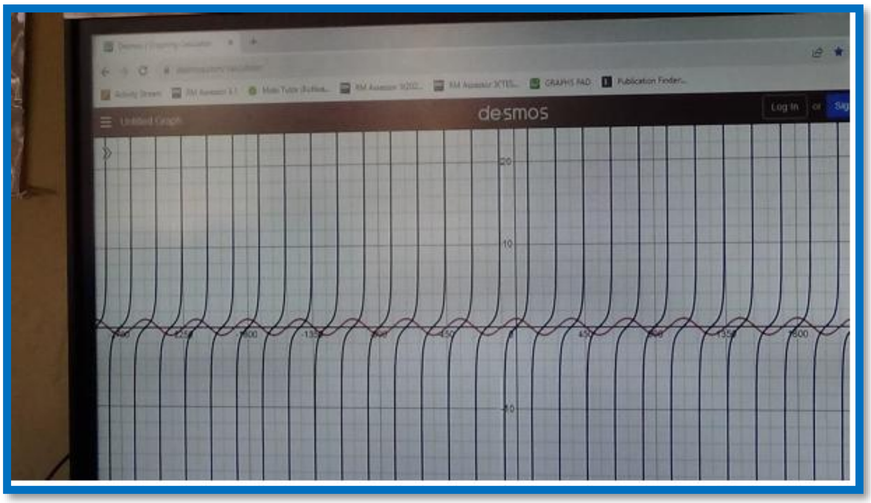
Basic Graphs Of Cosine And Tangent In Medium Zoom (Demonstrating The Concept Amplitude)



From this display (Figure 5.18) learners gave the amplitude of both the cosine and the tangent graphs. Teacher F then zoomed out as shown in Figure 5.19 and learners were convinced that the tangent graph had amplitude while its minimum and maximum values were negative and positive infinity, respectively. Teacher F's demonstrations and illustrations suggested the evolvement of KCT.

Figure 5.20

Basic Graphs Of Cosine And Tangent In Zoomed Out (demonstrating the concept amplitude).



The two figures vividly portrayed the concepts, and this could have helped those learners who struggle to understand the tangent graph's maximum, minimum and amplitude.

5.3.2.3 Specialised Content Knowledge (SCK)

Teacher F employed SCK to be able to detect an error where a learner gave the range of the basic sine graph as y∊[1;-1]. The second learner corrected it to y∊[-1;1]. This prompted the teacher to further execute SCK by demonstrating, explaining, and justifying the idea of range and amplitude using Desmos software (see Figure 5.21 and Figure 5.22).

Figure 5.21

Coordinates Associated With The Range Of The Basic Sine Graph (Demonstrating The Concept Range).

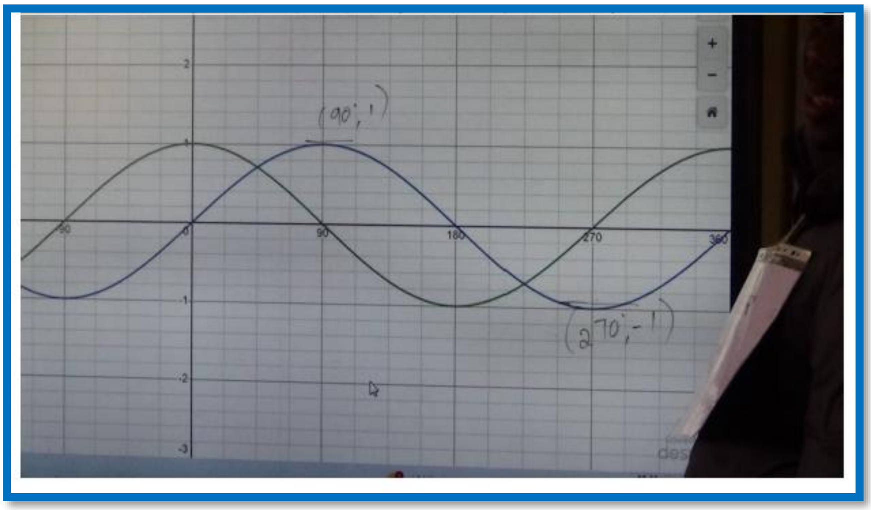
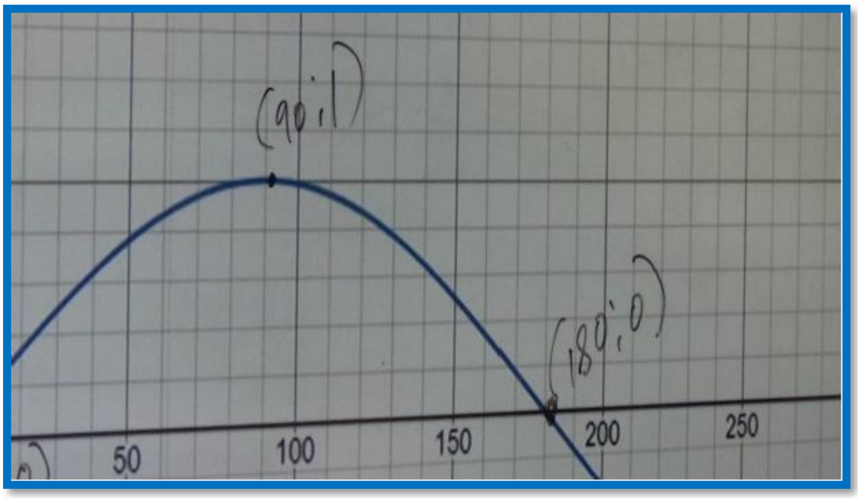


Figure 5.22

Coordinates Associated With The Amplitude Of The Basic Sine Graph (Demonstrating Amplitude).

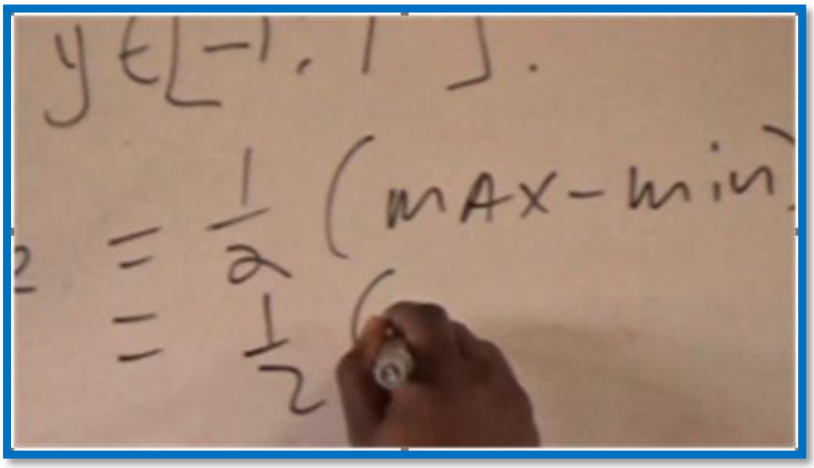


Ni Shuilleabhain and Clivaz (2017) view this as demonstrating mathematical representations of a topic (trigonometric functions) effectively. This also qualifies as

KCT. The teacher and the learners ended up coming up with a general way of calculating the amplitude of the sine function (and trigonometric functions in general) as shown in Figure 5.23.

Figure 5.23

The Formula Of Finding The Amplitude Of Sinusoidal graphs.



The way these concepts were developed differed from the traditional approach where teachers started by giving learners the formula.

5.3.3 Presentation and observation of research lesson 2

My observation of lesson 2 yielded KCT and SCK. The tenets of these two domains were identifiable during the teaching and learning sessions.

5.3.3.1 Knowledge of Content and Teaching (KCT)

The presenting teacher posted the equation of the graphs from homework on the whiteboard while projecting the graphs on the smart-board using the Desmos software (see Figures 5.24 and 5.25).

Figure 5.24

Homework Posted On The Whiteboard For Discussion

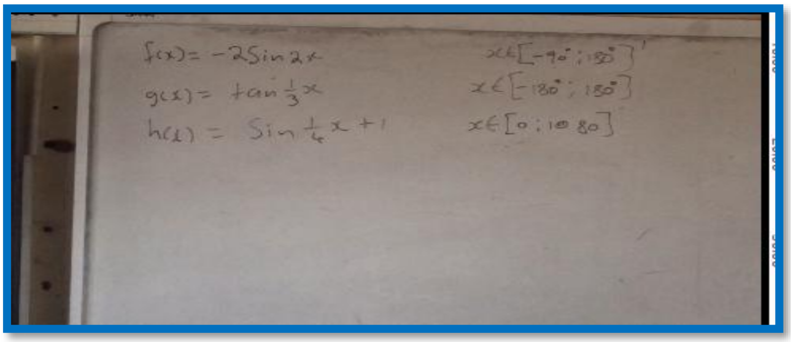
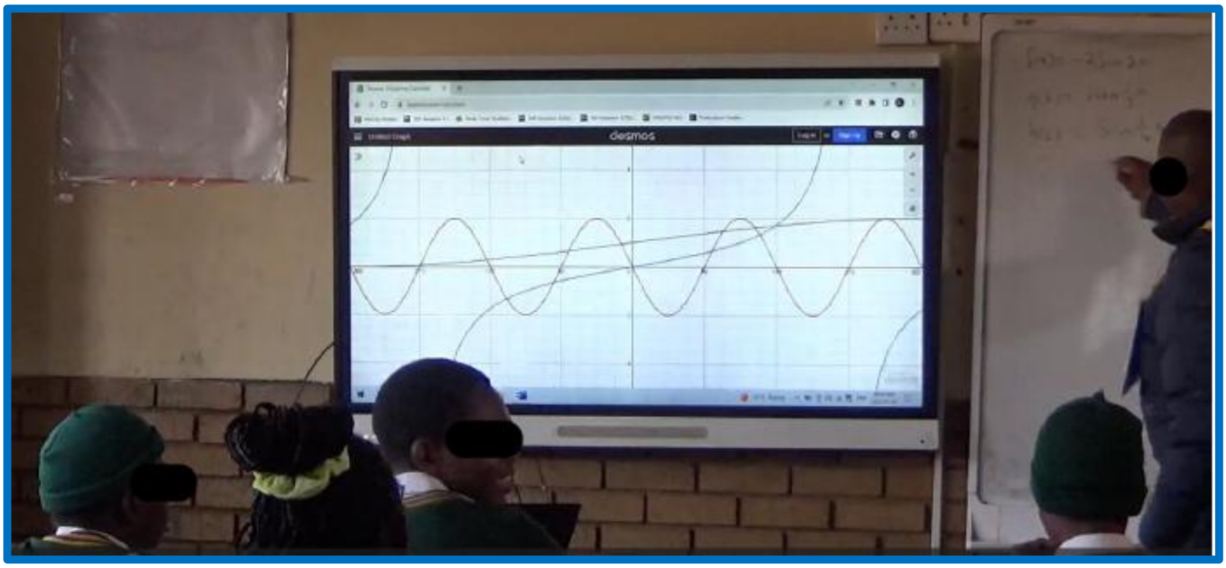


Figure 5.25

Sketches Of Functions

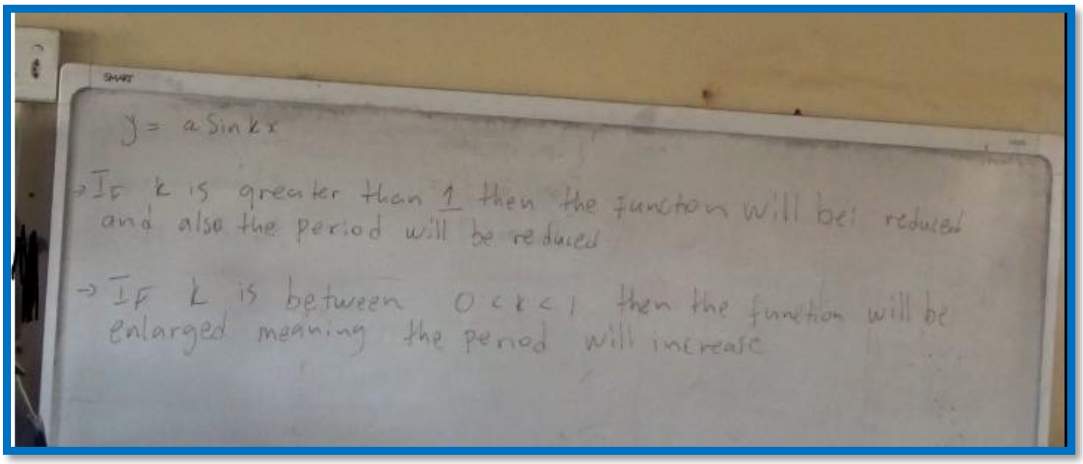


After agreeing on the solutions to homework questions, the teacher and the learners further discussed the effects of k. It emerged from the learners that for $y=-2\sin 2x$ the period of the graph had been reduced/compressed while $y=-2\sin\left(\frac{1}{2}x\right)$ the period had been extended/expanded. The class discussion ended with the conclusion shown in Figure 5.26.

The phrases 'the function will be reduced' and 'the function will be enlarged' seem to have been coined during this discussion and made sense to both the teacher and learners.

Figure 5.26

Conclusion On The Effects Of Parameter k.



Teacher F then introduced parameter p by projecting the graphs of $y = \sin x$ and $y = \sin\left(x - 30^{\circ}\right)$ followed by asking the learners to state the relationship between the two graphs (see Figure 5.27).

Figure 5.27

Effects Of Parameter $\boldsymbol{p}$ *On The Sine Graph.*

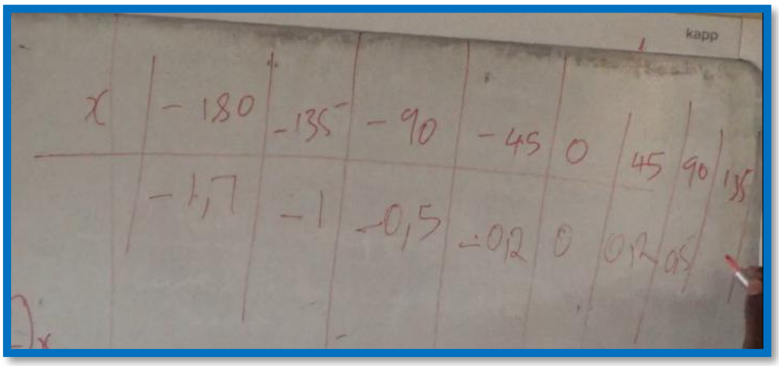
The learners were able to state that the blue graph [$y = \sin\left(x - 30^\circ\right)$] had shifted 30 degrees to the right. Learners were then given an individual class activity of drawing the graphs of $y = \tan x$ and $y = \tan\left(x + 45^\circ\right)$. During feedback, the teacher asked for the relationship between the two tangent graphs without projecting the sketch. The learners responded very well. The teacher went on to ask for the maximum, minimum, amplitude, range, and period of the transformed tangent graph. Learners gave the correct responses/answers. The teacher then gave the learners some homework. After that, he used the Desmos software to drill learners in amplitude, following one learner's response that still took -1 as the amplitude for the tangent graph. The teacher carried out this instructional activity in response to the learner's misconception and this signalled teacher's professional noticing.

5.3.3.2 Specialised Content Knowledge (SCK)

A learner asked if $y = -0.2$ in the table of values for a tangent could be taken as an asymptote. The teacher, with the help of the class, constructed the table of values shown in Figure 5.28.

Figure 5.28

Table Of Values For $y = \tan\left(\frac{1}{3}x\right)$



The teacher went on to plot the graph to demonstrate and illustrate that -0.2 and 45° formed a point on the graph and not an asymptote. The teacher's response to this question was typical of teacher/professional noticing because he had engaged in pedagogical reasoning which had not been initially planned for by the LS team. Again, this learner's question could be signalling a common misconception in asymptotes by many learners.

5.3.4 Presentation and observation of research lesson 3

In my observation of the lesson presentation and observation of research lesson 3, I anticipated the existence of all six domains to feature. At the end of the lesson, only KCT and SCK were observable.

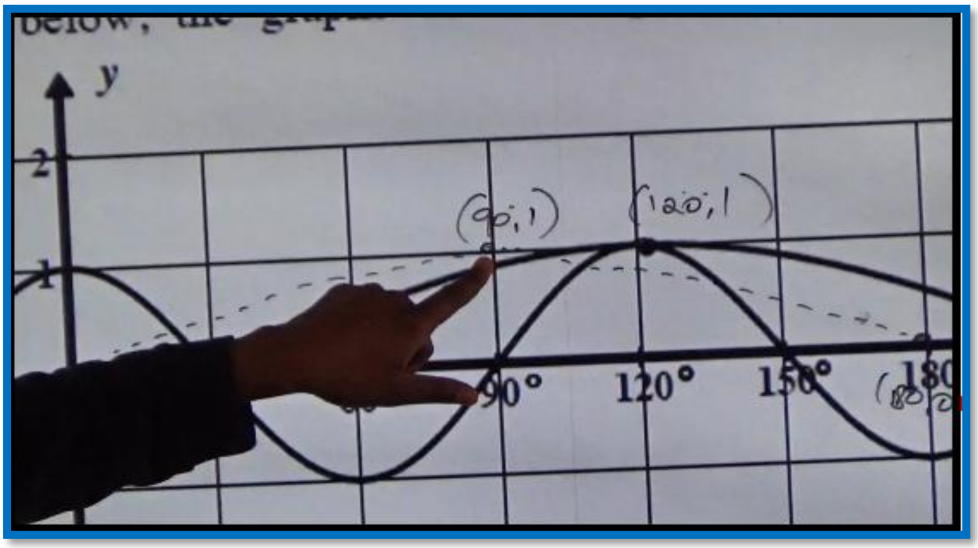
5.3.4.1 Knowledge of Content and Teaching (KCT)

The presenter started by posting the following paired functions on the whiteboard $f(x) = -2\sin(x - 30^\circ)$ and $y = -2\sin x$; $g(x) = -3\cos(x + 45^\circ)$ and $y = -3\cos x$; and $h(x) = -\tan(x - 60^\circ)$ and $y = -\tan x$. The teacher then asked the learners to describe the relationship within each pair. The learners were able to give the descriptions. This skill of describing linked the previous research lesson 2 to lesson 3 for the day.

The teacher also displayed a KCT skill by scanning the example to be used and projecting it on the smart-board. The approach made it easy for the teacher to superimpose the anticipated basic or mother graph(s) using whiteboard markers (see dotted graph in Figure 5.29).

Figure 5.29

Finding The Equation Of A Drawn Function Or Graph.



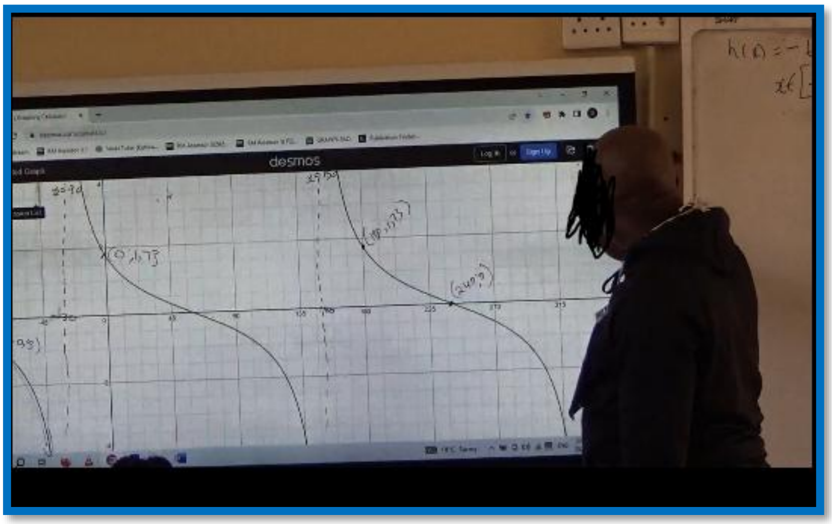
This was the problem that the teachers had spent more time on during planning (see Section 5.2.4). From Figure 5.29 Identifying the coordinates (90;1) and (120;1) served as a critical step towards finding the solution. The knowledge of mother graphs was pivotal in coming up with the equation of the drawn graphs (Makandidze, 2020). It became easy for learners to come up with the equations of the graphs during class discussions, yet the teachers had expected the learners to struggle.

5.3.4.2 Specialised Content Knowledge (SCK)

The presenting teacher (Teacher F) moved around checking learners' homework and discovered that some learners were not indicating turning points and intercepts. Addressing the class he said: “When we sketch a function there must be what...coordinates, especially where the graph turns and also where the graph intersects.” The teacher went on to demonstrate using the homework solution as shown in Figure 5.30.

Figure 5.30

Showing Coordinates Of Turning Points And Intercepts.



Thereafter, the teacher emphasised to the learners that coordinates of key points were important to the lesson of the day that had to do with finding the graph of a function. Practices associated with Figure 5.30 served as bases for handling questions like the one in Figure 5.29.

5.4 HONING OF MATHEMATICAL KNOWLEDGE FOR TEACHING TRIGONOMETRIC FUNCTIONS THROUGH REFLECTIVE PRACTICE.

Findings from data collected from post-lesson reflection of lessons 1 to 3 sought to answer the third research question: How do teachers’ reflective practice hone their mathematical knowledge for teaching trigonometric functions? By honing, I meant the sharpening, improving, and or enhancing of mathematical knowledge for teaching trigonometric functions.

5.4.1 Context of the lesson post-lesson reflection session

Reflective practice is where teachers discuss the strengths and weaknesses of research lessons presented, thereby spontaneously crafting ways of improving the lessons (Moumou, 2021; Sekao, 2023). The participating teachers held their post-lesson reflection sessions in the Computer Laboratory after the school lessons had ended. Scheduling the post-lesson reflection after school lessons had ended was a way of avoiding unnecessary disruption of the school timetable. Three sessions for lessons 1, 2 and 3 were held on days 3, 4 and 5 at 15:00 to 15:41, 14:44 to 15:02, and 14:54 to 15:35 respectively (Table 4.4 in Chapter 4, sub-section 4.7.2). The general setting or environment that characterised the sessions was as shown in Figure 5.31, where teachers sat around the table as per LS prescriptions (Sekao, 2023).

Figure 5.31

Teacher E Explaining A Point While Others Listened, During A Post-Lesson Reflection.

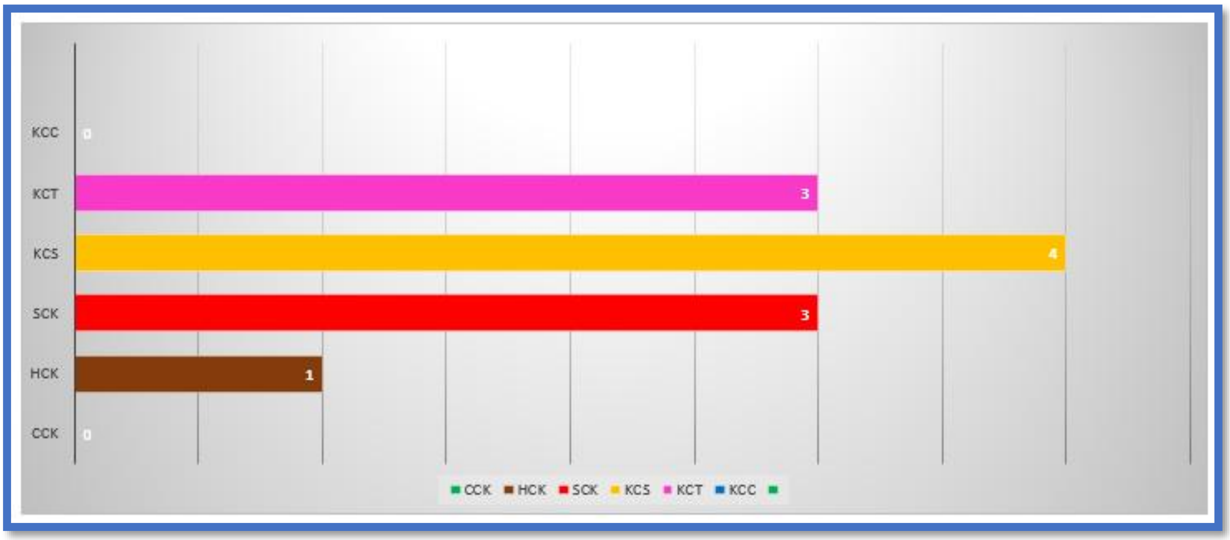
The post-lesson reflection sessions did not have a specific teacher chairing (the reflection facilitator, according to(Sekao, 2023)). Instead, teachers took turns to reflect, with the presenting teacher being the first to air his views on the lesson at the beginning of each session. Teacher F brainstormed and reminded other teachers on the basics of post-lesson reflection at the beginning of the first session. The post-lesson discussions focused mainly on the articulation of planned objectives (by the presenter), the presenter's handling of learners' written and oral responses (mathematical thinking), and the general observations by other teachers (Sekao, 2023). It is important to note that post-lesson reflection is characterised by teachers discussing their observation experiences (mental snapshots) of teacher-learner interactions, which is a portrayal of teacher/professional noticing (Choy, 2016).

All the teachers attended the reflection of research lesson 3. Teacher A could not attend post-reflection of lesson 1 because she was attending a Natural Sciences workshop. Teacher E missed the post-lesson reflection of lesson 2 because he was occupied with the school's assessment activities. However, their absence did not have much bearing on the reflection process since most of the LS team members were there.

From my observation of the three post-lesson reflection sessions, the teachers' collaborative conversations and activities revealed tenets of KCT, KCS, SCK, and HCK at differing occurrences as shown in Figure 5.32.

Figure 5.32

Occurrences Of Domains During Post-Lesson Reflection.



The bar graph shows teachers deliberated on KCS most followed by SCK, KCT and HCK.

5.4.2 Post-lesson reflection of research lesson 1

During my observation of post-lesson reflection of research lesson 1, I found that by reflecting on the presented lesson, teachers honed their mathematical knowledge for teaching trigonometric functions in several ways. The sub-domains of SMK and PCK that featured quite prominently in teachers' reflective process were HCK, SCK, KCS, and KCT.

5.4.2.1 Horizon Content Knowledge (HCK)

While the presenter (Teacher F) was reflecting on experiences in research lesson 1, he pointed out that learners were able to identify the use/applications/existence of trigonometric functions in real-life such as in water waves. Again, he noted that most learners were able to draw the basic functions: cosine, sine, and tangent. Teacher F noted: “Prior knowledge is very very important.” All teachers agreed and Teacher C commented, “*Yaaa isisizile kakhulu.”* [Yes, it helped us a lot.]. In agreement with the former speakers, Teacher E said, “More of especially if you link it to real life situations and then move to the content. It makes a lot of sense now.” Teacher C added: “I remember *ma esekaqala* some of them were lost. But immediately *ebuza* that other girl *ukuthi* ‘were you not hospitalised?’ they all came back to the lesson*. They all came back ukuthi oooooh”*. [I remember when he started the lesson some learners were lost. But immediately he asked that girl ‘were you not hospitalised’, all learners could now follow the lesson. They all came back to exclaim ‘oooooh’.]. Prior knowledge and applications of mathematics in real-life situations are characteristic of HCK (Moumou, 2021; Ní Shúilleabháin & Clivaz, 2017; Proctor, 2019). The teachers' discussions here signal their noticing and appreciation of the importance of employing prior knowledge and real-life mathematical applications in the teaching and learning of trigonometric functions.

5.4.2.2 Knowledge of Content and Students (KCS)

The teachers reported that dealing with the definition of concepts had a positive impact on research lesson 1. In line with this, Teacher F averred that: "When we give this lesson in Grade 10, we must give them an explanation of the period, amplitude, range, domain." Again, Teacher F pointed out that most learners used their calculators (the table method) to draw the graphs. The rest of the teachers agreed that they also observed the trend. This finding is interesting because teachers honed their mathematical knowledge for teaching trigonometric functions by identifying learners' learning experiences (of the four above-mentioned concepts and use of a calculator) by identifying and interpreting the mathematical meaning associated with learners' responses (Ní Shúilleabháin & Clivaz, 2017).

5.4.2.3 Knowledge of Content and Teaching (KCT)

Teacher F reflected on his struggle to teach the concept range as a weakness and requested other teachers to help him teach the concept better. The teacher revealed his desire to effectively teach this concept when he said:

Excerpt 5.4.2.3(a)

Teacher F: …So I am not sure *ukuthi* how can we, how can we teach this. [so, I am not sure how best this concept could be taught].

Teacher B:

> *Mina* I think *lendaba ye* range at the end *uyilungisile.* I think *lalo obekalahlekile sekale clarity* because *... zonke izinto ebezibatshaya ucine uzilungisile. njengoba ububabuza ukuthi* what is a period what is a what what. *Sebeyabona manje. Njengakusasa iactivity yakusasa if inayo irange* they will be able to identify because *uclarifayile ama misconceptions abo."* [I think you were able to address the learners' challenges with the concept range. I think even the learners who were struggling are now fine since you could explain and clarify most concepts (like the period) to them. The learners will be able to handle the concept range in tomorrow's activities because you handled their misconceptions very well."]

Teacher C: ...*And this thing eyokuthi udifayine it helped njengoba ubutshilo yesterday. It did help a lot.* [The idea of the definition of terms and concepts that you did (as planned yesterday) helped the learners. It helped a lot.]

Everybody agreed with Teacher C and Teacher B went on to say:

Teacher B: *Yikho lapho esebonakhona ukuthi oooooooh sikhuluma ngento eso. Uyabona njengaku tan ube ukhuluma nge range uyabatshela uthi lapheqala lapho ephelela khona. iqalelaphi iphelelaphi uyabona. Sebeyabona ukuthi ok ibracket leli* *liya includa leli ali includi babe le clarity...*[The learners could follow up on the concept range. Especially with the tangent graph, you were able to demonstrate to them where it starts and where it ends. The learners now know about the including/square bracket and the excluding/round bracket because they got clarity…]

Teacher C (interjects): ... of what is happening and Teacher F (who is the presenter), could only agree.

Teacher E :

> I will agree with you. You know the issue of definitions for the concept, it is very very much important because it is what makes learners to understand the content because I realised when you were saying the period they were giving you the formula how to calculate the period instead of defining the period. Because that is where we will be able to pick as to whether these learners understand what you are talking about or not and this will help them when they understand even if they answer questions, they will start answering them well.

Learners had misconceptions with range. So, here, the presenter (Teacher F) was made aware of his strengths by the observers. Indeed, he had doubted whether he had made justice to the concept ‘range’ in his teaching. There is evidence that observers were closely following the teaching and learning that was taking place. I hold the notion that the observers, in turn, developed KCT as the lesson unfolded.

The teachers also reflected on the achievement of the planned objectives in the following excerpt.

Excerpt 5.4.2.3(b)

Teacher F: Did we finish the lesson?

Teacher E: We don't know. Now it comes in. If we had objectives, we would be able to know as to whether we finished or not.

Teachers B and C (simultaneously): We did not touch *kahle kahle* the main objective, the main goal was not.... *Besi introducer uk.* We were supposed to introduce *uk.*

[We did not achieve our main objective of introducing k.]

Teacher B: We did not achieve our objective.

Teacher F: Ok. The issue of k . So, tomorrow I must start by introducing the issue of k. Here it says sketch the three graphs in the same set of axes.

Teacher E: It seems *re jampile* [we omitted].

Teacher C interjects: *sabanika ihomework*. [and we gave them homework]

Teacher F: I skipped that on purpose and gave them homework related to that so that tomorrow it will be continuation to that. (All teachers laughing)

Teacher E: Ok it comes back to me to say you give them then you intervene later. Good. That is another way, teaching strategy.

Teacher F: I saw that the time has beaten me. There is no other way. I had to give them homework that is related to tomorrow's work.

Simultaneously Teacher E and F: So tomorrow we will be able to see whether we have achieved our objectives or not from the homework.

Teacher F: We are not going to draw the graphs on the board, but we will project them on the smart-board.

Teacher E: So that we don't spend time on corrections.

From the ongoing discussion, I observed that teachers could not fit plotting basic graphs (sine, cosine and tangent), and introducing parameter k within research lesson 1. Sekao (2023) warns teachers against over planning. The teachers' experiences in research lesson 1 helped them to sharpen their planning skills in matching objectives with the duration of the period. I also noted that learners were given questions as homework on parameter k before being introduced to it by the teacher. Again, it follows that the achievement of an objective can be checked against the homework assigned.

5.4.2.4 Specialised Content Knowledge (SCK)

Although the observing teachers reported that Teacher F had addressed the misconceptions surrounding the concept range, the presenter's utterances indicated that he had considered the concept to be a challenge to learners. From my observation, the learners were struggling with the concept of range. This was supported by Teacher F who pointed out that some learners in class were giving it in degrees i.e., 30°, 60°, 90°, etc. while others gave it as +1 to -1. It follows then that Teacher F honed his SCK (and that of others) by being able to trace and identify patterns in learners' errors and misconceptions in range.

5.4.3 Post-lesson reflection of research lesson 2

Of the six MKT domains, my observations identified KCS and SCK.

5.4.3.1 Knowledge of Content and Students (KCS)

The teachers' reflective utterances that were indicative of honing KCS pertained to the way learners drew the tangent graph. Firstly, the teachers observed that some learners were drawing a straight line instead of a curve and struggled to correctly position the asymptote. This was revealed by the teachers' conversations in the following excerpt.

Excerpt 5.4.3.1(a)

Teacher B:

...kule ntombazana le ebengihlezi duzane layo for the tan graph *ubhala istraight line, udrawa istraight line*. I think maybe *nga singa emphasiza lapho* like *umkhombise ma uyi drawer ukuthi mbona (*moving her finger*) ukuthi ngi drawer kanjani. Mhlawumbe*

uzokhona ukuyi drawer. I think *ulenkinga yokuyi drawer.* [There was a girl seated next to me who drew the tangent graph as a straight line. I think we should improve the situation by demonstrating (moving her finger) to the learner how the smooth curvature is drawn. Maybe the learner will manage the graph. I think she has a problem of drawing the graph.]

Teacher F: She is not the only one, I saw a few.

Teacher D: (signalling an asymptote with his hand) Most of them *ungathi abazi ukuthi kule asymptote they just draw.* [It appears most of them are not aware of the existence of the asymptote, they just draw.]

Teacher C interjects: *Babhala* sebaqedile*.* [They insert the asymptote after drawing the graph.]

Secondly, teachers observed that other learners could not show the continuity of the tangent function as portrayed in Excerpt 5.4.3.1 (b).

Excerpt 5.4.3.1 (b)

Teacher F: Learners do not use arrows to show that the tangent graph is a continuous function.

Teacher B: But I think *uclarifayile but ukuze sizebona ukuthi bazwile sizabona nge homework.* [But I think you clarified it to learners. We will see from the homework if they understood.]

Teacher F: At least I tried to clarify it so that it becomes at least *easiyanyana* [easily understandable]. And then that issue it came up again. The issue of saying minimum and maximum for tan, the other learner said it's -1. It means that learner was not paying attention.

All the teachers agreed with the two teachers. Furthermore, the other teachers commended the presenter for revisiting and emphasising the concept of amplitude by identifying the rest point/line of a trigonometric function.

5.4.3.2 Specialised Content Knowledge (SCK)

The teachers were able to diagnose/observe the reasons why learners could not draw the other two graphs assigned as homework. The intervals given by the teachers had limitations that the learners could not handle. The discussion below revealed this.

Excerpt 5.4.3.2

Teacher F: I saw something. The one that they did not write because of the domain that was 1040°. The other one the tan graph domain was... (opening his arms wide). *But all in all, bona they get the issue of a, the issue of k, p and q.* [Overall, the learners seem to be understanding the effects of parameters a, k, p, and q.]

Teacher C: *And nayo ikusizile ukuthi kube le issue of a ma inegative of positive uemphasize. Already ukhona uku balancisa labangakhoni. so isizile.* [The activities that you used involving negative and positive a were very effective. The approach accommodated struggling learners. It helped.]

Teacher F agrees: yaaa, yaaa. [yes, yes.]

Teacher C's utterances suggest that the group was able to appropriately select purposeful examples for lesson 2. The teachers then reviewed and re-aligned what was expected of research lessons 1 and 2 in future. They agreed that research lesson 1 should deal with parameters a and q whilst research lesson 2 with parameters k and p. In this excerpt, teachers sharpened their SCK by highlighting and deliberating on the learners' challenges with homework and their possible causes (a domain of 1040°). This suggests that, in future, teachers would choose purposeful activities better. In actuality, 1040° was the period of the function and not the domain as Teacher F put it. Since no teacher corrected this, it revealed that these teachers were also confusing the period with the domain.

5.4.4 Post-lesson reflection of research lesson 3

While teachers were reflecting on lesson 3, their conversations were characterised by KCS, KCT and SCK.

5.4.4.1 Knowledge of Content and Students (KCS)

Since this was the last reflection session, Teacher C wanted to know the overall comments on all the lessons from other teachers. The following discussion in Excerpt 5.4.4.1 ensued.

Excerpt 5.4.4.1

Teacher C: *bengicanga ukuthi ama lessons ethu wona ahambe kanjani so.* [I wish to hear your views on how we did in all our lessons.]

Teacher F: *Mina ama lessons ethu ngicabanga ukuthi ahambe right but there are still learners who are still behind.* [I think all our lessons went well, but some learners are still struggling.]

Teacher C: *Ma ubheke ama lessons ethu aright ama lessons ethu. Siwaqale kuhle ama objectives ethu and ayalandelana it's not uqala elinye le ulande elinye le siphinde sibuyele emuva ahamba sharp nge order. Iproblem kulabantwana lana ama slow learners.* [All our lessons were fine. We sequenced our objectives very well. The challenge is that there are slow learners.]

Teacher F: Those are the learners that we want. So am just going to develop a set of graphs and give them, 'please give me *ama* equations' that's it. And if they do that consistently they will understand because *huri* [that]the next lesson is going to focus on finding the equation *kuphela* [only]. Finding the equation only because if you do not do that you are going to lose a lot of learners. I can see most of them are already discouraged.

It seems teachers are of the view that slow learners need to be taught trigonometric functions one aspect at a time. In this regard, the teachers honed their KCS by identifying learners' difficulties in terms of their pace of learning trigonometric functions (Ní Shúilleabháin & Clivaz, 2017).

5.4.4.2 Knowledge of Content and Teaching (KCT)

The teachers also reflected on the achievement of objectives that were planned for research lesson 3 (see Excerpt 5.4.4.2)

Excerpt 5.4.4.2

Teacher F: So, it means for today we did not reach our... objectives (Teacher B and C say 'objectives' simultaneously with Teacher F - in agreement) because we supposed to identify the types of graphs. Ok we have identified them only and also writing the equation. But other things like transformations applied to trig function; graphically determining the values of x for which ...we did not do that; when is the graph increasing when is the graph decreasing. So, we did not do that, so we did not reach our objective. So, it means our objectives were too much. So today we were supposed to say finding the equation only.

Teacher B: But *itransformation* to trig functions and their description in words *ubayenzisile* because *uze wa emphasiser ukuthi ku important ukuthi itransformation* *likhone ukuyi explainer* in words. [But you covered the transformation of trigonometric functions. You even stated to the learners the importance of being able to describe the transformation in words.]

Teacher C: *I see ukuthi ik ingathi iyabatshaya kancane. The effects of* k *.* [I see that learners have a bit of a challenge with the effects of parameter k .]

Teacher F: *Yaaa the effects of k abekho shuwa*. [Yes, they do not have confidence in parameter k .]

Teacher C: *Into engiyibonileyo ukuthi bayakhona ukuyi* answer *yi* p . [I observed that they can handle parameter p .]

Other teachers agreed that most learners had understood the effects of the parameter p , the horizontal movement. However, the teachers also raised the concern that a handful of learners were still struggling with the value of q , the vertical movement. Furthermore, Teacher C raised concern about the teaching of the tangent graph for learners' understanding: "*I don't know mani itan kahle kahle singayiyenza njani ukuthi ibe simply fo abantwana."* [I don't know, how can we teach the tangent graph to be understood by learners.]. Teacher F had a remedy for the situation that he expressed as follows: "No *ngizobayenzisa bayenze bayenze bayenze*." [I will make them work on the tangent graph repeatedly.]. Teacher C in support of teacher F said: "Again, and again and again."

From this excerpt, it appears teachers have raised concerns of over-planning of objectives, learners' challenges with parameters q and k , and learners' difficulties with the tangent function. The teachers' realisation that they over planned the objectives honed their skills by compelling them to break the objectives down and spreading them over several periods. It was not clear what Teacher F meant by suggesting having learners do the tangent graph over and over again. Furthermore, the teachers did not deliberate on ways/skills that could help in the teaching of parameters q and k better.

5.4.4.3 Specialised Content Knowledge (SCK)

Teachers agreed that learners still experienced the challenge of sketching the tan graph. Teachers also observed that learners were not considering the given intervals when they drew graphs/functions. The presenter and the observers noted that learners were not writing coordinates when drawing graphs that were given as homework. Identifying and deliberating on these challenges in the learners' written work is an indication of the sharpening of SCK by teachers. All these also qualify as honing of KCS (Ní Shúilleabháin & Clivaz, 2017).

5.5 CONSOLIDATION OF THE FINDINGS REGARDING TEACHERS' DEVELOPMENT OF MATHEMATICAL KNOWLEDGE FOR TEACHING TRIGONOMETRIC FUNCTIONS IN A LESSON STUDY CONTEXT

A summary of findings for research questions (RQs) 1 to 3 is presented in Table 5.2. The summary is a consolidation of teachers' development of MKT trigonometric functions during LS. The table accommodates these three RQs because they are directly linked to stages 2 to 4 of the South African LS model (which are collaborative lesson planning, lesson presentation and observation, and post-lesson observation). The codes are in colours that correspond to Figures 5.2, 5.17, and 5.32.

The first column of Table 5.2 shows the knowledge bases of the MKT which are Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). The second column contains the respective domains of each knowledge base. These domains are the codes that I used to analyse data with the help of their characteristics (tenets or indicators) in the third column of Table 5.2. Columns 4 to 6 are accommodating stages 2 to 4 of the South African LS model that corresponds to RQs 1 to 3. In each of the LS stage columns (which correspond to themes in Table 5.1), I captured scenarios of LS activities that indicated tenets of any one code. I further indicated the section in which the scenario is presented in brackets. For instance, row 2 column 4 of Table 5.2 shows that the indicator of CCK (application of trigonometric functions in real-life) is presented in Section 5.2.2.1.

Table 5.2

Summary Findings Of Research Questions 1 To 3

MKT	Domain(codes)	Characteristics (tenets or indicators) (adapted from (Ní Shúilleabháin & Clivaz, 2017) and (Clivaz & Ni Shuilleabhain, 2019)	Stage 2 (Collaborative Lesson Planning)	Stage 3(Lesson Presentation and observation)	Stage 4(Post-lesson reflection)
Subject Matter Knowledge	CCK	*-Performing mathematical task* *-Use of notations and vocabulary* *-determining if a solution, a definition, a representation is correct*	Application of trigonometric functions in real-life (5.2.2.1)		
	HCK	*-considering other uses of mathematical knowledge* *-considering later purpose of mathematical knowledge*	• Assumed knowledge in the form of point-by-point plotting of basic graphs trigonometric functions (5.2.2.2).	• Introduction of trigonometric functions using the electrocardiogram (real-life application) appeared effective (5.3.2.1).	• Teachers confirmed the importance and effectives of using prior knowledge and real-

MKT	Domain(codes)	Characteristics (tenets or indicators) (adapted from (Ní Shúilleabháin & Clivaz, 2017) and (Clivaz & Ni Shuilleabhain, 2019)	Stage 2 (Collaborative Lesson Planning)	Stage 3(Lesson Presentation and observation)	Stage 4(Post-lesson reflection)
			• Application of trigonometric functions in real-life (5.2.2.2).		life applications to introduce trigonometric functions (5.4.2.1).
	SCK	*-looking for patterns in student errors* *-unpacking of mathematics content* *-understanding different interpretations of a concept or techniques appreciating the differences* *-talking explicitly about how mathematical language is used*	• Identification of suitable purposeful instructional activities when teachers agreed to pick questions from Exercise 2 (5.2.2.6). • Teacher D explaining the difference between sketching and point-by-point plotting to other teachers (5.2.2.6). • Written notes on parameter k by Teacher C (5.2.2.6). • Teachers agreeing to introduce parameter p alone into the basic	• Teacher F detected an error where a learner gave the range of the basic sine graph as y∈[1;-1] (5.3.2.3). • Teacher F demonstrated, explained, and justified the idea of range and amplitude using Desmos software (see Figure 5.21 and figure 5.22) (5.3.2.3). • Teacher F explained the asymptote of $y = \tan\left(\frac{1}{3}x\right)$ with the help of the table of values (see Figure 5.19) (5.3.3.2). • Teacher F discovered challenges of turning points and intercepts in the	• Teacher F reported traced and identified patterns in learners' errors and misconceptions in the concept range (5.4.2.4). • Teachers observed and diagnosed possible reasons why some learners could not attempt some questions given as homework (see Excerpt 5.4.3.2) (5.4.3.2). • Teachers noticed that learners are not considering the given

MKT	Domain(codes)	Characteristics (tenets or indicators) (adapted from (Ní Shúilleabháin & Clivaz, 2017) and (Clivaz & Ni Shuilleabhain, 2019)	Stage 2 (Collaborative Lesson Planning)	Stage 3(Lesson Presentation and observation)	Stage 4(Post-lesson reflection)
		-choosing making and using mathematical representation effectively *-explaining and justifying mathematical ideas* *-analysing or building examples having mathematical characteristics*	graph to avoid confusing learners (5.2.3.3). • Use of coloured pens to promote visualisation of trigonometric functions (5.2.4.6).	learners' homework and used Desmos to explain (see Figure 5.30) (5.3.4.2).	intervals and not labelling coordinates when drawing graphs/functions (5.4.4.3).
Pedagogical Content Knowledge	KCT	*-sequencing mathematical content* *-identifying or developing learning activities* *-selecting models, representations, examples,*	• teachers discussing the importance of selecting suitable textbooks for trigonometric functions (5.2.2.5). • Teacher D explained the teaching of the period of the tangent graph (5.2.2.5).	• The presenting teacher sequenced the content by starting with the plotting of basic graphs followed by identifying their properties (5.3.2.2). • the presenting teacher displayed KCT when he used the software to explain the	• Teachers B and C confirmed they observed that Teacher F articulated the concept range (and others) very well to learners (5.4.2.3 Excerpt 5.4.2.3(a)).

MKT	Domain(codes)	Characteristics (tenets or indicators) (adapted from (Ní Shúilleabháin & Clivaz, 2017) and (Clivaz & Ni Shuilleabhain, 2019)	Stage 2 (Collaborative Lesson Planning)	Stage 3(Lesson Presentation and observation)	Stage 4(Post-lesson reflection)
		and procedures that support the development of mathematical understanding *-anticipating or analysing teacher's reaction to learners' response or difficulties* *-anticipating or analysing teacher's action in relation to mathematical content* *-sharing or comparing representations and procedures in teaching*	• teachers improving the class activity on introducing the parameter p from what is shown in Figure 5.7 to what appears in Figure 5.8 and Figure 5.9 (5.2.3.2). • Incorporating the use of Desmos software (by teachers) in guiding learners towards the effects of parameters on trigonometric functions (5.2.3.2). • Sequencing lessons and concepts where teachers agreed to teach parameter k in lesson 2 since	concept of amplitude, maximum and minimum values of the tangent graph and/or (compared to) the cosine graph (see Figure 5.19 and Figure 5.20) (5.3.2.2). • Teacher F demonstrated the effects of parameter k using Desmos software (5.3.3.1). • Teacher F demonstrated the effects of parameter p using Desmos software (see Figure 5.20) (5.3.3.1). • The presenting teacher linked the previous research lesson 2 to lesson 3 for the day where learners were asked to	• Teachers admitting that they could not achieve all the objectives in lesson 1 because of over planning (5.4.2.3 Excerpt 5.4.2.3(b)).

MKT	Domain(codes)	Characteristics (tenets or indicators) (adapted from (Ní Shúilleabháin & Clivaz, 2017) and (Clivaz & Ni Shuilleabhain, 2019)	Stage 2 (Collaborative Lesson Planning)	Stage 3(Lesson Presentation and observation)	Stage 4(Post-lesson reflection)
		-selecting appropriate mathematical language, analogies, and metaphors	lesson 3 needed the application of the parameter (5.2.3.2). • Critiquing the use of $y = sin(x + 1°)$ in Desmos to demonstrate the effects of parameter p (5.2.3.2). • Teachers discussed the steps involved in interpreting graphs (determining or coming up with equations of given/drawn graphs) in preparation for lesson 3 (5.2.4.2). • Discussing the notation used in trigonometric functions (5.2.4.2). • Teachers solved an exercise on determining the equation(s) of	describe the relationship between pairs of graphs (5.3.4.1). • Teacher F scanned the example to be used and projected it on the smartboard using Desmos software (see Figure 5.21) (5.3.4.1).	

MKT	Domain(codes)	Characteristics (tenets or indicators) (adapted from (Ní Shúilleabháin & Clivaz, 2017) and (Clivaz & Ni Shuilleabhain, 2019)	Stage 2 (Collaborative Lesson Planning)	Stage 3(Lesson Presentation and observation)	Stage 4(Post-lesson reflection)
			already drawn graphs and simultaneously discussing how to teach learners the approaches (5.2.4.2).		
	KCS	*-identifying learners' knowledge or learning* *-identifying learners' difficulties or misconceptions* *-anticipating learners' mathematical responses* *-noticing and interpreting the mathematical meaning associated with learners' responses.*	• Teachers noted that learners should not be rushed but smoothly introduced to parameter p since some learners were still struggling with basic graphs (5.2.2.4). • Teachers agreed to test learners' prior knowledge on point-by-point plotting (5.2.2.4). • Teachers came up with a statement in the lesson plan that describes the		• teachers reported that dealing with definition of concepts (period, amplitude, range, domain) had a positive impact in research lesson 1(5.4.2.2). • Teachers reported their observations of learners' struggles in drawing the tangent graph especially the shape and

MKT	Domain(codes)	Characteristics (tenets or indicators) (adapted from (Ní Shúilleabháin & Clivaz, 2017) and (Clivaz & Ni Shuilleabhain, 2019)	Stage 2 (Collaborative Lesson Planning)	Stage 3(Lesson Presentation and observation)	Stage 4(Post-lesson reflection)
		-choosing an example that learners will find interesting and motivating *-selecting questions and tasks that seek out the presence of misconceptions*	properties of parameter k and it was to be shared with learners (5.2.2.4). • Anticipating Grade 11 learners' difficulties in sketching $y = sin(x+p)$ using the tabular method compared to $y = sin(x+q)$ that was easier in Grade 10 (5.2.3.1). • Teachers highlighting and anticipating that handling of the tangent graph could be more difficult for learners compared to sine and cosine during the introduction of parameter p (5.2.3.1).		the asymptote (see Excerpt 5.4.3.1(a)) (5.4.3.1). • Teachers noted their observation of learners not showing the continuity of the tangent graph when drawing it (see Excerpt 5.4.3.1(b)) (5.4.3.1). • Based on their observations, teachers suggested that slow learners be taught a single aspect in trigonometric functions at a time (see Excerpt 5.4.4.1) (5.4.4.1).

MKT	Domain(codes)	Characteristics (tenets or indicators) (adapted from (Ní Shúilleabháin & Clivaz, 2017) and (Clivaz & Ni Shuilleabhain, 2019)	Stage 2 (Collaborative Lesson Planning)	Stage 3(Lesson Presentation and observation)	Stage 4(Post-lesson reflection)
			• Editing of purposeful instructional activities by teachers in parameter a mixing positive and negative values for learners to be able to practice handling both (5.2.3.1). • Teachers opting to use the term 'basic' graph instead of 'original' graph to avoid confusing learners (5.2.4.1). • Teachers agreeing to use the term 'roots' for values of x for which a function is defined (5.2.4.1). • Practicing more activities on the tangent function by learners was		

MKT	Domain(codes)	Characteristics (tenets or indicators) (adapted from (Ní Shúilleabháin & Clivaz, 2017) and (Clivaz & Ni Shuilleabhain, 2019)	Stage 2 (Collaborative Lesson Planning)	Stage 3(Lesson Presentation and observation)	Stage 4(Post-lesson reflection)
			anticipated to be useful in easing the difficulties associated with the function (5.2.4.1).		
	KCC	*-linking mathematical knowledge to the syllabus* *-linking mathematical knowledge to a specific task available in the textbook* *-lateral curriculum knowledge* *-vertical curriculum knowledge*	• teachers consulted curriculum materials or pedagogical documents (ATP and CAPS) (5.2.2.3). • building Grade 11 content on Grade 10 parameters a and q (5.2.2.3). • teachers debated that q was learnt in Grade 10 and p should be introduced in Grade 11 (5.2.2.3). • Teacher D reminded other teachers about the domain of trigonometric functions $[-360°; +360°]$ that		

MKT	Domain(codes)	Characteristics (tenets or indicators) (adapted from (Ní Shúilleabháin & Clivaz, 2017) and (Clivaz & Ni Shuilleabhain, 2019)	Stage 2 (Collaborative Lesson Planning)	Stage 3(Lesson Presentation and observation)	Stage 4(Post-lesson reflection)
			should be used in Grade 11 (5.2.2.3).		

From the table, the rows show the occurrence, incorporation, and development of each knowledge domain during collaborative lesson planning, lesson presentation, observations, and post-lesson observation stages. For instance, Horizon Content Knowledge (HCK) was developed during collaborative lesson planning when teachers learnt that prior knowledge and real-life applications could help them to introduce a lesson in trigonometric functions. During lesson presentation and observation, the example of an electrocardiogram yielded positive results according to learners' responses. The importance of using real-life applications of trigonometric functions was acknowledged and confirmed by the presenter and the observing teachers during the post-lesson reflection of lesson 1.

Specialised Content Knowledge (SCK) pervaded all three LS stages with a total approximate occurrence of 12 (see row 4 of Table 5.2). The domain was developed mostly when teachers explained content to one another during collaborative lesson planning and post-lesson reflection. In addition, teachers developed SCK by learning content from the presenter while he/she was explaining it to learners.

Knowledge of Content and Teaching (KCT), like SCK, seems to be developed throughout stages 2 to 4 of the South African LS model and occurs 16 times (see row 5 of Table 5.2). The skill of sequencing objectives and trigonometric functions aspects, for instance, has been developed in each of the three stages. Again, most of the teachers' interactions during LS dwelt on how to teach trigonometric functions.

Knowledge of Content and Students (KCS) was developed bulkily during collaborative lesson planning. The possible reason could be that teachers always planned with the learners in their minds. KCS was also developed during post-lesson reflection.

Knowledge of Content and Curriculum (KCC) seems to have been developed greatly during collaborative lesson planning when teachers consulted the CAPS and ATP. It is however possible that the KCC developed influenced the presentation of the lesson when the presenting teacher linked Grade 11 work to what was learnt in Grade 10.

5.6 TEACHERS' PERSPECTIVES ON USING LESSON STUDY IN DEVELOPING MATHEMATICAL KNOWLEDGE FOR TEACHING TRIGONOMETRIC FUNCTIONS.

In this section, I present findings from semi-structured interviews to answer the secondary research question: What are the teachers' perspectives on the use of Lesson Study in the development of mathematical knowledge for teaching trigonometric functions? The purpose of the interviews was to elicit teachers' benefits, challenges, and views on the use of LS in the development of mathematical knowledge for teaching trigonometric functions, based on their participation and experiences in the study. Six categories emerged when I was analysing the semi-structured interviews. The categories are: teachers' experiences in collaborative lesson planning; affordances of implementing and observing the collaboratively planned lesson; teachers' experiences in post-lesson reflection dialogical discussions; trigonometric functions content acquired by teachers from participating in LS; pedagogic or teaching strategies learnt from the LS processes; and teachers' closing remarks on the use of Lesson Study in developing mathematical knowledge for teaching trigonometric functions. All the teachers attended the interviews that were held in the Computer Laboratory, two days after the LS processes had been completed (see Figure 5.33).

Figure 5.33

Teachers Responding To Interview Questions.

The interviews started at 15:10 and ended at 16:25. I asked questions to the group while video recording them, and the teachers took turns to respond. I also made follow-up questions to get clarity from individuals and /or group.

5.6.1 Teachers' experiences in collaborative lesson planning

I sought for what and how the teachers benefited from participating in the collaborative lesson planning sessions. A range of responses from all the teachers indicated that the collaborative lesson planning benefited them. Teacher A stated that:

> As a novice teacher, I have benefited a lot especially on the study lessons where we were planning how we will start our lessons and approach, approaches that we were going to do in class. So, ummm in lesson study ummm in lesson study 3 where we were preparing for lesson 3 where we were supposed to determine the equation on when we are given certain graphs, so we were needed to find the equation. So, I did not know that this method that we used was aaah it was new to me because on high school in high school we were not doing a similar approach. So, on drawing the original graph on the graph that we were given so that's not how we were doing approaching it. This method made it much easier for the learners and also for me in future if I am going to teach trig functions. So, *yaaa* [yes].

The response by Teacher A indicates that she learnt how to introduce and approach lessons such as lesson 3 which needed the determination of equations of drawn graphs. I hold that Teacher A gained KCT in this respect. In line with Teacher A's utterances, Teachers B, C, D, and F also reported that they learnt varied ways of approaching aspects of trigonometric functions during the collaborative lesson planning seatings. In particular, Teachers C and D noted that they came to know the amplitude better from the conversations. This is exemplified by what Teacher C said:

> Because like we have Teacher E telling us *ukuthi* [that] ok when the graph moves from the x-axis (demonstrating using her left hand) to this part it no longer has your resting point at the x-axis. Myself I thought the resting point is always on the x-axis… Again, we were doing that amplitude the top part

> is called your crest I didn't know that so as we sat down and planned our lesson it was very beneficial for us all.

According to the ATP, Grades 10 and 11 trigonometric functions are taught in the same week(s). The most interesting finding was that Teacher D went to teach the amplitude to his Grade 10s the way he had learned from the ongoing LS sessions, and he reported that the learners understood the effects of parameter q better than in the previous years. It entails that Teacher D applied KCT acquired (or developed) from LS collaborative activities to teach Grade 10s.

A further novel finding is that teachers simultaneously gained subject matter knowledge as well as planning and teaching skills. This was expressed by Teachers D, E, and F. Teacher D:

> The second thing was that the ownership it’s like because all of us prepared the lesson then it's quite intriguing to ... it's like to see whatever you are sitting down and planning it comes into action as *mam* [madam] said. The last one it actually enriches your content as a teacher because some of the content we didn't know them you find someone who knows better who can help you. So, I think it was very beneficial just to plan as a group and then come up with a lesson and then look at it and then see it grow as you refine it. So, I say I think it was very very beneficial.

Teacher F:

> And also, again to say the way in which during the planning process I could see that the way I was teaching trigonometry all along and the way now we plan it, it was different in such a way that we approached it in a perception *yokuthi* [that] we are only going to focus on such an aspect. That was the other part that made me to see oh LS is working for me because now I do not have to look at everything and rush, rush. No, I have to do what I have to look at a specific aspect and then deal with that aspect. If that specific aspect is addressed, then I go to another aspect.

Teacher D's utterances confirmed the teacher's development of KCT and SCK. In a similar vein, Teacher F's response affirmed his growth in KCT that involved sequencing of trigonometric functions during planning. As a follow-up question, I requested Teacher F to elaborate on how he used to plan his trigonometric functions lessons and he responded as follows:

> I was planning my trig functions in such a way that I will look at the ATP its 1, 2, 3 weeks and then I will say alright within this 3 week I just write some notes there and there and then look at some examples in the textbook and give learners just to sketch. Number 1, I did not give learners a proper introduction that's the first thing that I saw on my side. I did not give learners a proper introduction to say nowhere does this thing of trigonometry work. I did not do that for the past 10, 11 years of teaching trigonometry I was not doing that. And the other part is focusing on a specific aspect so I would say give them notes for Grade 10 and then go to Grade 11 straight. So that part also not

> considering whether learners understood Grade 10 or not I was skipping that part to say no according to my focus this is my focus. So at least LS has taught me that no say go back check whether the learners have understood previous knowledge.

It is clear from what Teacher F said that the teacher learnt the importance of considering and incorporating prior knowledge and real-life applications, which are indicators of CCK and HCK, when planning trigonometric functions lessons. Again, Teacher F revealed that he had challenges with the concept of period of the three trigonometric functions in the CAPS.

Teacher F:

> And the other thing that I saw is that I did not know that the difference between the period of tan the period of the sine and cosine graph… but within the preparation I saw that no the reason why the graph is 180° is because it completes a full graph within a period of 180° on all the intervals but I did not know that previously.

Teacher F:

> The other thing that I saw on my side is this thing of plotting and sketching. I just took it as one thing because I just said when I went to class, I just said sketch but when you look at the ATP the ATP says start by plotting first point by point. So, I omitted I did not know that plotting is point by point because

> that concept if you check … it comes from earlier grades *kubo* [in] grade 9 grade 8 where kids are plotting the graphs using the table. Yes, so so *mina* [myself] I just said sketch and then I just gave them the shape and then I did not consider this part of plotting which was very key because I just took them as the same thing. Now at least now I know that no there is this part that I must do first plotting *ukuthi* [that] kids must do it practically so that they take up the shape by themselves…

In his response(s), Teacher F mentions that he was developed in SCK by other teachers with respect to period of the tangent function. Furthermore, Teacher F admits that he used to skip the point-by-point plotting of trigonometric graphs because he could not differentiate it from sketching. The skipping of aspects and or concepts in trigonometric functions is common amongst the teachers that I support in the District. Teachers E and A bolstered the importance and benefits of collaborative lesson planning as uttered below.

Teacher E:

> And the other thing that I have realised is that as much as we are maths teachers, so we are able to learn from each other. There were certain things that I didn't know them better. So, when we are sitting together like this then I am able to understand certain things better and it will help me as well when I am alone in class to explain them better. So, I should think this kind of planning we need it so that we can develop further. Thank you.

Teacher A:

> Also, I saw that the lesson ....remember we were introducing p while we forgot about the k. So, if we were not together, we would go to class and not introduce k. Now because we were able to sit together it was easier for one of us to realise that we did not introduce a k. Working as a group helped us a lot because you see some of the mistakes that you are doing from others.

The two teachers' inputs indicate that collaborative lesson planning results in lesson plans of better quality than individual plans.

During the interviews, teachers also highlighted some challenges and or limitations that they faced during collaborative lesson planning. The participating teachers revealed that they had limitations with handling objectives, which signals a challenge in KCT. In the first instance, the group overlooked the stating of objectives in their write-up of the lesson 1 plan. Teacher C put it this way: "I can say that because remember the first planning that we did we had no objectives... I have learnt that when planning the first thing that I have to do is stating the objectives." Although the objectives were not calibrated, they existed in the form of activities for both learners and teachers. Teacher B argued that: "And another thing that I saw is our objectives. So, our challenge was that we had lots of objectives in lesson 1 we had lots of objectives, and we didn't meet all the objectives we didn't consider *itimeframe* [the time frame]." Talking about the time frame issue Teacher C commented: "And also the other thing when considering the time frame when giving learners maybe something to do in that plan maybe we should include … maybe I am going to give this example for 5 minutes."

In general, objectives inform activities and timing them during planning could help teachers to fit them within the given lesson time. Adding to the limitations above Teacher F averred that their planning could have had a smart-board for them to practice presenting before going to class. Teacher E held that learner activities could have been prepared into worksheets during the planning sessions. He contended that this could have saved time for the teacher of writing the activities on the board. It seems the participating teachers' interactions during interviews extended their quest to improve their planning of trigonometric functions lessons.

5.6.2 Affordances of implementing and observing the collaboratively planned lessons

Teacher F presented all the three lessons in the study. During the interviews, the teacher indicated how he benefited from his role in the following way:

> On the presentation side firstly, I benefited presenting in front of the camera. The other is to say when you present trigonometry you don't need to draw all these functions is what I picked up when presenting and showing the learners with regards to the software that I was using it was easier for me... I was able to present a lot of work within a lesson. Learners could easily see the movement of graphs. On the other class that I am not using the smart-board, it is difficult for me to explain to the learners the same stuff.

It is evident from Teacher F's words that the use of curriculum materials such as the Desmos software facilitated the teaching and learning of the effects of parameters in trigonometric functions. Teacher F's reported experiences are typical of teacher noticing and portraying KCT (Copur-Gencturk & Tolar, 2022; Dick, 2017; Sekao, 2023). Another finding emerging from Teacher F's experience as a presenter was that the use of suitable and correct language associated with trigonometric functions is very important. When concluding the effects of parameter k with the class, Teacher F reported that he found it important to write the conclusion in the learners' language. He held the notion that learners would understand it better, and that is a tenet of KCS. Teacher F further bolstered his assertion by saying: " ... I learnt that we must use the correct language and correct words when teaching because we gave learners an ambiguous definition of period. When we came back for reflection, we said 'look at this definition'."

Findings pertaining the teachers who observed during the lesson presentation and observation sessions suggest that these teachers gained content knowledge and teaching skills. They took the role of both a learner and an observing teacher. This is found in Teachers B's and A's inputs:

Teacher B:

> For me, I think I benefited a lot as an observer because as I was observing I was also a learner. So, whatever the presenter was presenting I was also receiving it like a learner. So, *la akula itrigonometry* […] lesson number 3 when we were discussing *lana* [there] I picked up you have to draw the basic graph

> on top of the sketched one. During the lesson, I could easily see even without drawing the basic graph...

Teacher A:

> To add on to Teacher B lesson 3 was the toughest the challenging lesson ever because finding the equations on functions it's very challenging and it's also challenging on us. So, it was Teacher F that made it easier for the learners and also because I was observing I was also a learner for us to understand the approaches that he used was easier... I think next time we should all be able to use the smart-board and the software it will be much easier...

Teachers A and B's observations of the research lessons suggest that they developed both content knowledge (SCK) and how to teach it (KCT). Teacher A's input shows that she has a desire to be able to use the Desmos software. Teachers C, E, and F reported that they captured the importance of prior knowledge during lesson presentation and observation stages.

Teacher E:

> Can I come in? Now when I was observing I have realised that you know it is very much key especially when you are going to teach a new concept to consider the prior knowledge because that is what makes these learners to understand the new concept that you are going to introduce and the way in which it was done it was perfect because we considered the knowledge of

these learners from Grade 10 when you are going to cons... that's what made those lessons to flow.

Teacher F:

... the other thing whilst you are teaching you need to always go back to prior knowledge maximum, minimum, amplitude it's very very important to always have an integrated approach to the functions.

Teacher C:

To my side as an observer, I think it did benefit me because number 1 when I am sitting there with those learners, I see that we are dealing with different learners in one class we don't have the same group that's where prior knowledge of grade 10 it picks them up ...

Another important finding is that of observing teachers learning the engagement and handling of learners' responses from the presenter (Teacher F) during lesson time.

Teacher D: *Mina* [myself] I learnt a lot from *meneer* [sir] the way he was handling learners because sometimes there is this challenge that you think these learners know this... e.g. when explaining the amplitude of tan after learners said it was 1. I enjoyed how he engaged the learners and handled the situation.

Teacher A: (agreeing with Teacher D): And choosing the learners at random 'give me the answer' 'give me the answer' so that we check if all the learners understand. [all the observers agreed with Teacher A]

Teacher C: The presenter handled the tan graph very well, especially the range which is an element of real numbers where learners could not be convinced.

The teachers' responses show that the teachers developed their mathematical knowledge for teaching trigonometric functions by professional noticing. According to Choy (2016) teachers take mental snapshots while seeing what is happening in any one LS stage and use them as a means of knowledge development. It appears the observing teachers developed SCK, KCS, and KC from Teacher F's presentations.

5.6.3 Teachers' experiences in post-lesson reflection dialogical discussions

It emerged from the teachers that post-lesson reflection helped teachers to improve on their lessons by correcting the shortcomings of the previous lessons, as purported by Teachers C and A.

Teacher C:

> The first thing that I saw especially when we reflect because now, we see the mistakes that we did when we planned the first time. Remember we came back and sat down and then that's when we actually noticed that we did not have any objectives. So, reflection actually helps that you can come up back

and say guys no this not how it should be done by the third time you know.... it's like concerted lesson...

Teacher A:

It also helped us to improve on how to improve our lessons whether we should keep them if for example that on tan graph learners had a challenge on drawing the tan graph so we were able to reflect and check *gore* [that] no most learners do not know how to draw the tan graph and we came up with solutions on how to improve for the learners to be able to draw the tan graph giving them more work and yeah ...

I further asked the teachers to clarify on the lessons that they were improving, and they responded as follows:

Teacher E:

You know you could see if you observed lesson 1 it was not flowing like the last lesson that I have observed where I thought maybe it was the interpretation of graphs and I know that part used to be difficult for the learners when we were teaching it. But I have realised *gore* [that] when we come back and reflect that's where we try to fix our mistakes and then hence the next lesson will flow better than the previous one. Because the third one it was just super, and it showed that we came back and fixed whatever was not right.

Teacher F:

> LS it helps especially the issue of reflection is very important. Because when I was presenting the lesson 2 I saw that ok we did not meet the target for lesson 1 and then for lesson 1 they had to sketch there was a point when we reflected that eeh those graphs were many because the kids spent a lot of time sketching *ku* [the graph] $y = \sin\left(\frac{1}{2}x\right)$ with a big interval (opening both his arms wide) so they spent a lot of time doing what sketching and then it was an eye opener to say no do not give learners too many things especially if you can just show them there *bona* [then they] do 1 or 2 illustrations because it will take time. So, reflection it was very beneficial.

Teacher E: [extending what Teacher F was saying]…

> More especially if you have objectives so your activities must be driven towards your objectives... as long as you can achieve your objectives then it's fine.

In line with this, there was also a novel finding where teachers refined and aligned the preceding lessons with the following lessons. From my observation, this practice was mainly informed by learners' responses to what teachers had presented. This ensured the compatible blending and sequencing of aspects from the ATP in favour of learners' conceptual understanding. Overall, Teachers A's, C's, E's, and F's reported experiences in the post-lesson reflection sessions honed and developed their KCT, KCS and SCK.

5.6.4 Trigonometric functions content acquired by teachers from participating in LS.

I went on to probe teachers' gains and or benefits in their LS experiences concerning content in trigonometric functions. The teachers responded in various ways that indicated how they had benefited as individuals and as a group.

Researcher: Overall, did you benefit anything in terms of content from participating in LS?

Teacher B: *Mina* [Myself]

> I benefited *ku* [from] interpretation lesson number 3 interpretation of the trig functions and how to write your formula the equation of the transformed graph.

Teacher C: The resting point on my side.

Researcher: What is it used for that resting point?

Teacher C:

> Remember like myself I had a misconception *ukuthi iresting* [of saying the resting] point is always on your *x*-axis even though sometimes it starts on 2 *mara* [but] for me if they were going to say write the resting point, I was going to say zero... it's not always the *x*-axis of your graph.

Researcher: Ok. How does it help learners to know the resting point?

Teacher F:

We use the resting point when we answer the question of amplitude. And also, the resting point also helps learners when we come to interpretation of graphs. [all the teachers agreed to what Teacher F was saying]

Teacher E:

Yes, when you interpret better from the previous lesson it helps the interpretation just flow.

Teacher D:

Mina [Myself] I was doing it *lama* [with] Grade 10 today not even last year, today. We were doing I think interpretation of graphs we were looking for *q* and they could see it very easily the *q*, by the resting point because the graph was now shifted one unit up so now when I told them ... it was very easy for them to see.

Teacher E:

> Even in lower grades you can see that lesson now the strategy has been applied in grade 10.

All laughed. The teachers' utterances are characteristics of pedagogy. This signals that content and pedagogy cannot be separated. Teachers reported development in SCK and KCT concerning interpreting the effects of parameters; finding equations of already drawn graphs; and amplitude.

5.6.5 Pedagogic or teaching strategies learnt from the LS processes.

Pedagogy and content continue to be blended in mathematics classrooms. I asked the teachers to reveal and elaborate on the pedagogy or teaching strategy that they had learnt during their participation in the LS process. Teacher F indicated that he found ascertaining key concepts and explaining them explicitly to learners important when teaching trigonometric functions.

Teacher F:

> When you are teaching trig functions, the first thing that you must do especially when you are explaining to the learners you must have the correct key words and the correct explanation for amplitude explain to them, period explain to the kids, the movement because some learners are still confusing the movement of horizontal and vertical. So, you need to explain very correctly to say when the graph shifts vertical, we only look at the y values when the graph

> shifts horizontally, we check at the x-axis. That is the other thing that I used to omit but now I saw that no after thorough preparation...It also helps when you are interpreting because you see that this graph has moved 2 units up and, in that case, it works out nicely.

In line with this, Teacher D observed that he realised that it was important to thoroughly teach the three basic graphs (sine, cosine, and tangent) to learners. The teacher noted that this made the interpretation of the transformation of the graphs by learners easier.

To get a deeper insight, I then asked the teachers' views on the sequencing of objectives and concepts when teaching trigonometric functions. Teachers C and F had similar sentiments that signalled the importance of sequencing concepts and objectives. Teacher C related the success of lesson 3 to proper sequencing (of the effects of parameters a, q, k and p respectively) that the group had done:

> We saw *ukuthi* [that] lesson 3 was flowing. So, having to sequence our objectives is very much important you get to see the most difficult part it is now going to be easier. So, for these learners if we say finding the equation is difficult they won't agree but *thina* [ourselves] we know that it is very difficult.

Teacher F elaborated on the importance of sequencing the combination of parameters that prepared learners for Grade 12 type of questions.

> The other thing *meneer* [sir] when it comes to sequencing of concepts we need to say when we teach trigonometry, we need to start by saying $y = a\sin kx$,

> again deal with that concept from there $y = a\sin kx + q$ vertical movement again $y = a\sin b(x+p)+q$ ... because you go to grade 12 now there are no longer going to ask them this simple question, they are going to give them everything on one equation. So, if they know the sequencing from basic to vertical to horizontal and integration it becomes easier...

All the other teachers agreed with teachers C and F. Teachers acknowledged the fact that LS's collaborative activities developed their KCT, especially in the teaching of basic graphs and concepts; and the proper sequencing of lesson objectives related to the effects of parameters.

5.6.6 Teachers' closing remarks on the use of Lesson Study in developing mathematical knowledge for teaching trigonometric functions

Teacher A held the view that Lesson Study would be useful for the improvement of the teaching and learning of trigonometric functions. This was because different teachers brought varied approaches of teaching the topic.

Teacher A observed that:

> It will help because if we are together then we will come with different approaches on how to make trigonometry easier. Because if I am alone especially, I am going to make an example on the lesson 3. I, as Teacher A, I will use my own method that I used at varsity and high school while there is an easier method that I could have used for learners to understand.

The observation by Teacher A suggests that she got exposed to an approach that was better than those that she had learnt at school and during pre-service training. In agreement with Teacher A, Teacher E said "…simply because when we are sitting like this, we are learning from each other we are capacitating each other and then it will be easy to teach even if you are alone." Teacher F saw LS as a potential intervention to any problematic topic in mathematics. The other interesting finding was that LS study had a positive influence in the teaching of the trigonometric functions in other grades as portrayed by what teacher D said, "*Mina* [myself] I think LS actually informs teachers' teaching in the lower grades. I am saying I am teaching grade 10, ... now I know the approach." The teacher was referring to that teaching of amplitude in Grade 10 that he had highlighted earlier. It, therefore, follows that it is advantageous to have a LS that is made up of teachers teaching different grades. I also asked the teachers about how participation in LS had affected their professional and personal relationships and the following conversation ensured:

Teacher D: Tr F said I think personally professional everyone who came it's like we are all equal ownership of a lesson not his lesson... we are agreeing.

Teacher E:

> And again, it also brings in the issue of communication. I will not be afraid if maybe I have a problem with a certain topic to come and stand before my colleagues or to approach one of my colleagues so that we can...

Teacher F: There is no judging... Professionally we are stronger together.

Teacher B:

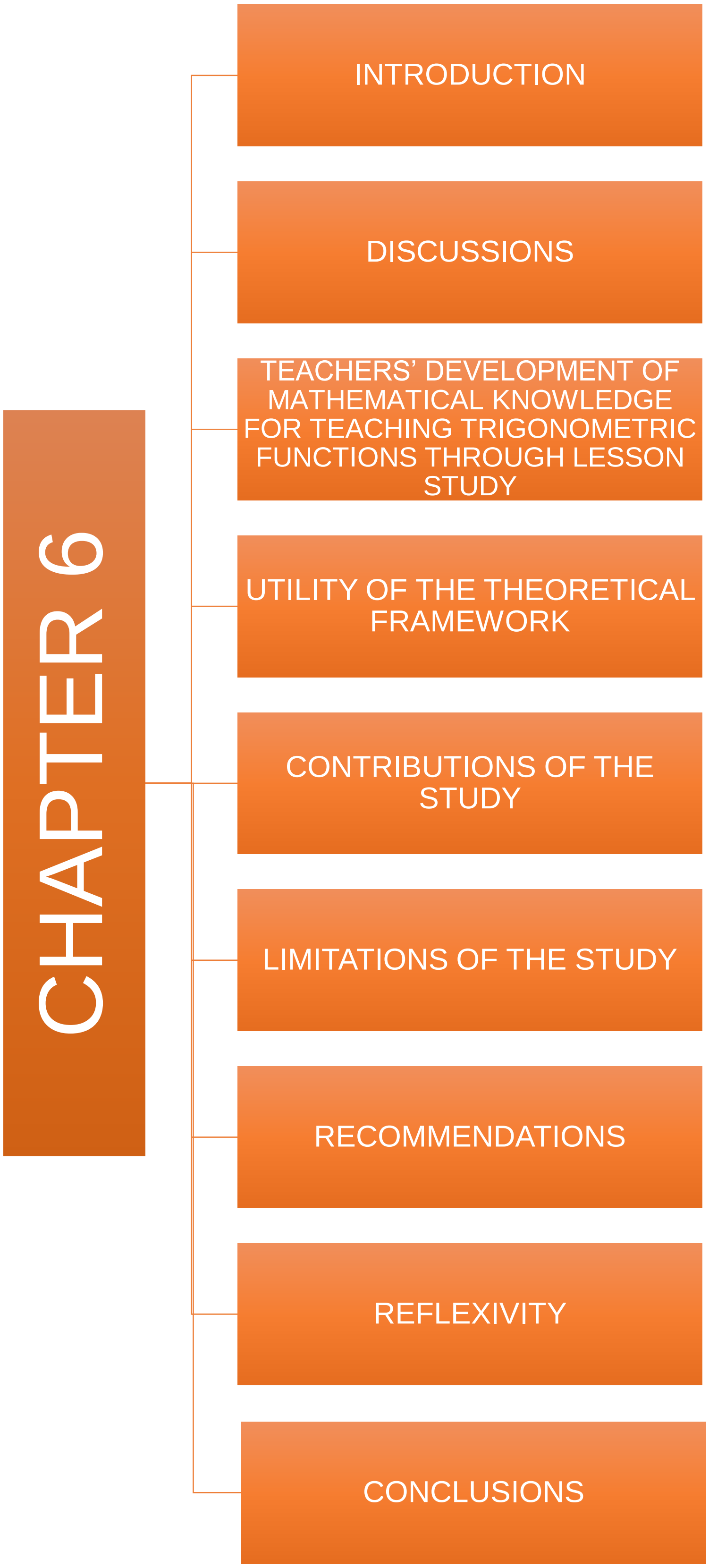
> I used to be afraid to ask content from my HOD thinking that he will say I do not know my work. Now I know I am free I can go and ask help from him.

All the teachers laughed with amusement. This showed that LS had cultivated long-term professional relations among the participating teachers towards the teaching and learning of mathematics. The participating teachers seemed to value their experience in LS collaborative activities than anything else.

5.7 CHAPTER SUMMARY

In this chapter, I presented the findings that highlighted how teachers developed mathematical knowledge for teaching trigonometric functions through Lesson Study. During planning, I found that teachers developed their mathematical knowledge for teaching trigonometric functions through their collaborative lesson planning conversations and by incorporating all the six MKT domains. Teachers' mathematics knowledge for teaching trigonometric functions evolved during the research lesson presentation when they engaged/incorporated HCK, SCK, and KCT. I also found that teachers' reflective practices were honed through post-lesson reflection conversations (that incorporated all the six knowledge domains) critiquing the achievement of objectives, and refining and re-aligning previous and future lessons. The fourth finding was that teachers gained content knowledge and teaching skills in trigonometric functions through participating in Lesson Study and they saw LS as a solution to address the challenges in the teaching and learning of trigonometric functions. In the following chapter, I present discussions, recommendations, and conclusions.

6. CHAPTER 6 DISCUSSION, RECOMMENDATIONS AND CONCLUSIONS



6.1 INTRODUCTION

In this chapter, I discuss the findings that I presented in the preceding chapter. The purpose of my study was to explore teachers' development of mathematical knowledge for teaching trigonometric functions through LS. To achieve this exploration, I pursued the main research question (RQ), namely; How do teachers develop mathematical knowledge for teaching trigonometric functions through Lesson Study? To answer the main question, I was driven by the following four research questions: RQ1: How do teachers develop mathematical knowledge for teaching trigonometric functions in the collaborative lesson planning stage? RQ2: How does teachers' mathematical knowledge for teaching trigonometric functions evolve during the research lesson presentation? RQ3: How do teachers' reflective practice hone their mathematical knowledge for teaching trigonometric functions? and RQ4: What are the teachers' perspectives on the use of Lesson Study in developing mathematical knowledge for teaching trigonometric functions?

Chapter 6 is made up of eight main sections which are: discussions; teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study; utility of the theoretical framework; contributions of the study; limitations of the study; recommendations; reflexivity, and conclusions. In section 6.2 (discussions), I discussed and responded to the four sub-questions. I then responded to the main research question in section 6.3.

6.2 DISCUSSIONS

In this section, my discussion of findings is structured per research question (RQ). Within each RQ, I start by stating the purpose of the RQ followed by a summary of findings, blended with a literature review. The discussion then ends with implications.

6.2.1 Development of mathematical knowledge for teaching trigonometric functions in collaborative lesson planning

The first research question sought to explore the development of mathematical knowledge for teaching trigonometric functions in collaborative lesson planning. To answer this research question, I collected data by observing teachers during their collaborative lesson planning in the second stage of the South African LS model. Teachers' collaborative lesson-planning activities and discussions showed that all six domains of MKT were incorporated, employed and developed with KCT, KCS, and SCK dominating (see Figure 5.1 in Chapter 5). This confirmed reports by Clivaz and Ni Shuilleabhain (2019) and Ní Shúilleabháin and Clivaz (2017) who found that all the six domains were incorporated in the planning stage.

In their planning, teachers agreed to introduce trigonometric functions to learners by discussing real-life applications of the topic, which is synonymous with CCK and HCK. The teachers anticipated that such an approach of infusing real-life applications in the teaching and learning of trigonometric functions might motivate and enhance learners' understanding. This finding was consistent with those of Roussouw et al. (1998) and Tatira (2021) who concur that linking prior knowledge to real-life applications of mathematical content (like trigonometric functions) during planning has the potential to enhance learners' understanding. This finding could be suggesting the importance of infusing real-life applications of trigonometric functions during planning by teachers. In line with KCC tenets, teachers consulted curriculum materials (the ATP and CAPS) on prior knowledge (content covered in Grade 10) and what should be covered in Grade 11. This was evident when teachers excavated that the effects of parameters a and q were done in Grade 10 while those of p and k were to be introduced in Grade 11; 0° to 360° are the trigonometric functions domains covered in Grade 10 whereas -360° to 360° are for Grade 11. The deliberations on the trigonometric functions' domains prompted the teachers to edit some purposeful activities in the textbook. For instance, the domain for $y = \tan\frac{1}{2}x$ was changed from $\left(0^\circ;360^\circ\right)$ to $\left(-180^\circ;180^\circ\right)$. This implies that employing KCC during planning guided teachers in compiling and refining trigonometric functions purposeful activities to suitability. Teachers tapped on KCS when they agreed and saw it as a smooth way for their learners to start lesson 1 by testing prior knowledge on point-by-point plotting of basic functions, followed by the introduction of the parameter p. In the process of doing so, teachers had to find an alternative way of introducing p since instructing learners to discover the effects of the parameter by plotting $y = \sin\left(x + p\right)$ using the calculator/tabular method was confusing even to teachers. KCS amongst the teachers compelled them to consider the impact of selected purposeful activities on learners' learning. This was achieved by generating questions like $y = \sin\left(x + 30^\circ\right)$ and $y = -\sin\left(x + 30^\circ\right)$ since the textbook only offered functions with positive parameter a like the former. The participating teachers seemed to incorporate, engage, and develop KCC and KCS by consulting curriculum materials and choosing, editing, and sequencing purposeful activities. The finding corroborated Proctor (2019) assertion that KCS transpires when teachers engage in sifting tasks/activities that will interest learners with possible misconceptions, difficulties and common errors in mind. Enama (2021) also notes that assumed/prior knowledge, class activities and their sequencing are pondered during lesson planning. Seemingly, KCS was essential in enabling teachers to choose purposeful examples that fostered learners' conceptual understanding of trigonometric functions.

It was clear that teachers engaged and developed their KCT towards selecting suitable curriculum materials (called *kyouzai-kenkyuu* in Japanese) for Grade 11 trigonometric functions by choosing two textbooks (*Classroom Mathematics* and *Mind Action Series*). Furthermore, teachers collaboratively used Desmos software to project graphs on the same set of axes that would show the horizontal shifts associated with parameter p. The software enabled the teachers to appreciate the illustration and elaboration $y=\sin\left(x-30^{\circ}\right)$ and $y=\sin\left(x+30^{\circ}\right)$ over $y=\sin x$ and $y=\sin\left(x+1\right)$. I discovered that only Teacher F was knowledgeable with Desmos, and the teacher capacitated others. As mentioned in the literature review, Sekao (2023) contends that curriculum materials are key to collaborative lesson planning as well as teachers' learning. From this finding, there was a possibility that collaborative lesson planning resulted in teachers gaining the skill of evaluating trigonometric functions content in the textbooks and integrating of technology in the topic.

One of the most interesting findings was that teachers grappled on how to plan for the effects of the parameter *k* (periodicity) and ended up admitting that they had misconceptions. Teacher D brainstormed the other teachers on how to teach parameter *k*. Teacher F also sought help from others on explaining (to learners) the periodicity of the tangent function (being 180° while those of sine and cosine were 360° each). Teacher D gave a clear explanation on that. This finding corresponds with that of Tatira (2021) where a teacher failed to clarify to the learners why the tangent function had a period of 180°. While challenges on periodicity by learners have been reported in studies by Fahrudin and Pramudya (2019) and Kamber and Takaci (2018), the findings of my study confirm Sekao (2023) assertion and Ogbonnaya and Mogari (2014) finding that learners' misconceptions could be related to teachers' knowledge and practices. The finding suggested that collaborative lesson planning provided teachers with the opportunity to develop one another on KCT in the periodicity of trigonometric functions.

My study also found several scenarios that depicted a conglomerate of KCT and SCK. Teachers engaged one another on the planning of effects of parameter p thereby coming up with logical sequential presentation that developed through steps (see Figures 5.7 to 5.9). Planning for determining equations of given graphs was approached through written steps (see Figure 5.11) and solving of selected purposeful activities (see Figure 5.14). Graphically determining the values of x was also planned through thorough solving of questions where Teacher D took the lead in brainstorming other teachers (see Figure 5.13). Teachers used SCK to identify, evaluate, and choose the purposeful activities in the form of exercise 2 (see Figure 5.4). The LS team also managed to summarise the effects of parameter k as shown in lesson plan 1 (see Appendix C) and in Teacher C's notes (see Figure 5.5). Significant time (more than 40 minutes) was spent on finding equations of given graphs (interpreting of graphs) signalling challenges on the part of teachers. The teachers facilitated the solution of the problem(s) by using different colours to indicate graphs involved in a way they believed afforded them visualisation of transformations (see Figure 5.14). This approach seemed to render it important for teachers to solve questions in the selected purposeful activities before going to class. I hold that solving questions during planning gave teachers confidence during research lesson presentation.

Responding to RQ1, ‘How do teachers develop mathematical knowledge for teaching trigonometric functions in the collaborative lesson planning stage?’, findings emerging from the collaborative lesson planning stage raises the possibility that teachers in my study developed mathematical knowledge for teaching trigonometric functions by scaffolding one another in content and pedagogy in the following ways: infusing prior knowledge and real-life examples, uses or applications of trigonometric functions (CCK, HCK); establishing anticipated prior knowledge from curriculum materials and creating purposeful activities that ascertain that knowledge in learners (KCC, KCT); editing/altering and sequencing purposeful activities from textbooks to align them with lesson objectives (KCT, SCK, KCS); collaboratively working out solutions for purposeful activities before going to class (KCT, SCK, KCS); integrating technology (i.e. Desmos) and coloured pens to promote visualisation of trigonometric graphs (KCT, SCK). Most of the findings of my current study are consistent with those of Sudejamnong et al. (2014) which showed that teachers developed knowledge during the planning stage by discussing and sharing knowledge from the textbooks; discussing lesson objectives and constructing open-ended questions; and anticipating learners’ thinking.

6.2.2 Evolvement of teachers' mathematical knowledge for teaching trigonometric functions in research lesson presentation

The second research question of my study examined how teachers' mathematical knowledge for teaching trigonometric functions evolved during the research lesson presentation. During lesson presentation and observation, I observed that the embodiment of HCK using real-life applications of trigonometric functions in the introduction of the first lesson by the presenting teacher aroused learners (see the picture of the electrocardiogram in Figure 5.3). Electrocardiogram machines display sinusoidal structures synonymous with sine and cosine functions. This finding is consistent with that of Magno (2022) who motivated learners in trigonometric functions by modelling the cosine graph using temperature and tidal movement recordings. Magno (2022) approach which involved recorded values and knowledge of computers was used to model. In my study, teachers gathered and/or considered sources that already depicted the concept of trigonometric functions. Considering findings from my study and the literature reviewed (Magno, 2022; Tatira, 2021), it appears real-life applications of trigonometric functions brought to light were only aligned to sine and cosine functions and not the tangent function. Teachers participating in my study reported learners' conceptual challenges in the tangent function during collaborative lesson planning (see Chapter 5 section 5.2.3.1), post-lesson reflection (see Chapter 5 section 5.4.3 Excerpts 5.4.3.1 (a) and 5.4.3.1 (b) and interviews (see Chapter 5 sections 5.5.3 and 5.5.4). I argue that the lack of tangible real-life applications for the tangent function in the teachers' and learners' spheres of lives could be exacerbating these challenges. These findings have important implications for the exploration of the real-life applications of the tangent function as well as other approaches of teaching the function by future researchers.

Another finding that emerged during lesson presentation and observation was that the presenting teacher articulated KCT by using Desmos software to teach/explain/define concepts of amplitude, maximum and minimum values of the basic tangent function in comparison to the basic cosine. The potential of the Desmos software allowed the teacher to zoom the basic functions in and out thereby vividly portraying the concepts to the learners (see Figures 5.18 and 5.19). The Desmos software also facilitated the projection of homework solutions thereby saving time. The Desmos software was further used by teachers to build generalisations on the effects of parameters k and p (see Figures 5.23 to 5.26). After these displays, learners were able to draw the functions of $y = \tan x$ and $y = \tan\left(x + 45^{\circ}\right)$ as well as describing the relationship between the two. KCT was executed when teachers blended the use of scanned graphs (from the textbook) on the smart-board and whiteboard to teach the finding of equations of the already drawn graphs. The use of technology in my study concurs with previous research where GeoGebra software (Ignacio, 2022) , Hawgent dynamic mathematics software (Li et al., 2022), and Desmos software (Loce, 2021) were integrated in the unit circle method of teaching trigonometric functions. All these studies reported that technology facilitated the teaching of trigonometric functions and enhanced conceptual understanding of the topic by learners. However, these integrations were not done in the context of LS like in my current study. Many researchers in general report that the integration of technology help in dealing with misconception in trigonometric functions (de Villiers & Jugmohan, 2012; Hamzah et al., 2021; Kamber & Takaci, 2018; Magno, 2022; Makandidze, 2020; Mosese & Ogbonnaya, 2021; Mulyono & Hapizah, 2021) These intriguing outcomes of my study suggest that integrating technology in the teaching and learning of trigonometric functions has the potential to facilitate/enhance the teaching and understanding of concepts. This possible beneficial use of Desmos software as an instructional material by teachers has implications for further research in the integration of technology in LS.

One of the important findings aligned to SCK was that of (the presenting teacher) being able to detect and/or address learners' misconceptions/errors/challenges during the research lesson presentations. In lesson 1, the presenting teacher noticed a learner who gave $y \in [1;-1]$ as the range of the basic sine function and the teacher then demonstrated and explained the concepts range and amplitude using Desmos software (see Figures 5.20 and 5.21). In the process of employing Desmos software, the teacher and learners were able deduce a general way of calculating the amplitude. Perhaps, this approach resulted in conceptual understanding compared to the traditional approach where teachers tell learners the formula through chalk and talk. During lesson 2 a learner asked a question that suggested a misconception in the asymptote of the tangent function. The presenting teacher dealt with the problem by engaging the whole class in using the calculator to tabulate the x and y values of $y = \tan x$ (see Figure 5.27). In my view, the teacher's actions signalled the learner-centredness of LS. SCK was also incorporated to detect omission of labelling turning points and intercepts of functions by learners in their homework. Elaborations using Desmos software were applied by the teacher (see Figure 5.29). This served as the background of the lesson/problems on finding equations of drawn graphs (like those shown in Figure 5.28). Taken together, these responses to the learners' reactions by the presenting teacher are ways of teacher/professional noticing, which is a means of knowledge development. It therefore suggests that SCK evolved through teacher/professional noticing in the context of LS.

In response to RQ2: 'How does teachers' mathematical knowledge for teaching trigonometric functions evolve in research lesson presentation?', teachers' mathematical knowledge for teaching trigonometric functions evolved in research lesson presentation when the teachers: used real-life applications of trigonometric functions to introduce the topic; used Desmos software to teach/explain/define concepts, build generalisations on the effects of parameters, present and explain solutions; detected and or addressed learners' (oral and written) misconceptions/errors/challenges.

6.2.3 Honing of teachers' mathematical knowledge for teaching trigonometric functions in post-lesson reflection

For the third research question, my study sought to explore how teachers' reflective practice honed their mathematical knowledge for teaching trigonometric functions. The discussions in this section are characterised by the synthesis of teachers' reflections on what transpired during lesson presentations in relation to what the teachers had planned.

One important finding was that teachers affirmed that HCK, in the form of real-life applications, links prior knowledge and new content well. This recognition by teachers of the benefit of infusing real-life applications in the teaching of trigonometric functions concurs with my observations during the presentation of research lesson 1. Sekao (2023) and Upa et al. (2023) perceive reflection as an environment that allows the whole LS group to identify its lesson planning strengths and weaknesses which, in turn, paves the way for improvement. It, therefore, follows that teachers could ease the abstraction and difficulty associated with the introduction of trigonometric functions to Grade 11 learners by infusing real-life applications of the topic.

In line with KCS, teachers honed their mathematical knowledge for teaching trigonometric functions when they highlighted that learner understood trigonometric functions better when concepts were well defined. As it emanated from the teachers' discussions, the concepts referred to include: period, amplitude, range, and domain. From my experience, understanding these concepts helps learners to interpret the effects of parameters on trigonometric functions. With such a skill, learners easily found the equations of already drawn graphs, the work that had been taught in lesson 3 by participating teachers. Another finding was that learners were conversant with drawing graphs using the calculator or tabular method. This leads to the next finding where learners struggled to draw the tangent graph, which was revealed by drawing straight lines instead of curves, not correctly positioning the asymptote, and leaving out the asymptote. Such a challenge in drawing graphs is reported by the DBE diagnostic report where candidates joined points of trigonometric graphs using a ruler (DBE, 2017). It was also found that teachers preferred that slow learners be taught trigonometric functions one aspect at a time. I see it as a positive stance for teachers to cater for all categories of learners. From the ongoing discussion, teachers in my study now know learners' learning of trigonometric functions better. Such knowledge informs teachers' ability to design and plan lessons that stimulate learners, choose purposeful activities that will interest learners with possible misconceptions, difficulties and common errors in mind (Ball et al., 2008; Chua, 2019; Hill et al., 2005; Moumou, 2021; Proctor, 2019).

One other powerful finding was that teachers became aware of the importance of planning and sequencing objectives (KCT) that fitted into a lesson period. Teachers had over planned, as Sekao (2023) warned, and so they agreed that: drawing basic graphs and the introduction of the parameter k should be split into two lessons, and that the same should be done with the equations of drawn graphs and the determining of the roots of x. Discussing of lesson objectives is typical of lesson improvement and development of knowledge as noted by (Sudejamnong et al., 2014). The presenting teacher was made aware of the achievement of objectives by observers. Such observations by teachers confirm Choy (2016) assertion that teachers take mental snapshots of what they see in class and then discuss it during reflection. In so doing, teachers honed or sharpened their practices.

Although the range had been defined in class, I found from the presenting teacher's articulation of SCK (during reflection) that learners had misconceptions about the range. He reflected that some learners gave it in degrees while others swapped the lower and the upper values. This concurs with Chua (2019) and Proctor (2019) assertion that SCK transpires when teachers can establish errors and misconceptions by learners and analyse the presentation of learners' solutions. Another important finding was that teachers were able to acknowledge their success in choosing purposeful activities. This led teachers to end up agreeing that research lesson 1 should deal with parameters a and q whilst research lesson 2 deal with parameters k and p. According to my experience and observation of the LS process undertaken by the teachers, each of the parameters k and p could be fitted better in two separate lessons. There would therefore seem to be a definite need for teachers to match aspects to be taught in trigonometric functions with the duration of a lesson. It would be important that future research investigate the time that could be needed to teach and cover the respective aspects of trigonometric functions.

From the above discussion I can respond to RQ3: How do teachers' reflective practice hone their mathematical knowledge for teaching trigonometric functions? Teachers' reflective practice hone their mathematical knowledge for teaching trigonometric functions by: discussing the successes and challenges of planning and teaching trigonometric functions concepts; coming up with better ways of breaking down and sequencing trigonometric functions aspects (in the form of objectives) to properly fit them in respective lesson periods; embracing and adopting (through collaborative discussions) new ways of selecting and editing purposeful activities; and re-aligning and refining following research lessons based on the experiences of previous lesson(s).

6.2.4 Teachers' perspectives on using Lesson Study in developing mathematical knowledge for teaching trigonometric functions.

In this section, I focus on teachers' perspectives on using LS in developing mathematical knowledge for teaching trigonometric functions, based on the teachers' responses to the interviews. During interviews, teachers reported that they benefited from collaborative lesson planning in the following ways: they learnt new methods of teaching trigonometric functions that were different from those they learnt at high school and university; they understood concepts like amplitude (that it is not always measured from the x-axis); they learnt from one another that trigonometric functions are better taught by handling one aspect or parameter at a time before combining them to avoid over-planning and or overloading learners; learnt to differentiate between the periodicity of the tangent graph and that of the sine and cosine graphs. Collectively, the teachers' perspectives on collaborative lesson planning are synonymous with the development of KCT, SCK, and KCS through dialogical interactions. The outcome agrees with the findings in Gutierez (2019) qualitative study reporting that collaborative lesson planning brings content and pedagogical scaffolding that results in improved teaching skills and professional relationships among participating teachers. In support, Sekao (2023) contends that collaborative lesson planning environments are the basis of extensive teacher learning. The perspectives of teachers in my study on collaborative lesson planning match with those of Druken (2023) study in which 70% of the teachers reported that they developed pedagogic skills and gained content through sharing instructional strategies. Interviews revealed the following shortcomings during collaborative lesson planning: not clearly stating lesson objectives (especially for lesson 1); over-planning objectives and purposeful activities; and not timing class activities. The teachers suggested that rehearsing lesson presentations on the smart-board during collaborative lesson planning could improve research lesson presentation. This corroborates Bolívar and Ortiz (2017) findings that say collaborative lesson planning provided environments where teachers improved their practices by discovering their strengths and weaknesses in content and pedagogy. In a similar vein, teachers in Druken (2023) study reported that discussing challenges and sharing views within LS cultivated collaboration and professional growth among them.

Teachers' utterances during interviews revealed that lesson presentation and observation sessions benefited them in the following ways: gained confidence and discovered the strength of Desmos software in presenting and teaching trigonometric graphs; learnt the importance of using appropriate wording when defining concepts to learners; gained content knowledge and teaching skills while observing lessons; appreciate the importance of prior knowledge; learnt how to handle learners' reactions and responses (how to respond to learners' questions). The findings in my study concur with those of Druken (2023) study where teachers observed that they had learnt and gained confidence in teaching the new aspects of the syllabus through actively engaging one another in LS rather than just passively reading them.

It emerged from the interviewees that post-lesson reflection helped them to improve their lessons by correcting shortcomings of the previous lessons. Teachers elaborated this assertion by citing the fact that they improved from lesson 1 (which did not have clear objectives) to lesson 3 which flowed smoothly (where aspects were well sequenced resulting in learners being able to determine the equations of given graphs without struggling). I argue that the teachers' responses imply their development of KCT, SCK, and KCS. This finding is in agreement with Proctor (2019) assertion that teachers develop mathematical knowledge for teaching when they blend their understanding of subject matter, teaching and learners. Furthermore, Moumou (2021) found that post-lesson reflection had a positive influence on the development of MKT domains. This combination of findings provides some support for the premise that post-lesson reflection results in refined lessons and lesson plans in the teaching of trigonometric functions.

In terms of content knowledge (CK) or subject matter knowledge (SMK), teachers revealed that they learnt and understood the concept amplitude that was measured from the resting point to the crest (and not always from the x-axis as their earlier misconception) and they grasped the interpretation of trigonometric graphs as well as finding equations of given graphs. SMK and PCK are two sides of the coin. Teachers uttered that they gained, developed, and acquired the following pedagogical skills and teaching strategies during the LS process: how to teach and explain the concepts of amplitude and period explicitly to learners, and that thoroughly teaching properties of basic graphs (sine, cosine, and tangent) facilitated the understanding of effects of parameters. The findings of my study are consistent with those of Alamri (2020) whose participating teachers reported that LS cultivated appreciation of LS and helped them to improve their content knowledge, pedagogical knowledge, and knowledge for learner learning.

On their overall perspectives on LS improving teaching and learning of trigonometric functions, teachers shared the following: teachers in the LS group bring varied approaches to teaching trigonometric functions that are new to others; learning from and capacitating each other results in individuals gaining confidence to teach the topic alone in class; what has been learnt in Grade 11 LS could easily be propagated and applied in Grade 10 trigonometric functions; viewing LS as a potential intervention to any problematic topic in mathematics; to cultivate long term professional relations amongst teachers in the teaching and learning of mathematics where they will freely consult one another on any problematic topic. These findings corroborate the views of participants in Almadi (2022) study who believed that participating in LS enhanced teachers' SMK and PCK as well as fostered viable school-based learning communities. This implies that mathematics teachers in South Africa could form school-based LS groups to tackle challenging topics like trigonometric functions and improve learner achievement.

From the participating teachers' spoken experiences and perspectives, LS has far more potential and benefits than challenges in providing environments that nurture the development of mathematical knowledge for teaching trigonometric functions. According to Druken (2023) such a finding can lure other teachers and convince stakeholders to support the LS practice. I, therefore, argue that the South African mathematics education community should consider adopting LS as one of its teacher developmental models. The Department of Basic Education could support the initiative by training LS teachers and gradually fostering school-based, district-based, provincial-based, and national-based LS groups.

To answer RQ4: 'What are the teachers' perspectives on using Lesson Study in the development of mathematical knowledge for teaching trigonometric functions?' I found teachers' perspectives as follows: Collaborative lesson planning (the second stage of the South African LS study model) exposed teachers to the stating of lesson objectives, understanding and teaching concepts in trigonometric functions; Lesson presentation and observation afforded them the opportunity to learn integration of technology in the teaching and learning of trigonometric function, and handling of learner reactions during the lesson; post-lesson reflection honed their skills in lesson improvement; the overall LS processes cultivated professional relationships amongst themselves that will see them continue with collaborative activities in the teaching of mathematics.

6.3 TEACHERS' DEVELOPMENT OF MATHEMATICAL KNOWLEDGE FOR TEACHING TRIGONOMETRIC FUNCTIONS THROUGH LESSON STUDY

The main purpose of my study was to answer the main RQ: 'How do teachers develop mathematical knowledge for teaching trigonometric functions through Lesson Study?' In sections 6.2.1 to 6.2.4, I answered RQ1 to RQ4 (which are sub-questions) respectively. The answers to the sub-questions feed into and shape the answer to the main RQ. My study found that teachers developed mathematical knowledge for teaching trigonometric functions through LS in the following ways: 1) during collaborative lesson planning, teachers, by collaboratively discussing and infusing prior knowledge and real-life examples, uses and applications of trigonometric functions (CC, HCK), establishing anticipated prior knowledge from curriculum materials and creating purposeful activities that ascertain that knowledge in learners (KCC, KCT), editing/altering and sequencing purposeful activities from textbooks to align them with lesson objectives (KCT, SCK, KCS), collaboratively working out solutions for purposeful activities before going to class (KCT, SCK, KCS), and integrating technology (i.e. Desmos) and coloured pens to promote visualisation of trigonometric graphs (KCT, SCK); 2) in lesson presentation and observation, teachers used real-life applications of trigonometric functions to introduce the trigonometric functions, used Desmos software to teach/explain/define concepts, build generalisations on the effects of parameters, present and explain solutions, and detected and/or addressed learners' (oral and written) misconceptions/errors/challenges; and 3) during post-lesson reflection, teachers developed knowledge by collaboratively discussing the successes and challenges of planning and teaching trigonometric functions and concepts, coming up with better ways of breaking down and sequencing trigonometric functions and aspects (in the form of objectives) to properly fit them in respective lesson periods, embracing and adopting (through collaborative discussions) new ways of selecting and editing purposeful activities, and re-aligning/refining following research lessons based on the experiences of previous lesson(s). RQ4 sought to triangulate the findings in RQ 1 to RQ3 through participating teachers' voices. The ways in which teachers developed their mathematical knowledge for teaching trigonometric functions through LS were reiterated in teachers' perspectives as follows: Collaborative lesson planning (the second stage of the South African LS study model) exposed teachers to stating lesson objectives, understanding and teaching concepts in trigonometric functions; lesson presentation and observation afforded them the opportunity to learn integration of technology in the teaching and learning of trigonometric function, and handling of learner reactions during the lesson, and post-lesson reflection honed their skills in lesson improvement; and the overall LS processes cultivated professional relationships amongst themselves that will see them continuing with collaborative activities in the teaching of mathematics.

6.4 UTILITY OF THE THEORETICAL FRAMEWORK

The MKT framework helped me to: frame and structure research questions; answer my research questions; review and contextualise related literature; collect and analyse data and discuss findings (Grant & Osanloo, 2014; Kivunja, 2018; Omodan, 2022). I am detailing and elaborating in these aspects in the following paragraphs. It appears the MKT framework pervaded almost all the sections of my study. This is line with Kivunja (2018) assertion that a theoretical framework is a structure or lens that helps the researcher to focus upon, understand and make meaning of data, thereby creating interconnections within a study.

When I was collecting data during collaborative lesson planning, lesson presentation and observation, and post-lesson reflection, I observed teachers' behaviours that portrayed tenets of MKT domains (CCK, HCK, SCK, KCC, KCS, and KCT). For example, considering, incorporating, and discussing real-life applications of trigonometric functions by teachers in stages 2 to 4 of the South African LS model was typical of HCK (and/or CCK) while selecting, editing and/or sequencing of purposeful activities mirrored SCK, KCS, and KCT. I also extracted data from lesson plans and teacher notes that was characteristic of MKT domains. The interview questions that I asked the participating teachers were leaned towards SMK/CK and PCK, the knowledge bases of the MKT.

The MKT domains assisted me to make sense of the data that I had collected. For instance, while playing video recorded LS sessions in Atlas.ti, I used the domains as codes during the analysis of findings. To elaborate on this: during planning teachers deliberated on using $y = \sin\left(x + 1^\circ\right)$ as an elaboration for the effects of parameter p. After employing Desmos software, they later learnt that the difference between graphs of $y = \sin\left(x + 1^\circ\right)$ and $y = \sin x$ was insignificant. This led teachers to opt for functions like $y = \sin\left(x + 30^\circ\right)$ and $y = \sin\left(x - 45^\circ\right)$ that interested learners during the lesson. Possibly, this signalled evolvement of KCT, SCK and KCS amongst teachers. The research questions of my study were aligned to the MKT framework and my discussions were structured according to these research questions. My discussions within each research question (coupled with implications), revolved around MKT domains.

6.5 CONTRIBUTIONS OF THE STUDY

My study is original since it appears to be one of the earliest studies in South Africa to explore teachers' development of mathematical knowledge for teaching trigonometric functions through LS. In essence, my study makes several noteworthy contributions to mathematics education theoretically and practically, in the light of the MKT framework, the South African LS model, and the teaching and learning of trigonometric functions. In the following paragraphs, I discussed the theoretical and practical contributions of my study although there seems to be a thin line separating the two.

Theoretical contributions – Zhou et al. (2017, p. 261) view theoretical contributions as "a process which is based on the theory development and advancement in existing theory with some logics and facts." My study contributes to theory by exploring teachers' development of mathematical knowledge for teaching trigonometric functions through LS that could impact positively on learners' performance. Additionally, my study is one of the few studies that amalgamate the MKT framework and the South African LS model. This has resulted in the advancement of knowledge through deeply understanding and explaining the strength of LS-MKT synergy in the teaching and learning of trigonometric functions. For instance, my study revealed that knowledge bases (and their corresponding domains) in each LS stage are not always incorporated in the same weightings and or occurrences. Furthermore, most research in LS is predominantly conducted in lower grades and/or middle schools. It follows that there is limited research involving the application of LS in high school settings. This, in turn, suggests that my current study is one of the earliest studies that may have been conducted at high school, that is, at FET level in the South African context. Overall and in general, the findings in my study add to a growing body of literature on the triad of LS, MKT, and trigonometric functions.

Practical contributions – In general, LS is collaborative. Participants in Venketsamy (2022) report that LS fostered in them collaborative skills like planning. Research reports that professional development programmes that are collaborative and conducted in school or classroom contexts coupled with reflection, like LS, are more effective than those that are expert or trainer-dominated and off-site (Bannister, 2018; Bett, 2016; Calleja et al., 2021; DBE, 2015b; Stoll et al., 2012). The findings of my study provide insights for teachers and researchers on the teaching of trigonometric functions in LS contexts. Findings in the planning and presentation stage have shown the gains of integrating technology (Desmos software) in the teaching and learning of trigonometric functions. The need for cognisance of breaking down and sequencing objectives or aspects of trigonometric functions during planning has been raised in my study. The poor performance in the teaching and learning of trigonometric functions could be bettered through LS. I also contend that it is important to be mindful of the MKT domains during the teaching and learning of trigonometric functions.

6.6 LIMITATIONS OF THE STUDY

My study is not without limitations. The findings could have been better if I had used a mixed method approach instead of exclusively qualitative. In the mixed methods, learners could have participated and written a pre-test before intervention and a post-test after intervention. The study could provide insight into the impact of LS on both learners and teachers. Also, I could have left the teachers to diagnose their learners' performance in trigonometric functions rather than using the DBE diagnostic reports. This could have incorporated stage 1 of the South African LS model thereby enriching the body of knowledge in the model. In my study, I focused only on three stages of the South African LS model. The study could have given more insight if I had included all its five stages in my exploration. The five stages have the potential to generate more data. I conducted this study within six days, a longer period could have yielded better results. Grade 11 trigonometric functions are allocated 15 days in the ATP. I could have observed more aspects of trigonometric functions being taught and this could have enriched data. My study is qualitative and hence its findings cannot be generalised to all schools/teachers. Nevertheless, the study could be replicated to other schools of similar settings. Despite the limitations that I have mentioned above, my study certainly adds to the knowledge on the teachers' development of mathematical knowledge for teaching trigonometric functions through LS.

6.7 RECOMMENDATIONS

In my study, I explored teachers' development of mathematical knowledge for teaching trigonometric functions through LS. The findings suggest that LS provides environments that nourish the development of teachers' MKT. I therefore recommend that LS be adopted as one of the major PD models in South Africa's mathematics education community. The DBE could support the initiative by training all the teachers in LS.

Further studies (in LS) could be pursued focusing on the tangent function only since it is challenging in the teaching and learning arena. As an opportunity for future research, a similar study involving all five stages of the South African LS model is also recommended. This would be a larger-scale study that could generate more data and corresponding studies.

6.8 REFLEXIVITY

Reflexivity refers to critical self-reflections/self-knowledge where researchers explore their experiences of conducting research and the influence those experiences may have on their personal and professional lives in the context of scholarship (Arora et al., 2023; Cherchari, 2021; Pillow, 2015). Scholars argue that reflexivity is common in qualitative studies and is growingly viewed as an ethical practice that seeks to acknowledge and hone researchers' roles/experiences in scholarship in an integral, transparent and objective manner (Arora et al., 2023; Cherchari, 2021; de Groot et al., 2019; Finlay, 2002; Pillow, 2015). In the current section, I present what motivated me to conduct this study and my scholarly gains of conducting the study.

Currently, I am working as a District Subject Advisor (DSA) or Senior Education Specialist (SES) within the education system in South Africa. My core responsibility is supporting high school teachers in content and pedagogics in mathematics. I have long been concerned about the teaching and learning of trigonometric functions in schools. Some teachers just facilitate the point-by-point plotting of basic trigonometric functions in a day or two (when the topic has been allocated 15 days of teaching time) and others omit the topic. In addition, most of these teachers have been attending Just In Time Training (JITT) for more than seven years, but this has not yielded positive results as portrayed in the DBE NSC diagnostic reports. Such observations and experiences in my line of duty motivated me to conduct the current study, after being introduced to LS by a local university.

Within the period that I have been engaged in this study, I have grown professionally by gaining deep insight of LS. The knowledge of LS has reinforced my support for teachers as a DSA or SES. The experience that I have acquired through this journey has equipped me with academic capabilities that will enable me to prosper in the field of research. Fundamentally, I have attained the knowledge and the skill to research which are indispensable in improving my personal and professional life.

6.9 CONCLUSIONS

There is a concerning decline in performance in trigonometric functions by learners in South Africa. Research has shown that there is a positive correlation between learners' performance and teachers' knowledge of trigonometric functions. Unfortunately, teachers are reported to be struggling with the content and pedagogics of the topic. This has been exacerbated by the teacher development of mathematics teachers in South Africa, JITT, which is characterised by a top-down trainer-dominated approach that renders teachers to be passive individuals or empty vessels to be filled in. LS, which is a research-based, classroom-based, and intervention teacher development model typified by learner-centredness seems to serve as a boost to teachers' content knowledge (CK) and pedagogical content knowledge (PCK) in trigonometric functions. Teachers who participated in LS in my current study have reported their content and pedagogical benefits.

The MKT framework which is concerned with the art of teaching mathematics is associated with growth in both instruction and learner achievement. The development of mathematical knowledge for teaching trigonometric functions by teachers flourishes in LS environments. Findings in my study show that teachers incorporated and developed all six MKT domains during collaborative lesson planning through collaborative discussions that led to scaffolding one another in CK and PCK skills in trigonometric functions concepts. During lesson presentation and observation, teachers developed the art of handling learners' reactions (thinking) as well as the integration of technology in the teaching of trigonometric functions concepts. The post-lesson discussions honed the teachers' skills in breaking down aspects or objectives in trigonometric functions to fit into a lesson period. Re-aligning and refining following lessons based on the experiences or outcomes of previous lessons is also the way teachers hone their mathematical knowledge for teaching trigonometric functions during post-lesson reflection.

Findings in my study also revealed that the teaching and learning of the tangent function is still a challenge. This calls for further studies in the function. The DBE should consider adopting LS as one of its key teacher development models. Schools should pave the way and accommodate LS in mathematics. My study has contributed to both theory and practice by exploring teachers' development of mathematical knowledge for teaching terrain.

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8. APPENDICES

Appendix A: Observation Tool (adapted from Ní Shúilleabháin and Clivaz (2017))

MKT domain	Elements of observation – indicators of MKT domains	Notes and elaborations
Knowledge of Content and Teaching (KCT)	Sequencing trigonometric functions content	
	Using instructional materials, representations, purposeful examples, and procedures that support the development of trigonometric functions understanding	
	Predicting teacher's reaction to learners' responses/thinking or difficulties	
	Using appropriate mathematical language, analogies and metaphors for teaching trigonometric functions.	
Specialized Content Knowledge (SCK)	Looking for patterns in learners' errors in trigonometric functions	
	Unpacking of trigonometric functions concepts/aspects	
	Using alternatives to solve trigonometric functions problems	
	Choosing, making, and using/demonstrating mathematical representations in trigonometric functions effectively	
	Explaining and justifying mathematical ideas in trigonometric functions	
	Determining if the concept or rule of a trigonometric function is mathematical necessity	
Knowledge of Content	Identifying students' knowledge or learning in trigonometric functions	

MKT domain	Elements of observation – indicators of MKT domains	Notes and elaborations
and Students (KCS)	Identifying students' difficulties or misconceptions in trigonometric functions	
	Noticing and interpreting the mathematical meaning associated with learners' thinking/responses	
	Using an example(s) that learners will find interesting and motivating	
	Using questions and tasks that seek out the presence of misconceptions in trigonometric functions	
Common Content Knowledge (CCK)	Use of notations and vocabulary in trigonometric functions	
	Determining if a solution, a definition, or a representation in trigonometric functions is correct	
Horizon Knowledge (HCK)	Considering other uses of trigonometric functions knowledge	
Knowledge of Content and Curriculum (KCC)	Linking mathematical knowledge in trigonometric functions to the CAPS syllabus	

Appendix C: Lesson Plans

LESSON ONE – 27 JULY 2023 [08:45 – 09:45]

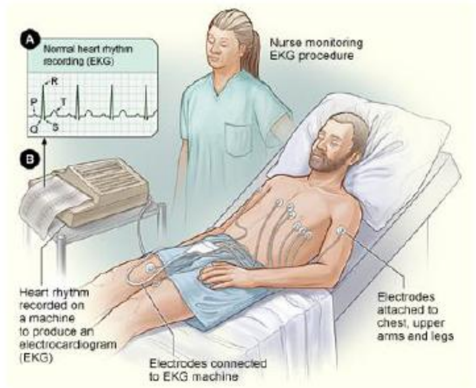
Lesson Objective

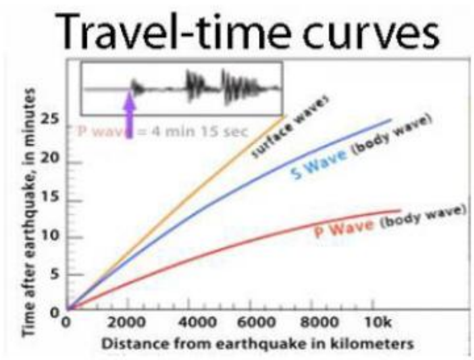
- Introductions of trigonometric functions within $x \in [-360; 360]$ and explanation of trig function on a real-life scenario
- Amplitude and vertical shift

Introduction

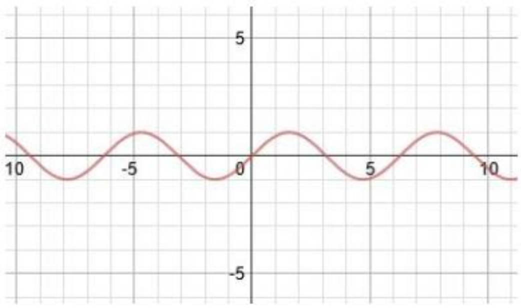
Electrocardiogram wave are trig functions that are used to calculate a pulse rate using an electrocardiogram (ECG)

We use trigonometric graphs for graphing an Earthquake wave.





Sinusoids
The graphs of the sine function and the cosine function are called sinusoids. They are used transmitting radio waves for communication such as phone calls, TV signals, internet browsing, electronic sirens are still used to create a loud modulated sound that can be transmitted over a very long distance.



In Grade 11 we start learning about basic trigonometric functions in the interval [-360;360]

Y=Sinx

Y=cosx

Y=tan x

Now that we understand our basic functions we now have to include the effect of a and k on the function.

Y=aSink(x)

Sketch the three graphs on the same system of axes

Y=Sinx Y=Sin2x $y = sin\frac{1}{3}x$ on the same system of axes $x \in [-360; 360]$ what do learners observe from the following functions.

Illustrations

1. Sketch the following functions on the same system of axes between $x \in [-360; 360]$

$$F(x) = Sin2x$$

$$g(x) = \frac{1}{4}Sin2x$$

$$h(x) = -3Sin2x$$

From the observation is that if k is greater than 1 then function will be reduced and also the period will be reduced

If k is between $0 < k < 1$ then the function will be enlarged meaning the period will increase; frequency will be reduced.

Period -The time interval or gab between two given points

Frequency -Is the number of waves.

Sketch the following graphs on different system of axes.

a) $Y = -2Sin2x$ $x \in [-90; 180]$
b) $Y = tan\frac{1}{3}x$ $x \in [-180; 180]$
c) $Y = Sin\frac{1}{4}x + 1$ $x \in [0; 1080]$

Write down the values of the following.

1.Minimum Value

2.Maximum Value

3.Range

4.Period

5.Amplitude

Lesson 2 – 28 JULY 2023 [08:45 – 09:45]

Lesson Objective

- *y* = *a* sin *x*; *y*= *a* cos *x*; *y* = *a* tan *x*
- Changing period: *y* = *a* sin bx ; *y* = *a* cos bx ; *y* = *a* tan bx
- Period = $\frac{360°}{b}$, b > 0 and for *y* = *a* tan b *x* , Period = $\frac{180°}{b}$ and the critical poin

  are determined $\frac{P}{4}$
- Vertical translation : *y* = sin bx + *q* ; *y* = cos bx +*q* ; *y* = tan bx + *q*
- Horizontal translation: *y* = sin (*x* + *p*); *y*= cos (*x* + *p*); *y* = tan (*x* + *p*).

Learners to Sketch the following functions on the same system of axes.

1.Sketch the following functions on the same system of axes $x \in [-30; 360]$

- $Y = Sinx$
- $Y = Sin(x - 30)$

2.Sketch the following functions on the same set of axes within $x \in [-90; 360]$

- $Y = tanx$
- $Y = \tan{(x + 45)}$

Activities

Sketch the following functions on different set of axes

1. $F(x) = -2\sin(x - 30)$ $x \in [-30; 390]$
2. $g(x) = -3\cos(x + 45)$ $x \in [-360; 90]$
3. $h(x) = -\tan(x - 60)$ $x \in [-90; 360]$

Write down the values of the following

1.Minimum Value

2.Maximum Value

3.Range

4.Period

5.Amplitude

Lesson 3 – 31 JULY 2023 [13:30 – 14:30]

Interpretation of Trigonometric functions

Lesson objective

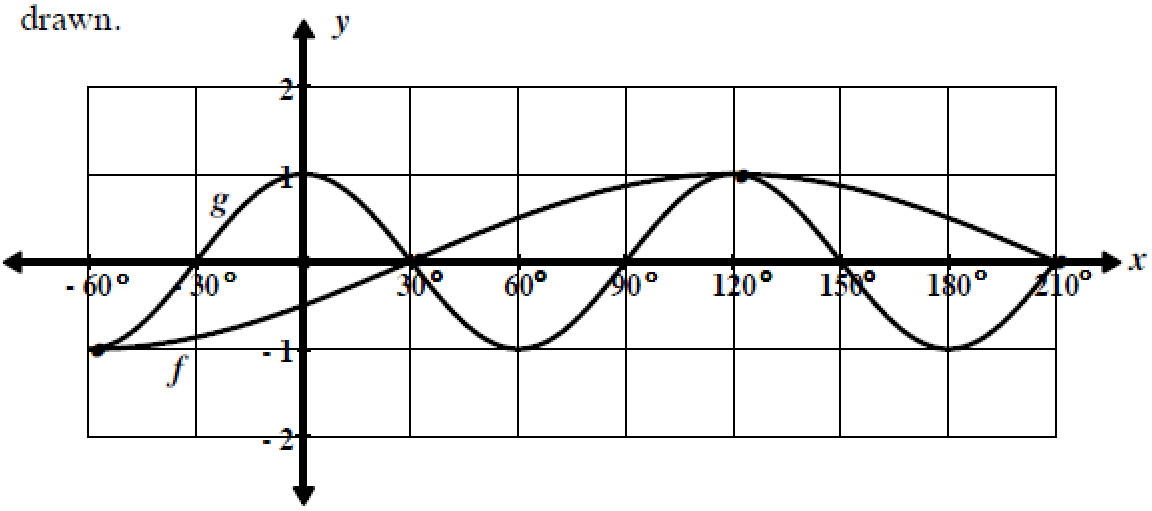
- Identify the type of graph in each sketch.
- Finding the defining equations for the given trigonometric function
- Transformation applied to trigonometric functions and their description in words.
- Graphically determine the values of x for which
  $$f(x) > 0;\ \ f(x) * g(x) < 0\ ;\ \ f(x) = 0; \textit{When is a function Increase and decrease}.$$

Steps of finding the equation of a given function are:

1. Draw the basic function on top of the sketched graphs with a different colour and observe the movements from the origin and the turning point.

Illustration

In the diagram below, the graphs of two trigonometric functions are drawn.



(a) Write down the equation for the graph of f.

(b) Write down the equation for the graph of g.

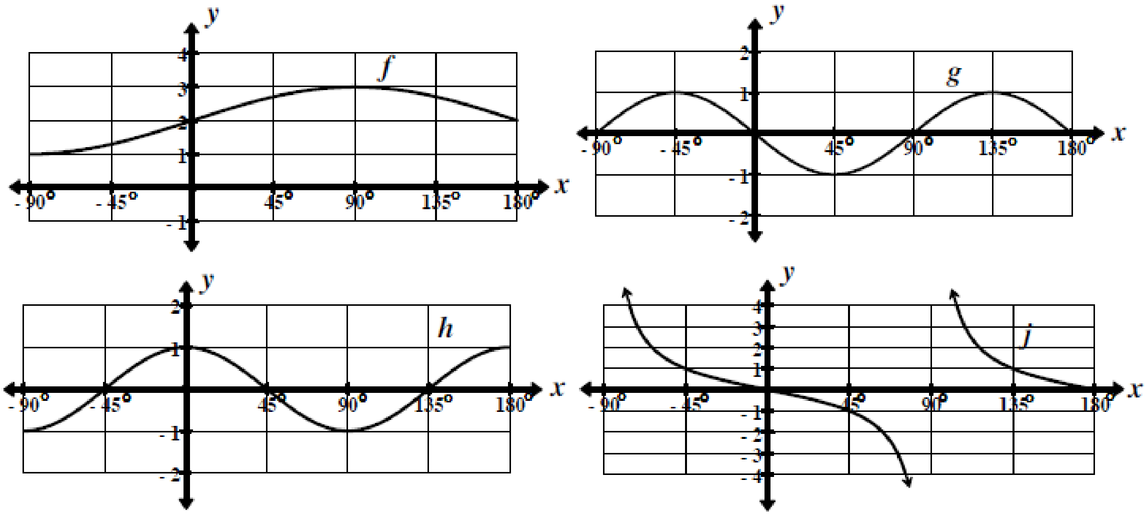
(c) Determine graphically the value(s) of x for which:

(1)	$f(x) = g(x)$	(2)	$f(x) < g(x)$
(3)	$f(x) > g(x)$	(4)	$f(x) \geq g(x)$
(5)	$f(x).g(x) \leq 0$	(6)	$f(x).g(x) \geq 0$
(7)	$f(x) = 0$	(8)	$f(x) \geq 0$
(9)	$g(x) = 0$	(10)	$g(x) < 0$

(d) Write down the period of the graph of $y = g\left(\frac{1}{6}x\right)$

Class Activity

The graphs of four different trigonometric functions are represented below.

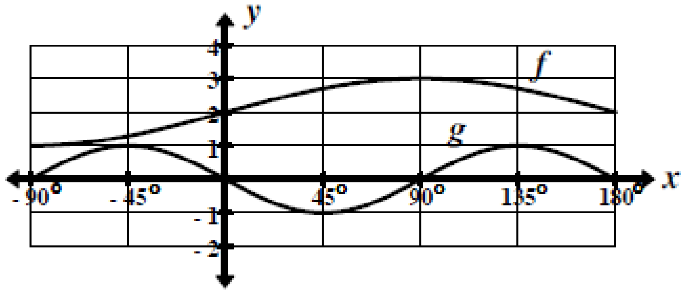


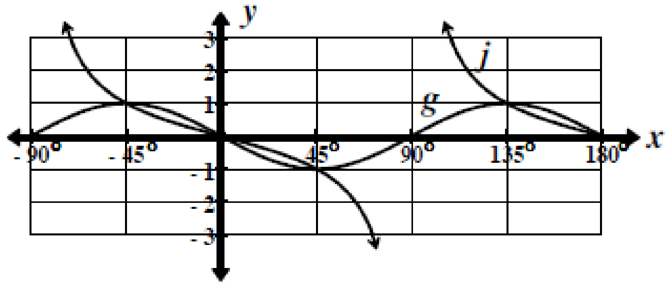
(a) Determine the equation of each graph.
(b) Write down the amplitude of graph f.
(c) Write down the period of graph f and g.
(d) Determine graphically the values of x for which:
   (1) the graph of h increases.
   (2) the graph of j decreases.

Determine graphically the values of x for which:

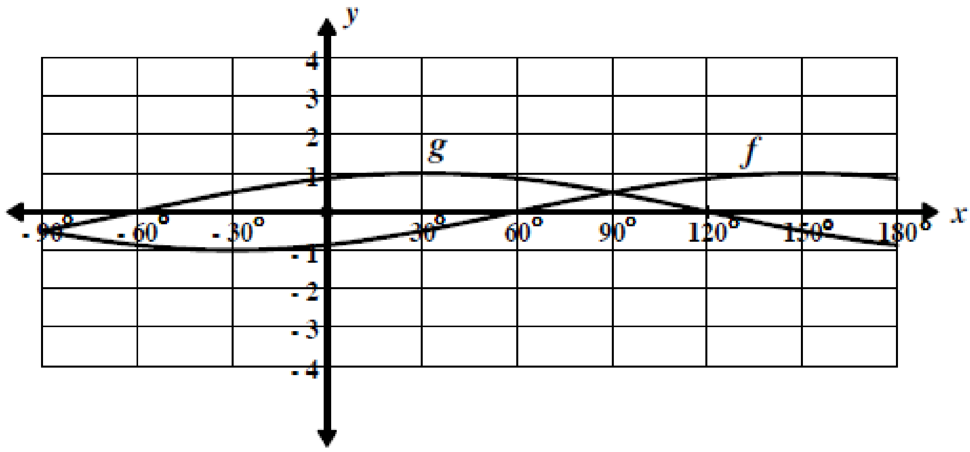
(a) $f(x) - g(x) = 2$

(b) $g(x) \geq j(x)$





Determine the equation of f and g



Appendix B: Interview Protocol

Interviewees: ______________________________

Date: ______________________________

Time: ______________________________

Good day, colleagues. I am going to ask you some questions about your experience in the LS process that you have undergone. All that will be said in this interview will be kept confidential and be used only for research purposes. Our conversation will be video recorded. You are warmly requested to speak clearly. You are free to withdraw from this session at any given time.

Lesson Study stage	**Question(s)**
2. Lesson planning	- Did you benefit from the lesson planning stage? Elaborate. -Did you face any challenges in the lesson-planning stage? Elaborate. - Suggest any improvements in this stage
3. Lesson presentation and observation	- Did you benefit from the lesson presentation and observation stage? Elaborate. -Did you meet any challenges in the lesson presentation and observation stage? Elaborate. -Suggest any improvements in this stage
4. Post lesson reflection	-Did you benefit from the post-lesson reflection stage? Elaborate. -Did you experience any challenges in the post-lesson reflection stage? Elaborate. - Suggest any improvements in this stage

Appendix D: Ethics-related documents

Enquiries: Lancelot Makandidze
Email: Lancelot.Makandidze@gauteng.gov.za

Dear District Director

REQUEST FOR PERMISSION TO CONDUCT RESEARCH STUDY

I am a PhD student at the University of Pretoria, and I am conducting a research study titled: **Teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study.** The purpose of the study is to explore teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study. This letter serves to request your permission to conduct research at … Secondary School where FET Phase learners and mathematics teachers will be involved.

This study involves observing teachers while they teach trigonometric functions to learners. I will be a passive participant who will do audio/video recordings and take field notes while the teacher and the learners are busy in class. I will observe three lessons. The lessons will be observed as they appear in the school timetable to avoid any disruption. Teachers will also be audio/video recorded during the collaborative lesson planning, lesson presentation and post-reflection stages of the Lesson Study cycle.

This research project will also involve unstructured interviews with mathematics teachers after school hours. The obtained information will be treated with the strictest confidentiality and used solely for this research.

I also would like to request your permission to use the data provided, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

For any additional information, you may contact me, Lancelot Makandidze, at 0633315506 or my supervisor, Dr RD Sekao at, 012 420 4640 or david.sekao@up.ac.za.

Yours Sincerely

____

Makandidze L.S.

Enquiries: Lancelot Makandidze
Email: Lancelot.Makandidze@gauteng.gov.za

Dear Principal

REQUEST FOR PERMISSION TO CONDUCT RESEARCH STUDY

I am a PhD student at the University of Pretoria, and I am conducting a research study titled: ***Teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study***. The purpose of the study is to explore teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study. This letter serves to request your permission to conduct the research in your school where FET Phase learners and mathematics teachers will be involved. The District Director has granted permission in this regard, and I have attached the letter of permission.

If permission is granted, the FET Phase mathematics teachers will be invited to participate in this study by:

a) being observed and audio/video recorded during the lesson study cycle (collaborative planning, lesson presentation, post-reflection stages).
b) availing their collaboratively prepared lesson plan for analysis.
c) being part of an interview session that will be audio/video recorded.

The learners will, after their parental consent is obtained, be invited to participate in this study by being observed during lesson presentations. In addition, note that *participation* of the teachers and learners is completely *voluntary* and if they agree to participate, I will ensure that the following ethical principles are adhered to:

- *Informed consent:* teachers' consent and learners' assent to participate will be based on their understanding of the purpose and process of the study as I would have explained them.
- *Safety in participation:* the teachers and learners will not be exposed to any risk or harm of any form because they will not be required to deviate from their day-to-day teaching and learning process.
- *Privacy*: The names and the data provided by both teachers and learners will be kept confidential and anonymous.
- *Trust*: teachers and learners will not be subjected to deception or betrayal in the research process or its published findings.

I also would like to request your permission to use the data provided, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

For any additional information, you may contact me, Lancelot Makandidze, at 0633315506 or my supervisor, Dr RD Sekao at, 012 420 4640 or david.sekao@up.ac.za.

Yours sincerely

Mr L.S. Makandidze (student)

Dr RD Sekao (Supervisor)

Dear Mr LS Makandidze

LETTER OF CONSENT TO CONDUCT THE RESEARCH STUDY

I,…………………………………….., principal of … Secondary School, voluntarily and willingly permit Mr L.S. Makandidze to conduct a research study titled: ***Teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study.*** I understand that the participation of both learners and the FET Phase teachers in the afore-mentioned study to which I am consenting, will involve:

a) teachers and learners being observed during the lesson presentation.
b) teachers availing the collaboratively prepared lesson plan for analysis.
c) teachers being part of the interview that will be audio/video recorded
d) teachers being observed and audio/video recorded during the lesson planning, lesson presentation and reflection stages.

I declare that I understand the purpose of the study and that you (the researcher) subscribe to the ethical research principles, including informed consent, safety, privacy (confidentiality and anonymity) and trust.

In addition, I grant the University of Pretoria permission to use the data provided for this study, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Given the above information, I permit you to conduct your study in our school.

____________________________	________________	______________
(**Name** and **surname**)	Signature	Date

Enquiries: Makandidze Lancelot

Email: Lancelot.Makandidze@gauteng.gov.za

Dear Teacher

REQUEST FOR PERMISSION TO CONDUCT RESEARCH STUDY

I am a PhD student at the University of Pretoria, and I am conducting a research study titled: ***Teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study***. The purpose of the study is to explore teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study. I, therefore, request you to participate in the aforementioned study.

If you agree to participate in my study, you will be requested to:

a) be observed and audio/video recorded during the Lesson Study cycle (planning, lesson presentation, reflection stages).
b) avail the collaboratively prepared lesson plan for analysis.
c) be part of an interview session that will be audio/video recorded.

The learners will be invited to participate in this study by being observed during the lesson presentation. In addition, note that *participation* of the teachers and learners is completely *voluntary* and if they agree to participate, I will ensure that the following ethical principles are adhered to:

- *Informed consent:* your consent to participate will be based on your understanding of the purpose and process of the study as I would have explained to them.
- *Safety in participation: you will not be exposed to any risk or harm of any form because you will not be required to deviate from your day-to-day teaching process.*
- *Privacy: your names and the data you provide will be kept confidential and anonymous.*
- *Trust*: you will not be subjected to deception or betrayal in the research process or its published findings.

I also would like to request your permission to use the data provided, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

For any additional information, you may contact me, Lancelot Makandidze, at 0633315506 or my supervisor, Dr RD Sekao at, 012 420 4640 or david.sekao@up.ac.za.

Yours sincerely

Mr L.S. Makandidze (student)

Dr R.D. Sekao (Supervisor)

Dear Mr Makandidze

LETTER OF CONSENT TO CONDUCT THE RESEARCH STUDY

I,…………………………………….., a teacher at ……….. Secondary School, voluntarily and willingly permit Mr L.S. Makandidze to conduct a research study titled: ***Teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study.*** I understand that my participation in the afore-mentioned study to which I am consenting, will involve:

a) being observed and audio/video recorded during the lesson planning, lesson presentation, and reflection stages.
b) being part of the interview that will be audio/video recorded.
c) availing the collaboratively prepared lesson plan for analysis.

I declare that I understand the purpose of the study and that you (the researcher) will subscribe to the ethical research principles, including informed consent, safety, privacy (confidentiality and anonymity) and trust.

In addition, I grant the University of Pretoria permission to use the data provided for this study, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Given the above information, I consent to participate in your study.

______________________	______________	____________
(**Name** and **surname**)	Signature	Date

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Enquiries: Lancelot Makandidze
Email: Lancelot.Makandidze@gauteng.gov.za

Dear Parent

REQUEST FOR YOUR CHILD'S PARTICIPATION IN THE RESEARCH STUDY

I am a PhD student at the University of Pretoria, and I am conducting a research study titled: ***Teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study.*** The purpose of the study is to observe how teachers teach trigonometric functions. I, therefore, request your permission to observe three mathematics lessons in your child's class.

Note that you are not compelled to grant permission, and your child is also not compelled to be observed, however, if you give me permission, which I will greatly appreciate, I will ensure that the following ethical principles are adhered to:

- *Informed consent:* I will provide any additional information you may need so that you clearly understand the purpose and process of the study I have explained herein.
- *Safety in participation:* your child will not be exposed to any risk or harm of any form because they will not be required to do anything outside their day-to-day teaching and learning activities taking place in the classroom.
- *Privacy*: your child's name and the information generated will be kept confidential and anonymous.
- *Trust*: your child will not be subjected to any act of deception or betrayal in the research process or its published findings.

I would also like to request your permission to use the data that your child will provide, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

For any additional information, you may contact me, Lancelot Makandidze, at 0633315506 or my supervisor, Dr RD Sekao at, 012 420 4640 or david.sekao@up.ac.za.

Yours sincerely

L.S. Makandidze (student) Dr RD Sekao (Supervisor)

Dear Mr L.S. Makandidze

LETTER OF CONSENT FOR MY CHILD TO PARTICIPATE IN THE RESEARCH STUDY

I, ……………..………………………………, parent of………….……..…………………………, voluntarily and willingly permit my child to participate in the research study titled: **Teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study.** I understand that the participation of my child in the afore-mentioned study to which I am granting permission, will involve being observed during the lessons taught by their teacher(s). I declare that I understand the purpose of the study and that you subscribe to the ethical research principles, including *informed consent, safety, privacy* and *trust.*

In addition, I grant the University of Pretoria permission to use the data provided for this study, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Given the above information, I give permission for my child's participation in the study.

_________________________ _______________ _____________

(**Name** and **surname**) Signature Date

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Enquiries: Lancelot Makandidze
Email: Lancelot.Makandidze@gauteng.gov.za

Dear learner

INVITATION TO PARTICIPATE IN THE RESEARCH STUDY

I am a PhD student at the University of Pretoria, and I am conducting a research study titled: **Teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study.** The purpose of my study is to understand how your teachers teach trigonometric functions. I, therefore, request your permission to observe the three mathematics lessons focusing on trigonometric functions.

Please note that you are not forced to grant permission to be observed; however, if you give me permission, which I will greatly appreciate, I will ensure that:

- you fully understand the purpose and process of the study as I would have explained them.
- you will not be exposed to any risk or harm of any form since I will observe the lessons during the time reflected on your timetable, therefore you will not be required to do anything outside of your day-to-day teaching and learning process.
- your names and the information you provide will be kept confidential and anonymous.
- you will not be subjected to any act of deception or betrayal in the research process or its published findings.
- you may choose to withdraw from being observed at any time without any consequences.

I would also like to ask for your permission to use the data you will provide, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

For any additional information, you may contact me, Lancelot Makandidze, at 0633315506 or my supervisor, Dr RD Sekao at, 012 420 4640 or david.sekao@up.ac.za.

Yours sincerely

____________________ ____________________
L.S. Makandidze (student) Dr RD Sekao (Supervisor)

Dear Mr L.S. Makandidze

LETTER OF ASSENT TO PARTICIPATE IN THE RESEARCH STUDY

I,…………………………..… a FET Phase learner at … Secondary School, voluntarily and willingly agree to participate in the study titled: **Teachers' development of mathematical knowledge for teaching trigonometric functions through Lesson Study.** I understand that as part of the study to which I consent, I will be observed during the lessons.

I declare that I understand the purpose of the study and that you subscribe to the ethical research principles, including informed consent, safety, privacy (confidentiality and anonymity) and trust as you explained to me.

In addition, I grant the University of Pretoria permission to use the data provided for this study, confidentially and anonymously, for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and use the data for teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Given the above information, I agree to voluntarily participate in the study.

______________________________	____________________	________________
(**Name** and **surname**)	Signature	Date