(15) can be stabilized by the proposed hybrid output feedback control scheme for a wide range of variations in the detecting time τ.

IV. Concluding Remarks

In this note, a new hybrid output feedback control scheme was proposed to stabilize a class of continuous-time LTI systems with single output. The arguments were based on the multirate sampling technique and the Multiple-Lyapunov-Function theorem. While this note focused only on the single output case, the proposed design procedure could be extended to the case of multi-output (i.e., $p > 1$) without essential changes. In addition, the multirate sampling scheme can be extended via detecting the output $y(t)$ more than once within a sampling period T_s, e.g., over a sequence of detecting time $0 < \tau_1 < \tau_2 < \cdots < \tau_k < T_s$. Then, with more information on $y(t)$, it becomes possible to further partition the state space and design more multiple-output feedback gains correspondingly, and hence improve the chance to stabilize the system. A natural question is how generic the method could be. We ask whether it is always possible to find a pair of sampling period T_s and detecting time τ (or a sequence of detecting time $0 < \tau_1 < \tau_2 < \cdots < \tau_k < T_s$) such that the system (assumed to be reachable and observable) can be stabilized by the proposed multiple SOF controller scheme. If not, what conditions the state matrices $\{A, B, C, D\}$ should satisfy?

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Adaptive Synchronization for Generalized Lorenz Systems

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***Abstract*—In literature it is conjectured that the states of the generalized Lorenz system with an unknown parameter can not be estimated by adaptive observers. In this paper we show that this unknown parameter and the states can actually be estimated simultaneously by some kind of adaptive observer. The proof is obtained by constructing some exponential observer to achieve chaotic synchronization for the generalized Lorenz system. The result implies that more work needs to be done to apply generalized Lorenz system in secure communication.**

***Index Terms*—Adaptive observer, chaotic synchronization, persistently exciting.**

I. Introduction

Chaotic synchronization has drawn much attention since the celebrated work [11] of Pecora and Carrol was published in 1990. It is motivated not only by scientific interest, but also by potential applications of

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chaotic synchronization in different fields, particularly in secure communication. Several chaos-based communication methods have been proposed, such as chaotic masking, chaotic modulation and chaos shift keying (see [5], [6], and [7]). However, many proposed schemes have a low degree of security ([12], [15], [18]). Some parameters of the chaotic transmitter system are used as the password. From the viewpoint of control theory, many robust and adaptive techniques can "decrypt" the parameter. To solve this problem, [3] suggests a new class of chaotic system, which is state equivalent to the generalized Lorenz system, first introduced in [4] and [17], through a change of coordinates ([2]). It is shown in [3] that a class of adaptive observers, which is widely used before, cannot be used to estimate the unknown parameter. Based on this fact, a conjecture is provided: generalized Lorenz system allows secure synchronization.

In this paper, we aim at finding more properties of the transformed generalized Lorenz system with an unknown parameter introduced in [3], and thus showing that its states and unknown parameter can actually be estimated by constructing a different kind of adaptive observer. In other words, we will use the kind of adaptive observer introduced in [19] to achieve synchronization. Reference [19] shows that this kind of observer is exponential if some function satisfies certain persistently exciting (PE) condition. To apply this result for the transformed generalized Lorenz system (8) in this paper, we first prove the output, which is defined as its first sate variable $\eta_1(t)$, to be PE (see Lemma 4). Then we further show that another function, $\Upsilon_1(t)$ in (21), is also PE. It is noted that such a proof for the PE property of the output $\eta_1(t)$ is still lacking in literature although [3] mentions that it may hold due to the transitivity property. In order to prove the PE property of $\eta_1(t)$, we need to consider the dynamics of the transformed generalized Lorenz system (8). By some analytic techniques, some general properties of the trajectories of (8) are obtained. After the above preparation, an exponential observer is successfully constructed to estimate the states and unknown parameter of the transformed generalized Lorenz system. The numerical results also convince that the unknown parameter can be estimated exactly. Thus the unknown parameter τ can not be a password, and more efforts need to be done besides the designing of an unknown parameter ([3]) in order to improve security. The idea in this paper is applicable to some other smooth chaotic systems too ([9], [16]), that is, if some PE property can be proved, then we can construct similarly an adaptive observer to deal with the robust synchronization problem.

The layout of the paper is as follows. In Section II, we recall some basics on the adaptive observer defined in [19]. In Section III, we give some properties of the trajectories of the transformed generalized Lorenz system. The adaptive observer for the transformed generalized Lorenz system with unknown parameter is given in Section IV, and it is shown to be an exponential observer by the results of Section III. An example in Section V shows the efficiency of our proposed observer. The last section is the conclusion.

II. Some Facts About Adaptive Observers

Definition 1: ([14]) A vector function $w : R \to R^{2n}$ is *persistently exciting (PE)* if there exist $\alpha_1, \alpha_2, T > 0$ such that

$$\alpha_1 I \leq \int_t^{t+T} w(s)w^T(s)ds \leq \alpha_2 I, \quad \forall\, t \geq 0. \tag{1}$$

Consider the following system:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + \Psi(t)\theta \\ y(t) = C(t)x(t) \end{cases} \tag{2}$$

where $x(t) \in R^n$, $y(t) \in R^m$, $u(t) \in R^l$ are the state, output, and input vectors, respectively, $A(t), B(t), C(t), \Psi(t)$ are known matrices of appropriate dimensions and continuous in time, and $\theta \in R^p$ is an unknown constant vector. The following conditions were introduced in [19] for (2).

Condition 1. There exists a bounded time-varying matrix $K(t) \in R^n \times R^m$ so that the system $\dot{\tau}(t) = [A(t) - K(t)C(t)]\tau(t)$ is exponentially stable.

Condition 2. The solution $\Upsilon(t) \in R^n \times R^p$ of $\dot{\Upsilon}(t) = [A(t) - K(t)C(t)]\Upsilon(t) + \Psi(t)$ is persistently exciting in the sense that there exist $\alpha_1, \beta_1, T_1 > 0$ such that

$$\alpha_1 I \leq \int_t^{t+T_1} \Upsilon^T(s)C^T(s)\Sigma(s)C(s)\Upsilon(s)ds \leq \beta_1 I$$
$$\forall t \geq t_0$$

for some $t_0 \geq 0$ and some bounded positively definite matrix $\Sigma(t) \in R^m \times R^m$.

Reference [19] shows that if Conditions 1 and 2 hold, then the following adaptive observer is a global exponential observer for system (2) [see (3), as shown at the bottom of the page], where $\Gamma \in R^p \times R^p$ is any symmetric positive definite matrix.

Reference [1] tells that the above result can be applied to a class of nonlinear system

$$\begin{cases} \dot{x}(t) = A(u(t), y(t))x(t) + \varphi(u(t), y(t)) + \Phi(u(t), y(t))\theta \\ y(t) = Cx(t) \end{cases} \tag{4}$$

where θ is an unknown constant or slow time-varying vectors, and the components of $A(u(t), y(t)), \varphi(u(t), y(t))$ and $\Phi(u(t), y(t))$ are continuous functions depending on u and y, and uniformly bounded.

III. The Complicated Behavior of the Generalized Lorenz Systems

The following generalized Lorenz system is defined in [3]:

$$\dot{x} = \begin{bmatrix} A & 0 \\ 0 & \lambda_3 \end{bmatrix} x + \begin{bmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{5}$$

where $x = [\, x_1 \quad x_2 \quad x_3 \,]^T$, $\lambda_3 \in R$, and A has eigenvalues λ_1, $\lambda_2 \in R$ such that

$$-\lambda_2 > \lambda_1 > -\lambda_3 > 0. \tag{6}$$

$$\begin{cases} \dot{\hat{x}} = [A(t) - K(t)C(t)]\hat{x}(t) + B(t)u(t) + K(t)y(t) + \Psi(t)\hat{\theta} + \Upsilon(t)\dot{\hat{\theta}}(t), \\ \dot{\hat{\theta}}(t) = \Gamma\Upsilon^T(t)C^T(t)\Sigma(t)[y(t) - C(t)\hat{x}(t)], \\ \dot{\Upsilon}(t) = [A(t) - K(t)C(t)]\Upsilon(t) + \Psi(t) \end{cases} \tag{3}$$

Moreover, the generalized Lorenz system is said to be *nontrivial* if it has at least one solution that goes neither to zero nor to infinity nor to a limit cycle.

Reference [2] shows that there exists a nonlinear change of coordinates, $z = Tx$, which transforms (5) into the generalized Lorenz canonical form

$$\dot{z} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} z + cz \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & \tau & 0 \end{bmatrix} z \quad (7)$$

where $z = [z_1 \quad z_2 \quad z_3]^T$, $c = [1 \quad -1 \quad 0]$ and the parameter $\tau \in (-1, \infty)$. System (7) is state equivalent to the following form (see [3]):

$$\frac{d\eta}{dt} = \begin{bmatrix} (\lambda_1 + \lambda_2)\eta_1 + \eta_2 \\ -\lambda_1\lambda_2\eta_1 - (\lambda_1 - \lambda_2)\eta_1\eta_3 - \frac{1}{2}(\tau+1)\eta_1^3 \\ \lambda_3\eta_3 + K_1(\tau)\eta_1^2 \end{bmatrix} \quad (8)$$

where $\eta = [\eta_1 \quad \eta_2 \quad \eta_3]^T$ and $K_1(\tau) = (\lambda_3(\tau+1) - 2\tau\lambda_1 - 2\lambda_2)/(2(\lambda_1 - \lambda_2))$. The corresponding coordinate change and its inverse are [3]

$$\eta^T = \left[z_1 - z_2 \quad \lambda_1 z_2 - \lambda_2 z_1 \quad z_3 - \frac{(\tau+1)(z_1-z_2)^2}{2(\lambda_1-\lambda_2)} \right] \quad (9)$$

$$z^T = \left[\frac{\lambda_1\eta_1+\eta_2}{\lambda_1-\lambda_2} \quad \frac{\lambda_2\eta_1+\eta_2}{\lambda_1-\lambda_2} \quad \eta_3 + \frac{(\tau+1)\eta_1^2}{2(\lambda_1-\lambda_2)} \right]. \quad (10)$$

From the above transformations and (7) and (8), we have an equivalent system

$$\begin{cases} \dot{\eta}_1 = \lambda_1\eta_1 + (\lambda_1 - \lambda_2)z_2, \\ \dot{z}_2 = \lambda_2 z_2 - \eta_1 z_3 \\ \dot{z}_3 = \lambda_3 z_3 + \eta_1^2 + (1+\tau)\eta_1 z_2. \end{cases} \quad (11)$$

The following assumption is needed in later text.

Assumption 1: The states of system (8) and their time derivatives are continuous and bounded.

Remark 1: The proofs of the boundness of Lorenz type systems are reported in [8] and [20]. As for some specific type of chaotic systems, the corresponding proof is given only for some special parameter region ([21]). Therefore, the above boundness hypotheses in Assumption 1 are reasonable. It is also helpful to note that, under Assumption 1, $\eta_1(t)$ is uniformly continuous by applying the Mean Value Theorem.

For the parameter τ, [2] shows that we need to consider the region $\tau < -\lambda_2/\lambda_1$ since (6) must be met. Therefore, we assume $\tau < -\lambda_2/\lambda_1$ from now on.

System (11) has three equilibria $O_0(0, 0, 0)$ and the equation shown at the bottom of the page. Obviously, O_0 is unstable. The characteristic polynomial for $O_{1,2}$ is

$$\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + \frac{\lambda_3\left(\lambda_2^2 + \tau\lambda_1^2\right)}{\lambda_2 + \lambda_1\tau}\lambda + 2\lambda_1\lambda_2\lambda_3 = 0.$$

It is possible to make O_1 and O_2 both stable or unstable; for example, they are stable when $\tau < \tau_0$, while unstable when $\tau > \tau_0$, where $\tau_0 = -\lambda_2^2/\lambda_1^2(\lambda_1 + \lambda_2 + \lambda_3) + 2\lambda_1/(\lambda_1 + \lambda_2 + \lambda_3) + 2\lambda_1$. Therefore, the following assumption is made.

Assumption 2: System (8) has three unstable equilibria.

Suppose system (8) is chaotic, then it satisfies the following obvious properties which will be used in the proofs of some lemmas:

- at least one solution of the system does not go to zero, or to infinity, or to a limit cycle;
- for any finite $T < \infty$, it is impossible that the derivatives of any state variable of system (8) keeps its signs, i.e., neither $\dot{\eta}_i(t) > 0$ for $t \geq T$ nor $\dot{\eta}_i(t) < 0$, $i = 1, 2, 3$ (see [20] and [21]);
- the states $\eta_i(t)$ do not always be zero on any interval (α, β), that is, $\eta_i(t) \not\equiv 0$ on any (α, β), $i = 1, 2, 3$ ([20] and [21]).

Lemma 1: For system (8), there exists a time t_1 such that $\eta_3(t_1) > 0(< 0)$ for $t \geq t_1$ if $K_1(\tau) > 0(< 0)$.

Proof: Conversely, for any $t_1 > 0$, there exists $t' > t_1$ such that $\eta_3(t') < 0(> 0)$ and $\dot{\eta}_3(t') = 0$ if $K_1(\tau) > 0(< 0)$. Now $\lambda_3\eta_3(t') > 0(< 0)$ which contradicts with the fact that $\lambda_3\eta_3(t') = -K_1(\tau)\eta_1^2(t') < 0(> 0)$. This ends the proof. □

The following Lemma 2 follows in a similar way as Lemma 4 of [20] or Lemma 1 of [21].

Lemma 2: Assume $\dot{\eta}_1(t) \not\equiv 0$ for $t \in (-\infty + \infty)$. If there exists β such that $\dot{\eta}_1(\beta) = \ddot{\eta}_1(\beta) = 0$, then $t = \beta$ is not an extreme value point of $\eta_1(t)$.

Let

$$\dot{F} = -aF + aF^2 - be^{-dt}, \text{ where } a < 0, \quad b > 0, \quad d > 0 \quad (12)$$

and the initial value $F(0) \in (1/2\ 1)$ and $\dot{F}(0) < 0$. Its solution is ([13])

$$F(t) = -e^{-1/2dt}\frac{\sqrt{-ab}}{a}\frac{J_{v+1}(x) + C_1 Y_{v+1}(x)}{J_v(x) + C_1 Y_v(x)}$$
$$\text{where } v = -\frac{a}{d},\ x = 2\frac{\sqrt{-ab}e^{-1/2dt}}{d}. \quad (13)$$

C_1 is determined by $F(0)$, $J_v(x)$ and $Y_v(x)$ are the first and second kind of Bessel function, respectively, and are defined by the formulas

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k}}{k!\Gamma(v+k+1)}$$
$$Y_v(x) = \frac{J_v(x)\cos\pi v - J_{-v}(x)}{\sin\pi v} \quad (14)$$

with $\Gamma(x)$ being the Gamma function. The formula for $Y_v(x)$ is valid for any non-integer v. For a nonnegative integer n

$$Y_n(x) = \frac{2}{\pi}J_n(x)\lg\frac{x}{2} - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{2}{x}\right)^{n-2k}$$
$$-\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\left(\frac{2}{x}\right)^{n+2k}\frac{\psi(k+1) + \psi(n+k+1)}{k!(n+k)!} \quad (15)$$

where $\psi(1) = -\mathcal{C}$, $\psi(n) = -\mathcal{C} + \sum_{k=1}^{n-1} k^{-1}$, $-\mathcal{C}$ is the Euler constant.

Lemma 3: Suppose $F(t) > 0$ for $t \geq 0$ and b is bounded in (12), then there exists $t_1 > 0$ independent of b such that $F(t_1) = F(0)$.

Proof: It is easy to know that $F(t) < 1$ for all $t > 0$. In fact, let $t_1 \in (0 + \infty)$ be the first point such that $F(t_1) = 1$, then $\dot{F}(t_1) < 0$. This is impossible since $F(0) < 1$.

$$O_{1,2}\left(\mp\sqrt{\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2 + \lambda_1\tau)}}, \pm\frac{\lambda_1}{\lambda_1 - \lambda_2}\sqrt{\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2 + \lambda_1\tau)}}, \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\right)$$

Let $x = 2\sqrt{-ab}e^{-1/2dt}/d$, then x tends to zero when t is sufficiently large. Now $J_v(x)$ tends to zero and $J_{-v}(x)$ tends to infinite since $v > 0$. If v is not an integer

$$\begin{aligned}\lim_{t\to\infty} F(t) &= \lim_{t\to\infty} -e^{-1/2dt}\frac{\sqrt{-ab}(\frac{x}{2})^{-(v+1)}}{a\sin(\pi(v+1))\Gamma(-v)} \\ &\quad\times \frac{\sin(\pi v)\Gamma(-v+1)}{(\frac{x}{2})^{-v}} \\ &= \lim_{t\to\infty} -e^{-1/2dt}\frac{\sqrt{-ab}}{a}\left(-\frac{a}{d}\right) \\ &\quad\times \frac{2d}{2\sqrt{-ab}e^{-1/2dt}} = 1.\end{aligned}$$

Now the result follows from the fact that $F(0) \in (1/2\ 1)$. The case that v is an integer follows from a similar proof. □

Theorem 1: Suppose system (8) is chaotic, then there exists a finite time Δt so that $\eta_1(t)$ has at least one extremum in the interval $(t_0,\ t_0 + \Delta t)$ for any $t_0 \geq 0$.

Proof: Conversely, for any increasing sequence $\{\Delta t_i\}_1^\infty$ with $\lim_{i\to\infty}\Delta t_i = +\infty$, there exists a sequence $\{t_i\}_1^\infty$ such that η_1 has no extreme on (t_i, T_i), where $T_i := t_i + \Delta t_i$. Note that $\eta_1(t)$ is monotonic on $[t_i,\ T_i]$, then without loss of generality we can suppose $\{t_i\}_1^\infty$ is increasing, $\lim_{i\to\infty} t_i = +\infty$, $\eta_1(t_i)$ is a minimum, and $\eta_1(T_i)$ a maximum. Since system (8) and system (11) are state equivalent, we consider the latter for convenience. Now there are the following two cases.

1) Case I: $\eta_1(t_i) - \eta_1(T_i)$ tends to zero when $i \to \infty$.

Since $\eta_1(t_i) - \eta_1(T_i)$ tends to zero when $i \to \infty$, we can suppose $\eta_1(T_i) - \eta_1(t_i) < \epsilon_i$, where ϵ_i is positive and tends to zero when $i \to \infty$. By Assumption 1 we know that $\dot{\eta}_1(t) < M_1\epsilon_i$ holds for all $t \in (t_i\ T_i)$, where M_1 is a positive constant. Then $|z_2(T_i) - z_2(t_i)| < M_3\epsilon_i$ by (11). Similarly, we have that both $\dot{z}_2(t)$ and $\dot{z}_3(t)$ tend to zero for $t \in (t_i\ T_i)$ when i tends to infinity. Thus, $(\eta_1(T_i), z_2(T_i), z_3(T_i))$ tends to one of the three equilibria. In the following, we consider only the case that $(\eta_1(T_i), z_2(T_i), z_3(T_i))$ tends to O_0. For the other cases, it can be proved in a similar way after a coordinate change $(\eta_1, z_2, z_3) - O_{1,2}$. Now we have the following three subcases.

I.I) If $\eta_1(T_i) = 0$, then $\dot{\eta}_2(T_i) = 0$. Hence $\ddot{\eta}_1(T_i) = 0$, and it contradicts with Lemma 2 since $\eta_1(T_i)$ is a maximum.

I.II) If $\eta_1(T_i) > 0$, then $\dot{\eta}_1(T_i) = 0$ and $\ddot{\eta}_1(T_i) = (\lambda_1 - \lambda_2)\dot{z}_2(T_i) < 0$. Thus, $\dot{z}_2(T_i) < 0$ for $\lambda_1 - \lambda_2 > 0$. However, by (11) we know that $z_3(T_i)$ becomes sufficiently small when Δt_i becomes sufficiently large, therefore $z_2(T_i) = -\lambda_1\eta_1(T_i)/(\lambda_1 - \lambda_2)$ and $\dot{z}_2(T_i) = -(z_3 + \lambda_1\lambda_2/(\lambda_1 - \lambda_2))\eta_1(T_i) > 0$. This is a contradiction.

I.III) If $\eta_1(T_i) < 0$ and $K_1(\tau) < 0$, then it follows from Lemma 1 that $\eta_3(t_i) < 0$ for sufficiently large i. By (8) we know $\dot{\eta}_2(t_i) < \eta_1(-\lambda_1\lambda_2 - (\lambda_1 - \lambda_2)\eta_3 - 1/2(\tau + 1)\eta_1^2) < 0$. However, $\ddot{\eta}_1(t_i) = \dot{\eta}_2(t_i) > 0$ since $\eta_1(t_i)$ is a minimum. It is a contradiction.

I.IV) If $\eta_1(T_i) < 0$ and $K_1(\tau) = 0$, then $\lim_{t\to\infty}\eta_3(t) = 0$, which is impossible since the system is chaotic.

I.V) If $\eta_1(T_i) < 0$ and $K_1(\tau) > 0$, it is obvious that $\eta_1(t_i) < 0$ and $\eta_3(t_i) > 0$. Let $f = z_2/\eta_1$, then

$$\dot{f}(t) = af(t) + af^2(t) - z_3(t), \quad f(t_i) = \frac{\lambda_1}{\lambda_2 - \lambda_1} \tag{16}$$

where $a = \lambda_2 - \lambda_1 < 0$. It is easy to obtain $z_2(t_i) > 0$ and $\dot{z}_2(t_i) > 0$ since $\ddot{\eta}_1(t_i) > 0$ and $\dot{\eta}_1(t_i) = 0$. Thus, $\dot{f}(t_i) < 0$. If $\dot{\eta}_1(t) = 0$ for some point t, then $f(t) = \lambda_1/(\lambda_2 - \lambda_1)$, that is, $f(t) = f(t_i)$. If we can prove that there exists an integer N so that the function $y = f(t)$ travels through the line $y = \lambda_1/(\lambda_2 - \lambda_1)$ in the $t - y$ plane for every $i > N$ when $t \in (t_i\ T_i)$, then $\eta_1(t)$ reaches the maximum before $t = T_i$. Now we prove this in the following.

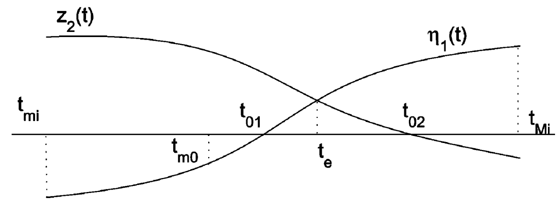

Fig. 1. Illustration for $\eta_1(t)$ and $z_2(t)$ in Case II.

Let $f = -1 + F$, then

$$\dot{F}(t) = -aF(t) + aF^2(t) - z_3(t), \quad F(t_i) = \frac{\lambda_2}{\lambda_2 - \lambda_1}. \tag{17}$$

From (8) and transformation (10), we know that $z_3(t) = \eta_3(t_i)e^{\lambda_3(t-t_i)} + \epsilon(t)$ holds for $t \in (t_i\ T_i)$, where

$$\epsilon(t) = \frac{(\tau+1)}{2(\lambda_1 - \lambda_2)}\eta_1^2(t) + K_1(\tau)e^{\lambda_3(t-t_i)} \times \int_{t_i}^{t} e^{-\lambda_3(s-t_i)}\eta_1^2(s)ds > 0.$$

We have the following equations for $F(t)$ and another function $F_1(t)$

$$\begin{aligned}\dot{F} &= -aF(t) + aF^2(t) - \eta_3(t_i)e^{\lambda_3(t-t_i)} - \epsilon(t) \\ \dot{F}_1 &= -aF_1(t) + aF_1^2(t) - \eta_3(t_i)e^{\lambda_3(t-t_i)}.\end{aligned} \tag{18}$$

Let the two equations have the same initial values, that is, $F_1(t_i) = F(t_i) = \lambda_2/(\lambda_2 - \lambda_1)$, then it follows from $\dot{F}(t) < \dot{F}_1(t)$ that $0 < F(t) < F_1(t)$. By (18), we have

$$\dot{F} - \dot{F}_1 \geq -a(F(t) - F_1(t)) - \epsilon(t). \tag{19}$$

Thus $0 > F(t) - F_1(t) > -e^{-a(t-t_i)}\int_{t_i}^{t} e^{as}\epsilon(s)ds$, for $t \in (t_i\ T_i)$. Then it follows from Lemma 3 that there exists a time $t_{i1} \in (t_i\ T_i)$ independent of $\eta_3(t_i)$ such that $F_1(t_{i1}) = F_1(t_i)$. In a similar way, we can prove that there exists a time $t_{i3} \in (t_{i1}\ T_i)$ independent of $\eta_3(t_i)$ such that $F(t_{i3}) = 1/2 + F(t_i)/2 \in (F(t_i)\ 1)$. Since $\epsilon(t)$ is sufficiently small, there exists a time $t_{i2} \in (t_{i1}\ t_{i3})$ such that $F(t_{i2}) = F(t_i)$, that is, there exists a time $t_{i2} < T_i$ for every $i > N$ such that $\eta_1(t_{i2})$ reaches its maximum, where N is a sufficiently large number. This contradicts the hypothesis that $\eta_1(t)$ is monotonic for $t \in (t_i\ T_i)$.

By the above four subcases, we conclude that Case I does not happen. Therefore, we consider the second case.

2) Case II: $\eta_1(t_i) - \eta_1(T_i)$ does not tend to zero when $i \to \infty$.

Since Δt_i tends to infinity, we choose $\Delta t_i \geq 2^{2i}$. Let $\eta_1(t_{m1}) = 1/2(\eta_1(T_i) - \eta_1(t_i))$, then either $t_{m1} - t_i$ or $T_i - t_{m1}$ is greater than 2^{2i-1}. Without loss of generality, let $T_i - t_{m1} \geq 2^{2i-1}$. Then there exists a time $t_{m2} \in (t_{m1}\ T_i)$ such that $\eta_1(t_{m2}) = (\eta_1(T_i) - \eta_1(t_{m1}))/2$. It is obvious that either $t_{m2} - t_{m1}$ or $T_i - t_{m2}$ is greater than 2^{2i-2}. After repeating the above process for i times, we obtain two times t_{mi} and t_{Mi} such that $\eta_1(t_{Mi}) - \eta_1(t_{mi}) < 1/2^i(\eta_1(T_i) - \eta_1(t_i))$ and $t_{Mi} - t_{mi} \geq 2^i$ (see Fig. 1 for illustration.)

Following the same way in Case I, $\eta_1(t_{Mi})$ and $\eta_1(t_{mi})$ tend to one of the three equilibria. For the same reason as Case I, we only consider the equilibrium O_0. From subcase I.I) we know that $\eta_1(T_i) > 0$. Thus we can suppose that $-\epsilon_i = \eta_1(t_{mi}) < 0 < \eta_1(t_{Mi}) = \epsilon_i$ and $\lim_{i\to\infty}(t_{Mi} - t_{mi}) = +\infty$.

Since $z_2(T_i) < 0$, there exists a time t_{02} such that $z_2(t_{02})$ reaches 0 for the first time. Firstly we prove that $z_2(t)$ is decreasing on $(t_{mi},\ \min(t_{Mi}, t_{02}))$. Since $\eta_1(t) < \epsilon_i$ for $t \in (t_{mi}\ t_{Mi})$, we can assume, without loss of generality, that $|\eta_3(t_{mi})| < -\lambda_1\lambda_2/(\lambda_1 - \lambda_2)$.

Let t_{01} be the time at which $\eta_1(t_{01}) = 0$. Then by $\dot{\eta}_1(t_{01}) > 0$ we have $\eta_2(t_{01}) > 0$. It follows obviously from (8) that $\dot{\eta}_2(t) < 0$ on $(t_{mi}\ t_{01})$ and $\dot{\eta}_2(t) > 0$ on $(t_{01}\ t_{Mi})$. Now by (10) we know that $\dot{z}_2 = (\lambda_2\dot{\eta}_1 + \dot{\eta}_2)/(\lambda_1 - \lambda_2) < 0$ for $t \in (t_{mi}, t_{01})$, and $\dot{z}_2 < \lambda_2(\lambda_2\eta_1 + \eta_2)/(\lambda_1 - \lambda_2) < 0$ for $t \in (t_{01}, \min(t_{Mi}, t_{02}))$ for $z_2(t) = (\lambda_2\eta_1 + \eta_2)/(\lambda_1 - \lambda_2) > 0$.

Since $z_2(T_i) < 0$, there exists a time t_e so that $\eta_1(t_e)$ is positive and reaches $z_2(t_e)$ for the first time. Let $\delta = \eta_1(t_e)$, then we claim that t_e must be less than t_{Mi}. In fact, if $z_2(t) > \epsilon_i$ on $(t_{mi}\ t_{Mi})$, then $\dot{\eta}_1(t) = \lambda_1\eta_1 + (\lambda_1 - \lambda_2)z_2 > -\lambda_2\epsilon_i$, thus $\eta_1(t_{Mi}) > \epsilon_i$ which contradicts $\eta_1(t_{Mi}) = \epsilon_i$.

Let $g = \eta_1/z_2$, after a simple computation we have the following formula from (11) for $t \in (t_{mi}\ t_{01})$

$$\dot{g} = -a - ag + g^2 z_3, \quad g(t_{01}) = 0. \tag{20}$$

If $t_{01} - t_{mi}$ tends to infinity with i, then $z_3((t_{01} + t_{mi})/2)$ and $\eta_3((t_{01} + t_{mi})/2)$ are sufficiently small on $(t_{mi}\ t_{01})$. Since $g(t) < 0$ for $t \in (t_{mi}\ t_{01})$ and $g(t_{01}) = 0$, there exists a time $t_{m0} \in ((t_{01} + t_{mi})/2\ t_{01})$ such that $g(t_{m0}) = -1/2$, that is, $\eta_1(t_{m0})/z_2(t_{m0}) = -1/2$. Hence, by (11), $\dot{\eta}_1(t_{m0}) = (2\lambda_2 - \lambda_1)\eta_1$ and $\dot{z}_2(t_{m0}) = (2\lambda_2 - z_3)\eta_1$. By coordinate change (10), we know $\dot{\eta}_2(t_{m0}) = (-4\lambda_2^2 + 3\lambda_1\lambda_2 - z_3(\lambda_1 - \lambda_2))\eta_1 > 0$,; however, (8) gives $\dot{\eta}_2(t_{m0}) = (-\lambda_1\lambda_2 - (\lambda_1 - \lambda_2)\eta_3 - 1/2(1+\tau)\eta_1^2)\eta_1 < 0$, which is a contradiction. Hence $\{t_{01} - t_{mi} : i = 1, 2, \ldots\}$ is bounded.

Since $t_{01} - t_{mi}$ is a finite time independent of i and $t_{Mi} - t_{mi}$ tends to infinity, $t_{Mi} - t_{01}$ tends to infinity too. By the same reason that we assume $|\eta_3(t_{mi})|$ is bounded, we can also assume $|z_3(t_{mi})| < -2\lambda_1\lambda_2/(\lambda_1 - \lambda_2)$. From (16) we know that $f(t)$ becomes small enough after a long time. Hence, $\dot{\eta}_1(t) > 2\lambda_1/\eta_1(t)$ for $t \in ((t_{mi} + t_{Mi})/2, t_{Mi})$. Then it follows from $\dot{\eta}_1(t_{mi}) > 0$ that $z_2(t_{mi}) > \lambda_1\epsilon_i/(\lambda_1 - \lambda_2)$. Now we can obtain $|\eta_1(t)| < (\lambda_1 - \lambda_2)z_2/\lambda_1$ for $t \in (t_{mi}, t_{01})$ because $\dot{\eta}_1(t) = \lambda_1\eta_1(t) + (\lambda_1 - \lambda_2)z_2(t) > 0$. From (11), we know that

$$\begin{aligned}\dot{z}_2(t) &= \lambda_2 z_2(t) - \eta_1(t)z_3(t) > \lambda_2 z_2(t) + \frac{\lambda_2}{2}z_2(t)\\ &= \frac{3}{2}\lambda_2 z_2(t).\end{aligned}$$

Thus, $z_2(t_{01}) > \exp(3/2\lambda_2(t_{01} - t_{mi}))z_2(t_{mi})$. Similarly we have $\delta = z_2(t_e) > e^{3/2\lambda_2(t_e - t_{01})}z_2(t_{01}) - \delta M$, where M is a positive constant. Then $\delta > 1/1 + Me^{3/2\lambda_2(t_e - t_{01})}z_2(t_{01}) > 1/1 + M\lambda_1/\lambda_1 - \lambda_2 e^{3/2\lambda_2(t_e - t_{mi})}z_2(t_{01})\epsilon_i$. Since $\dot{\eta}_1 > \lambda_1/2\eta_1$, we have $\eta_1(t_{Mi}) > e^{\lambda_1/2(t_{Mi} - t_{mi})/2}\eta_1(t_{mi} + (t_{Mi} - t_{mi})/2) > e^{\lambda_1(t_{Mi} - t_{mi})/4}\delta > \epsilon_i$, which contradicts $\eta_1(t_{Mi}) = \epsilon_i$. □

IV. Adaptive Synchronization With PE Condition

Consider system (8) with the output $y = \eta_1(t)$, its state cannot be estimated by a class of adaptive observers if the parameter τ is unknown [3]. By (3) and (4), we construct the following adaptive observer for system (8):

$$\begin{aligned}\frac{d\hat{\eta}}{dt} &= \begin{bmatrix} l_1 & 1 & 0\\ l2 & 0 & (\lambda_2 - \lambda_1)\eta_1\\ 0 & 0 & \lambda_3\end{bmatrix}\hat{\eta} + \begin{bmatrix}\lambda_1 + \lambda_2 - l_1\\ -\lambda_1\lambda_2 - l_2\\ 0\end{bmatrix}\eta_1\\ &\quad + \begin{bmatrix}0\\ -\frac{1}{2}\eta_1^3\\ \frac{\lambda_3 - 2\lambda_2}{2(\lambda_1 - \lambda_2)}\eta_1^2\end{bmatrix} + \begin{bmatrix}0\\ -\frac{1}{2}\eta_1^3\\ \frac{\lambda_3 - 2\lambda_1}{2(\lambda_1 - \lambda_2)}\eta_1^2\end{bmatrix}\hat{\tau} + \Upsilon(t)\dot{\hat{\tau}}\\ \dot{\hat{\tau}}(t) &= \Upsilon^T(t)C^T(t)[\eta_1(t) - C(t)\hat{\eta}(t)]\\ \dot{\Upsilon}(t) &= \begin{bmatrix} l_1 & 1 & 0\\ l2 & 0 & (\lambda_2 - \lambda_1)\eta_1\\ 0 & 0 & \lambda_3\end{bmatrix}\Upsilon(t) + \begin{bmatrix}0\\ -\frac{1}{2}\eta_1^3\\ \frac{\lambda_3 - 2\lambda_1}{2(\lambda_1 - \lambda_2)}\eta_1^2\end{bmatrix}\end{aligned} \tag{21}$$

where $l_i < 0$, $i = 1, 2$. The synchronization between (8) and (21) is achieved if $\lim_{t\to\infty}|\eta(t) - \hat{\eta}(t)| = 0$, that is, the above observer is an asymptotically stable observer for system (8). As mentioned in Section II, the observer is asymptotically stable under Condition 1 and 2. Condition 1 is proved in [3]. The difficulty is to prove that Condition 2 holds, that is $\Upsilon_1(t)$ is PE. In order to prove Condition 2, we need some properties of $\eta_1(t)$ and PE conditions.

Lemma 4: Suppose that system (8) is nontrivial and there exists a finite time Δt so that $\eta_1(t)$ has at least one extremum in the interval $(t_0,\ t_0 + \Delta t)$ for any $t_0 \geq 0$, then $\eta_1(t)$ is PE.

Proof: For any $t_0 > 0$, if there exist a positive constant $\alpha > 0$, a finite time Δt_0, and a time t' in $[t_0,\ t_0 + \Delta t_0]$ such that $|\eta_1(t')| > \alpha$, and α and Δt_0 are independent of time t_0, then there exists $\delta > 0$ independent of time t_0 such that $|\eta_1(t)| \geq \alpha/2$ for all $t \in [t' - \delta,\ t' + \delta] \subset [t_0,\ t_0 + \Delta t]$ since the derivation of $\eta_1(t)$ is bounded according to Assumption 1. Hence the PE condition (1) is satisfied. If the above α and Δt_0 do not exist, then for any positive integer i and $M/2^i$, and any increasing sequence $\{\Delta t_i\}$ with $\Delta t_1 > 4\Delta t$ and $\lim_{i\to\infty}\Delta t_i = +\infty$, there exists a sequence $\{t_i\}$ such that $|\eta_1(t)| < M/2^i$ on $[t_i, T_i]$ for all i, where $M := \sup_{t\geq 0}|\eta_1(t)|$ and $T_i := t_i + \Delta t_i$. Note that $\Delta t_i/2 > 2\Delta t$ and η_1 has at least one extremum in the interval $(t_0,\ t_0 + \Delta t)$ for any $t_0 \geq 0$, there exists a $t' \in [t_i + \Delta t_i/2,\ T_i]$ such that $\eta_1(t')$ is a maximum, therefore we have $\ddot{\eta}_1(t') = (\lambda_1 + \lambda_2)\dot{\eta}_1(t') + \dot{\eta}_2(t') = \dot{\eta}_2(t') < 0$. Note also that $|\eta_1| < M/2^i$ on $[t_i, T_i]$ and $\dot{\eta}_3 = \lambda_3\eta_3 + K_1(\tau)\eta_1^2$, therefore, when Δt_i is sufficiently large, $\eta_3(t)$ will become sufficiently small such that $-\lambda_1\lambda_2 - (\lambda_1 - \lambda_2)\eta_3(t) - 1/2(1+\tau)\eta_1^2(t) > 0$ for all $t \in [t_i + \Delta t_i/2,\ T_i]$. If $\eta_1(t') \geq 0$, then by (8) we have $\dot{\eta}_2(t') \geq 0$, which contradicts the previously got $\dot{\eta}_2(t') < 0$. In case of $\eta_1(t') < 0$, it follows from $\Delta t_i/2 > 2\Delta t$ that there must exists a $t'' \in [t_i + \Delta t_i/2,\ t')$ or $(t', T_i]$ such that t'' is the nearest minimum point to t'. A similar proof leads also to a contradiction. □

Lemma 5: ([10], [14]) Let $a > 0$, and the input $u(t)$ be PE in the 1-D system $\dot{x} = -ax + u(t)$, then the solution $x(t)$ is also PE.

Lemma 6: Let $x(t)$ be a scalar function of time, and suppose $x(t)$ and $\dot{x}(t)$ are continuous and bounded, then $x^2(t)$ is PE if $x(t)$ is PE.

Proof: Since $x(t)$ is PE, there exist $\alpha_1, \alpha_2, T > 0$ such that $\alpha_1 I \leq \int_{t_1}^{t_1+T} x^2(s)ds \leq \alpha_2 I$, for all $t_1 \geq 0$. Then there exists a time $t_2 \in (t_1, t_1 + T)$ such that $x^2(t_2) \geq \alpha_1/T$. Since $\dot{x}$ is bounded, we have $|x(t') - x(t'')| = |\dot{x}(\xi)||t' - t''| \leq M|t' - t''|$, where $M > 0$ and $\xi \in (t',\ t'')$. Thus there exists $\delta > 0$ independent of time t_1 such that $x^2(t) \geq \alpha_1/2T$ for all $t \in [t_2 - \delta,\ t_2 + \delta]$. Then it is obvious that

$$\int_{t_1}^{t_1+T} x^4 ds \geq \int_{t_2-\delta}^{t_2+\delta} x^4 ds \geq 2\delta\left(\frac{\alpha_1}{2T}\right)^2.$$

Now, we rewrite $\Upsilon(t)$ in the following form:

$$\dot{\Upsilon}(t) = \begin{bmatrix} l_1 & 1 & 0\\ l2 & 0 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\Upsilon(t) + \begin{bmatrix}0\\ (\lambda_2 - \lambda_1)y\Upsilon_3 - \frac{1}{2}y^3\\ \frac{\lambda_3 - 2\lambda_1}{2(\lambda_1 - \lambda_2)}y^2\end{bmatrix} \tag{22}$$

where $y(t)$ is the output of system (8), that is, $\eta_1(t)$. Apply the following transformation to (22):

$$\begin{aligned}\zeta &= \begin{bmatrix}1 & 1 & 0\\ -a_2 & -a_1 & 0\\ 0 & 0 & 1\end{bmatrix}\Upsilon = P\Upsilon\\ a_1 + a_2 &= l_1, \quad -a_1 a_2 = l_2\end{aligned} \tag{23}$$

then

$$\dot{\zeta} = \begin{bmatrix}a_1 & 0 & 0\\ 0 & a_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\zeta + \begin{bmatrix}(\lambda_2 - \lambda_1)y\Upsilon_3 - \frac{1}{2}y^3\\ -a_1\left[(\lambda_2 - \lambda_1)y\Upsilon_3 - \frac{1}{2}y^3\right]\\ \frac{\lambda_3 - 2\lambda_1}{2(\lambda_1 - \lambda_2)}y^2\end{bmatrix}. \tag{24}$$

Remark 2: From Lemma 5, it is easy to know that $\Upsilon_3(t)$ or $\zeta_3(t)$ is PE since Lemma 4 has already proved that $\eta_1(t)$ is PE.

Note that we can make $a_1, a_2 < 0$ by choosing properly l_1 and l_2 in (23), therefore it is assumed from now on that $a_1 < 0$ and $a_2 < 0$.

Lemma 7: There exist $\alpha_1, \alpha_2, \Delta t > 0$ such that

$$\alpha_1 I \leq \int_t^{t+\Delta t} \Upsilon_3^2(s)ds \leq \alpha_2 I, \quad \forall\, t \geq 0$$

and there is at least one local maximum of Υ_3 in $[t,\ t+\Delta t]$.

Proof: Since $\Upsilon_3(t)$ is PE, we can always find appropriate $\alpha_1, \alpha_2, \Delta t > 0$ such that

$$\alpha_1 I \leq \int_t^{t+\Delta t} \Upsilon_3^2(s)ds \leq \alpha_2 I, \quad \forall\, t \geq 0.$$

Now we prove the lemma by contradiction. If the result does not hold, then by following the same way as in Theorem 1, there exist two increasing sequences $\{t_i\}_1^\infty$ and $\{T_i\}_1^\infty$ such that t_i and $\Delta t_i = T_i - t_i > 0$ tend to infinity, and $\Upsilon_3(t)$ is monotonic on $[t_i,\ t_i + T_i]$. For the same reason in case II of Theorem 1, we can assume similarly that $|\Upsilon(T_i) - \Upsilon(t_i)| < \epsilon_i$ where ϵ_i is a sufficiently small positive number when i is sufficiently large. It follows from (22) and Assumption 1 that $\Upsilon_3(t)$ and $\dot{\Upsilon}_3(t)$ are bounded, therefore $|\dot{\Upsilon}_3(t)| < M\epsilon_i$ on $(t_i,\ T_i)$, where M is a positive constant. However, by Theorem 1 there exist $t_i < t', t'' < T_i$ such that $y^2(t'') - y^2(t') = \theta$, where θ is a positive constant. Let $b = (\lambda_3 - 2\lambda_1)/2(\lambda_1 - \lambda_2) < 0$, then $|\dot{\Upsilon}_3(t'')| = |\lambda_3 \Upsilon(t'') + by^2(t'')| = |\dot{\Upsilon}_3(t') + \lambda_3(\Upsilon(t'') - \Upsilon(t')) + b(y^2(t'') - y^2(t'))| > |b\theta| - (M - \lambda_3)\epsilon_i > M\epsilon_i$ when ϵ_i is sufficiently small. This contradiction ends the proof. □

Lemma 8: Let $a = (\lambda_2 - \lambda_1) < 0$, then $f(t) = ay\Upsilon_3(t) - 1/2y^3(t)$ is PE.

Proof: Let $b = \lambda_3 - 2\lambda_1/2(\lambda_1 - \lambda_2) < 0$, then from $\dot{\Upsilon}_3(t) = \lambda_3\Upsilon_3(t) + by^2(t)$ we obtain

$$f(t) = \frac{a}{\lambda_3} y \dot{\Upsilon}_3(t) - \frac{2ab + \lambda_3}{2\lambda_3} y^3 = \frac{a}{\lambda_3} y \dot{\Upsilon}_3(t) - \frac{\lambda_1}{\lambda_3} y^3$$

and $\Upsilon_3(t) < 0$ from Lemma 1. From Lemma 7, there exists $t_1 \in (t, t + \Delta t)$ such that $\dot{\Upsilon}_3(t_1) = 0$ and

$$y^2(t_1) = \frac{1}{b}(\dot{\Upsilon}_3(t_1) - \lambda_3\Upsilon_3(t_1)) \geq \frac{\lambda_3}{b}\sqrt{\frac{\alpha_1}{\Delta t}}$$

where Δt and α_1 are defined in Lemma 7. Since $y(t)$ and $\dot{\Upsilon}_3(t)$ are uniformly continuous, there exists $\delta > 0$ independent of time t such that $|a\dot{\Upsilon}_3(t)| < \lambda_1\lambda_3\sqrt{\alpha_1/\Delta t/(4b)}$ and $\lambda_1 y^2(t) > \lambda_1\lambda_3\sqrt{\alpha_1/\Delta t/(2b)}$ for $t \in [t_1 - \delta, t_1 + \delta]$. Therefore $|f(t)| = |y(a\dot{\Upsilon}_3(t) - \lambda_1 y^2(t))/\lambda_3| > \varepsilon$ for all $t \in [t_1 - \delta, t_1 + \delta] \subset [t, t + \Delta t]$, where ε is a positive constant. Hence

$$\int_t^{t+\Delta t} f^2(s)ds > \int_{t_1-\delta}^{t_1+\delta} f^2(s)ds > 2\varepsilon\delta.$$

Theorem 2: Under Assumption 1 and Assumption 2, observer (21) is an exponential observer for system (8) under the output $y = \eta_1(t)$.

Proof: By the transformation (23) we have $\Upsilon_1(t) = (a_1\zeta_1 + \zeta_2)/(a_1 - a_2)$ and $a_i < 0, i = 1, 2$, thus

$$\dot{\Upsilon}_1 = \frac{(a_1(a_1\zeta_1 + f) + (a_2\zeta_2 - a_1 f))}{a_1 - a_2} = a_1\Upsilon_1 - \zeta_2. \qquad (25)$$

By Lemma 8, we know that $f(t) = (\lambda_2 - \lambda_1)y\Upsilon_3 - 1/2y^3$ is PE, then from Lemma 5 and (24) we have $\zeta_2(t)$ is PE. Similarly $\Upsilon_1(t)$ is also PE. As mentioned above, Condition 1 is proven in [3]. Condition 2 holds from the fact that $\Upsilon_1(t)$ is PE. From Theorem 1 in [19], we know that observer (21) is an exponential observer for system (8). □

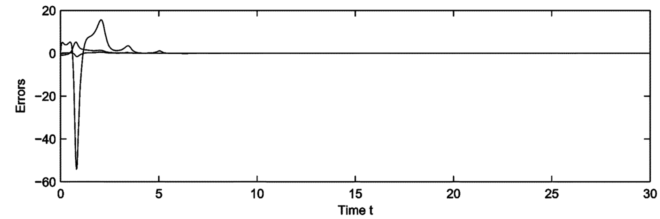

Fig. 2. Synchronization errors between (8) and (21).

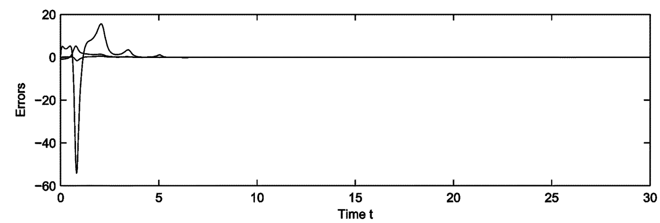

Fig. 3. Parameter estimation value.

Remark 3: For system (8) with output $\eta_1(t)$, the authors of [3] proved that it cannot achieve synchronization by certain kind of observer, owing to the unknown parameter τ. Now, without additional conditions, we prove that observer (21) can estimate the states and the unknown parameter at the same time.

V. Numerical Illustration

In [3], the authors illustrated that system (8) cannot be synchronized without knowing the exact τ with the parameters $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$ and $\tau = 0.5$. Now, selecting $l_1 = -28$, $l_2 = 180$, we show the efficiency of the observer (21) with the same parameters. First, we compute the three equilibria and the eigenvalues of the corresponding Jacobian matrices. Obviously, $O_0(0, 0, 0)$ is unstable since $\lambda_1 > 0$. After a simple computation, the other equilibriums are $O_1(3.266, 26.128, 5)$ and $O_2(-3.266, 26.128, 5)$ respectively. The Jacobian matrices corresponding to O_1 and O_2 have the eigenvalue 15.434 and $-12.217 \pm 10.385i$. Fig. 2 shows the synchronization errors between system (8) and (21). Fig. 3 shows that the unknown parameter τ is estimated exactly, which implies the parameter τ can not be a password. The initial values of system (8) and (21) are [1 2 3] and [2 3 4 3 2 − 1 2] respectively. The two figures show that both the state and the unknown parameter of (8) can be estimated.

VI. Conclusion

In this paper, we achieve synchronization for a class of chaotic system with unknown parameter, whose parameter is believed to be difficult to estimate. The key idea is to use a different kind of adaptive observer from literature to the transformed generalized Lorenz system. Such an idea and the kind of observer we used in this paper will be further applied in the synchronization problem of other chaotic systems.

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Receding Horizon Controls for Input-Delayed Systems

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***Abstract*—This paper presents a receding horizon control (RHC) for an unconstrained input-delayed system. To begin with, we derive a finite horizon optimal control for a quadratic cost function including two final weighting terms. The RHC is easily obtained by changing the initial and final times of the finite horizon optimal control. A linear matrix inequality (LMI) condition on two final weighting matrices is proposed to meet the cost monotonicity, under which the optimal cost on the horizon is monotonically nonincreasing with time and hence the asymptotical stability is guaranteed only if an observability condition is met. It is shown through simulation that the proposed RHC stabilizes the input-delayed system effectively and its performance can be tuned by adjusting weighting matrices with respect to the state and the input.**

***Index Terms*—Cost monotonicity, final weighting matrix, input delay, quadratic cost function, receding horizon control (RHC).**

I. Introduction

In many industrial and natural dynamic processes, time delays on states and/or control inputs are frequently encountered in the transmission of information or material between different parts of a system. The representative examples of time-delay systems are chemical systems, transportation systems, communication systems, and biological systems. As one of time-delay systems, an input-delayed system is easily found and preferred for easy modelling. Much research on input-delayed systems has been made for decades in order to compensate for the deterioration of the performance due to the presence of input delay [1]–[5].

For ordinary systems without time delay, predictive controls have received much attention as a powerful tool for the control of industrial process systems. One of predictive controls, called receding horizon control (RHC), moving horizon control, or model predictive control (MPC), has been widely investigated as a successful closed-loop control strategy for industrial fields such as chemical process controls in petrochemical, pulp, and paper industries. The basic concept of the RHC is to solve an optimization problem on the finite future horizon at the current time and implement only the first solution as a current control. This procedure then repeats at the next time. Since the RHC is based on the cost function on the finite future horizon, it presents many advantages such as a simple computation mechanism, good tracking performance, input/state constraint handling, time-varying and nonlinear system handling, and so on, compared with other popular steady-state infinite horizon controls [6]–[9].

For time-delay systems, there are only a few results for the RHC in [10]–[13]. A simple receding horizon control with a special cost function was proposed for state-delayed systems by using a reduction method [10]. The general cost-based RHC for state-delayed systems was introduced in [11]. Recently, the constrained MPC for uncertain state-delayed systems and the receding horizon H_∞ control for state-

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