HEFAT2014
10th International Conference on Heat Transfer, Fluid Mechanics and Thermodynamics
14 – 16 July 2014
Orlando, Florida

ANALYTICAL AND NUMERICAL METHODS FOR SOLVING PROBLEMS OF HEAT CONDUCTION FOR BODY WITH GRADED COATING

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ABSTRACT

In the present paper an axisymmetric heat condition problem for a non-homogeneous half-space heated by a heat flux is considered. The half-space consists of a homogeneous substrate and non-homogeneous coating. The solutions are obtained using Hankel integral transform technique. Compared:

1) the analytical solutions of the problems for coating, which has a heat conductivity coefficient is described by continuous function of the distance to the boundary surface, and the solution of the problem in which the non-homogeneous coating is modeled by the package of homogeneous layers;
2) the solution for the multilayered coating with a periodic structure and the solution of the problem in which this coating is described by a homogenized uniform layer.

INTRODUCTION

The progress of coating technology is the reason for wide employments of coatings for improvement in various technical fields. In the last decade, significant attention is focused on the problem of inhomogeneous medium formed by a homogeneous isotropic substrate and an inhomogeneous coating whose thermal and mechanical properties vary over its thickness. These problems come down to solving partial differential equations with variable coefficients. For the power (or exponential) law of variation of heat conduction coefficient (or Young's modulus) the analytical methods of solutions are known [1-5].

Parallel with the application of analytic methods for the solution of partial differential equations, inhomogeneous layers are also modeled by using an approach according to which coating is replaced with a package of homogeneous or inhomogeneous layers [5-11].

The special type of graded coating is multilayered coating with periodic structure [12-13]. In modeling laminated half spaces or coatings with periodic structures, it is customary to use two different approaches. The first of these approaches the layers are considered as separate continuous media. The second approach is based on the analysis of a homogenized uniform coating whose properties are determined on the basis of the material properties and geometric characteristics of the strip of periodicity [14–15]. The solutions obtained for the laminated half-space or layer are compared in [16–18].

In the present work, we consider an axisymmetric problem of heat conductivity for half-space with graded coating heated by a given heat flux. Examined: 1) coating, which has a heat conductivity coefficient is described by continuous function of the distance to the boundary surface, 2) multilayer coating with a periodic structure composed of the two-layer laminae repeated periodically. In the first case the two approaches was considered: 1) analytical method of solving differential equations with variable coefficients; 2) analytical-numerical method based on modeling non-homogeneous coating by package of homogeneous layers, between which the conditions of ideal thermal contact is assumed. The multilayer coating with periodic structure is described by means of two models: 1) the homogenized model with micro-local parameters [14-15], 2) based on a classical heat conduction model, in which the components of the coating are considered as separate homogeneous medium.

We analyze the difference between the temperature and heat flux in the non-homogeneous half-space caused by the use of two different models of nonuniform coatings.

NOMENCLATURE

a	[m]	radius of circle heating zone
c, α,	[-]	dimensionless parameters described the dependence of heat conductivity coefficient on the coordinate z
h_I, h_{II}	[-]	dimensionless thickness of layers of strip of periodicity
H	[m]	thickness of the coating
$H(r)$	[-]	Heaviside step function
H_i	[m]	thickness i-th layer of coating
$I_\nu(r)$	[-]	modified Bessel functions of first kind
$J_\nu(r)$	[-]	Bessel function of first kind

K^*	[W/(mK)]	parameter described the dependence of heat conductivity coefficient on the coordinate z
K_i	[W/(mK)]	heat conductivity coefficient i-th layer of coating
K_0	[W/(mK)]	heat conductivity coefficient of substrate
K_I,K_{II}	[W/(mK)]	heat conductivity coefficients of layers of strip of periodicity
K_S	[W/(mK)]	heat conductivity coefficient on the surface of the inhomogeneous half-space
$K_v(r)$	[-]	modified Bessel functions of second kind
r, z	[-]	dimensionless cylindrical coordinate system
s	[-]	parameter of Hankel integral transform
T	[K]	temperature
$q(r)$	[W/m^2]	given heat flux on boundary half-space
q_r	[W/m^2]	radial heat flux
Q_0	[W/m^2]	given maximal intensity of heat flux on boundary half-space
n	[-]	number of layer in coating

FORMULATION OF THE PROBLEM

Assume that the surface $z = h$ of a non-homogeneous half-space is heated by a heat flux $q(r)$ on the circle of radius a, where r, z are dimensionless cylindrical coordinates referred to the linear size a, $h = H/a$, H is the thickness of the coating.

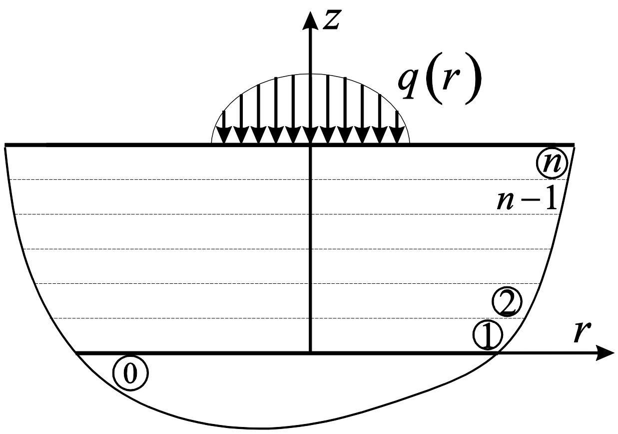

Figure 1 The scheme of the body

The non-homogeneous half space is formed by the homogeneous half-space with the heat conductivity coefficient K_0 and a system of non-homogeneous layers with thicknesses H_i and the heat conductivity coefficients K_i, $i = 1, 2, \ldots, n$, respectively, where the value of the parameter n corresponds to the number of the layer in the package. Assume that the conditions of perfect thermal contact are realized between the layers of the coating and between the coating and the base.

The analyzed problem of the theory of heat conduction is reduced to the solution of the following partial differential equations with constant coefficients:

$$\frac{\partial^2 T_i}{\partial r^2}+\frac{1}{r}\frac{\partial T_i}{\partial r}+\frac{1}{K_i}\frac{\partial}{\partial z}\left(K_i\frac{\partial T_i}{\partial z}\right)=0,\ i=0,2,\ldots,n\,, \tag{1}$$

with boundary conditions imposed on the surface of the non-homogeneous half-space

$$\frac{\partial T_n}{\partial z}=\frac{q(r)a}{K_n}H(1-r),\ z=h\,, \tag{2}$$

conditions of perfect thermal contact between the components of considered half-space

$$T_i=T_{i+1},\quad K_i\left(h_i^*\right)\frac{\partial T_i}{\partial z}=K_{i+1}\left(h_i^*\right)\frac{\partial T_{i+1}}{\partial z},\quad z=h_i^*,\ i=0,\ldots,n-1 \tag{3}$$

and conditions imposed at infinity

$$T_i\to 0,\ r^2+z^2\to\infty,\ i=0,1,\ldots,n\,, \tag{4}$$

where T_i is the temperature in the i-th component of non-homogenous medium, the index $i = 0$ describe the parameters and functions of state in the homogeneous half-space, h_i^* is the coordinate z upper surface i-th component of non-homogenous half-space, $h_0^*=0$, $h_i^*=h_{i-1}^*+h_i$, $h_i = H_i/a$, $i = 1, \ldots, n$, $h_n^*=h$, $H(r)$ – Heaviside step function.

METHOD OF SOLUTION

The general solution of the boundary value problem is sought by applying the Hankel integral transformation:

$$\tilde{T}_i(s,z)=\int_0^\infty T_i(r,z)\,rJ_0(sr)\,dr\,, \tag{5}$$

where $J_0(sr)$ is Bessel function. The Hankel transform of temperature in homogeneous half-space can be written in the form:

$$\tilde{T}_0(s,z)=t_0(s)\exp(sz)\,, \tag{6}$$

where $t_0(s)$ is the unknown function.

We considered the following cases.

Case a). Let $n = 1$. The dependence of the heat conductivity coefficient on the coordinate z is described by the formula

$$K_1(z)=K_0\exp(\alpha z),\ \alpha=\frac{1}{h}\ln\left(\frac{K_S}{K_0}\right),\ 0\le z\le h\,, \tag{7}$$

where K_S is heat conductivity coefficient on the surface of the inhomogeneous half-space. The general solution of the differential equation (1) specified in the coating can be written in the form:

$$\tilde{T}_1(s,z)=t_1(s)\exp\left(\alpha^{(-)}z\right)+t_2(s)\exp\left(\alpha^{(+)}z\right), \tag{8}$$

where $2\alpha^{(\pm)}=-\alpha\pm\sqrt{\alpha^2+4s^2}$, $t_1(s)$ and $t_2(s)$ are the unknown functions.

Case a′). Let $n = 1$. The dependence of the heat conductivity coefficient on the coordinate z is described by the power function

$$K_1(z)=K^*(c\pm z)^\alpha,\ c=\pm h\left(\left(\frac{K_S}{K_0}\right)^{1/\alpha}-1\right)^{-1},\quad K^*=\frac{K_0}{c^\alpha},\ 0\le z\le h. \tag{9}$$

In equation (9) for a case $K_0 < K_S$, we take sign „+"; when $K_0 > K_S$ – sign „–".

The Hankel transform of temperature in coating can be written in the form:

$$\tilde{T}_1(s,z) = t_1(s)\zeta^p I_p(s\zeta) + t_2(s)\zeta^p K_p(s\zeta), \quad (10)$$

where $2p + \alpha = 1$, $\zeta = c \pm z$, $I_p(s\zeta)$, $K_p(s\zeta)$ – modified Bessel functions.

Moreover, if analytical solution of partial differential equation with variable coefficient (1) is not known the non-homogeneous coating can be replaced by multilayered system of homogeneous layer. Their thermal properties are described by their heat conductivity coefficients:

$$K_i = \frac{1}{h_i}\int_{h_{i-1}^*}^{h_i^*} K_1(z)dz\,. \quad (11)$$

Case b). Coating composed of n homogeneous layer. The general solution of equations (1) expressed in the Hankel transform domain takes the form:

$$\tilde{T}_i = t_{2i-1}(s)\sinh(s(h_i^* - z)) + t_{2i}(s)\cosh(s(h_i^* - z)),\\ i = 1,2,...,n, \quad (12)$$

where $t_i(s)$, $i = 1,\ldots,2n$ are the unknown functions.

Case c): multilayered coating with periodical structure. Assume that a repeated fundamental layer comprise two homogeneous elastic sublayers with different thicknesses (h_I and h_{II}) and thermal conductivities (K_I and K_{II}). A large number of equations and boundary conditions on interfaces complicates solution of the problem. Another approach is using homogenized model [14-15] in which properties of the homogenized coating are determined on the base of properties of the components.

Applying the homogenized model to the coating, we solve the boundary value problem described by equation:

$$\frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r} + \frac{1}{p_1^2}\frac{\partial^2 T_1}{\partial z^2} = 0\,, \quad (13)$$

where T_1 is the temperature in the homogenized coating,

$$p_1^2 = \tilde{K}K_c^{-1}, K_c = \frac{K_I K_{II}}{(1-\eta)K_I + \eta K_{II}},\\ \tilde{K} = \eta K_I + (1-\eta)K_{II}, \eta = \frac{h_I}{h_I + h_{II}}, \quad (14)$$

boundary condition imposed on the surface of the non-homogeneous half-space

$$\frac{\partial T_1}{\partial z} = \frac{q(r)a}{K_c}H(1-r),\ z = h\,, \quad (15)$$

and boundary conditions of perfect thermal contact between the homogenized coating and the substrate:

$$T_0 = T_1,\quad K_0\frac{\partial T_0}{\partial z} = K_c\frac{\partial T_1}{\partial z},\ z = 0 \quad (16)$$

The boundary conditions (4) are stay without change. The general solution of equation (13) in Hankel transforms takes the form:

$$\tilde{T}_1 = t_1(s)\sinh((h-z)sp_1) + t_2(s)\cosh((h-z)sp_1)\,. \quad (17)$$

The equations (6), (8), (10), (12) and (17) contain unknown functions $t_i(s)$. These functions are obtained, satisfying boundary conditions (2)–(3) (or (15)-(16) in case c). Satisfying boundary conditions, the functions $t_i(s)$ may be written as

$$t_i(s) = \frac{\tilde{q}(s)a}{\breve{K}s}\breve{t}_i(s)\,, \quad (18)$$

where the functions $\breve{t}_i(s)$ are obtained solution of linear equations (see Appendix A), $\tilde{q}(s)$ is Hankel transform of heat flux, $\breve{K} = K_n$ in case a, a′, b) and $\breve{K} = K_c p_1$ in case c).

Applying the inverse Hankel transform to equations (6), (8), (10), (12) and (17), temperature can be find at the desired location

$$T_i(r,z) = \int_0^\infty s\tilde{T}_i(s,z)J_0(sr)ds\,. \quad (19)$$

At internal points of the non-homogeneous half space ($z < h$) the integrals were evaluated numerically using the Gaussian quadrature. On the surface $z = h$, we take into account the asymptotic behavior of the functions $t_{2n-1}(s)$ and $t_{2n}(s)$ as $s \to \infty$. The continuity of results when $z \to h$ was verified.

NUMERICAL RESULTS AND DISCUSSION

Assume that the heat flux is elliptical distributed as follows

$$q(r) = Q_0\sqrt{1-r^2}H(1-r),\ \tilde{q}(s) = \sqrt{\frac{\pi}{2}}Q_0\frac{J_{3/2}(s)}{s^{3/2}}\,, \quad (20)$$

where $J_{3/2}(s)$ is a Bessel function.

The analysis of the original relations in case a) (or a′) shows that the solution in the problem of modeling of inhomogeneous coating by the package of homogeneous layers depends on three (case a) or four (case a′) dimensionless parameters: the thickness of the coating h, the ratio of heat conductivity coefficients on the surfaces of non-homogeneous half space and the substrate K_S/K_0, the parameter α (only for case a′) and the number of layers in the package n. A similar solution obtained for an inhomogeneous coating with regard for the continuous dependence of the thermal properties on the coordinate is independent of parameter n. In what follows, we assume that $h = 0.4$, $K_0/K_S = 2$, 4, or 8, $\alpha = 1$ (only for case a′), and $n = 10$, 20, 40, or 80.

The temperature and the heat flux in radial direction r on considered nonhomogeneous surface are shown in Figs. 2 and 3 (a: case a; b: case a′). In this figures, the continuous lines correspond to the solution of the problem with continuous variation of the thermal properties. The rhombi correspond to the results obtained for the package formed by 40 homogeneous layers. The results of calculations presented in this Figs. show good agreement between the solutions obtained using the analyzed two models of the coating. As follows from Figs. 2 and 3 the maximum absolute error in finding the radial heat flux is obtained on the boundary of the heated zone.

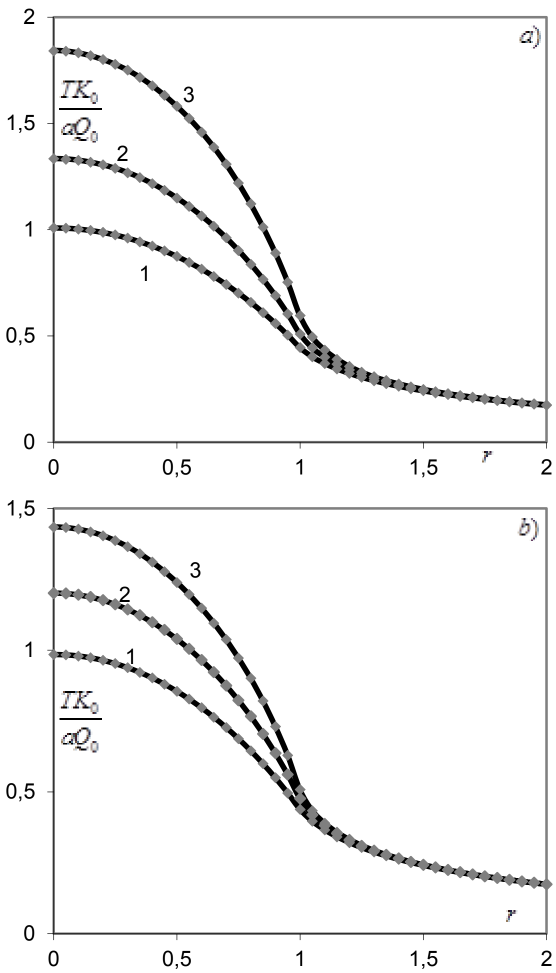

Figure 2 The dimensionless temperature distribution on the surface $z = h$: a) –case a; b) – case a′; 1: $K_0/K_S = 2$; 2: $K_0/K_S = 4$; 3: $K_0/K_S = 8$; $h = 0.4$

K_0/K_S	$n = 10$	$n = 20$	$n = 40$	$n = 80$	$n \to \infty$
case a					
2	2.79%	1.43%	0.68%	0.29%	0.6169
4	5.40%	2.73%	1.29%	0.54%	0.5021
8	7.79%	3.90%	1.82%	0.71%	0.4236
case a′					
2	3.95%	2.03%	0.98%	0.43%	0.6022
4	11.00%	5.68%	2.72%	1.14%	0.4562
8	23.00%	11.97%	5.65%	2.18%	0.3435

Table 1 The dimensionless radial heat flux on the boundary of the heated zone

The values of the radial heat flux in this point are presented in Table 1. The analytical solution is presented in the rows of Table 1 with $n \to \infty$. The relative error of their evaluation with the help of modeling of the inhomogeneous coating by the package of n homogeneous layers is given in columns with n = 10, 20, 40, and 80. It is easy to see that, as the number of layers becomes twice larger, the corresponding error becomes almost twice lower. In the case where there are 80 layers in the package and $K_0/K_S \le 8$, the error of finding the heat flux at the point $r = 1$, $z = h$ does not exceed 2.2%.

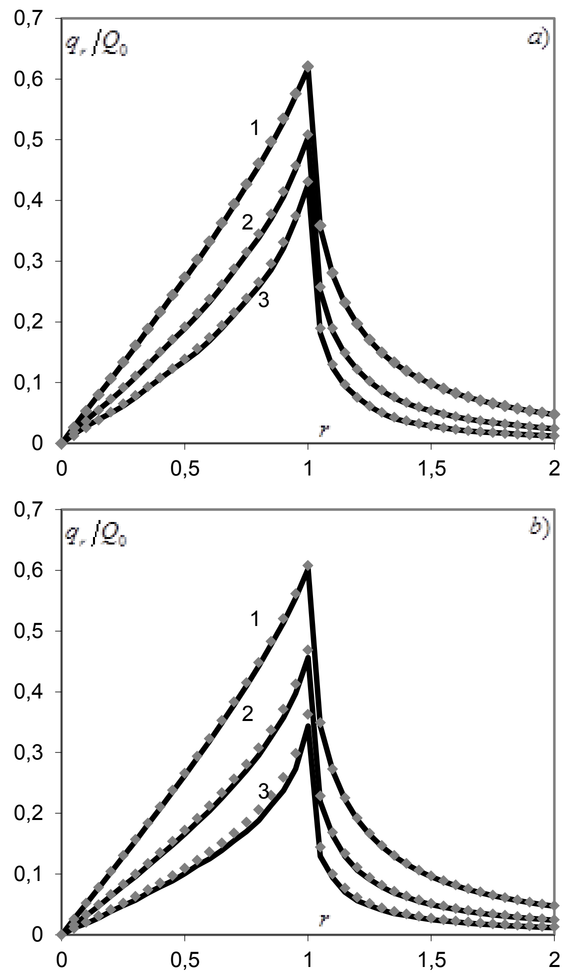

Figure 3 The dimensionless radial heat flux on the surface $z = h$: a) –case a; b) – case a′; 1: $K_0/K_S = 2$; 2: $K_0/K_S = 4$; 3: $K_0/K_S = 8$; $h = 0.4$

Estimating the original relations, we conclude that the distributions of temperature and heat flux in the problem of homogenized coating (case c) depend on four dimensionless parameters: the thickness of the coating h , the ratios of heat conductivity coefficients K_I/K_0 and K_{II}/K_0 and the ratio of the thicknesses of layers in the strip of periodicity h_I/h_{II}. Similar distributions for the non-uniform coating additionally depend on the number of layers in the coating n. To decrease the number of input parameters, we assume that the thermal properties of one layer in the strip of periodicity coincide with the thermal properties of the base ($K_I/K_0 = 1$ or $K_{II}/K_0 = 1$) and the thicknesses of all layers in the stack are identical ($h_I/h_{II} = 1$). We also assume that K_0/K_I (or K_0/K_{II}) = 4, h = 0.2, 0.4, or 0.8, and n = 10, 20, 40, or 80.

h	$n = 10$	$n = 20$	$n = 40$	$n = 80$	$n \to \infty$
0.2	1.12%	0.56%	0.28%	0.14%	1.0841
	-1.13%	-0.56%	-0.28%	-0.14%	
0.4	3.35%	1.67%	0.83%	0.41%	1.2481
	-3.26%	-1.64%	-0.83%	-0.42%	
0.8	7.99%	3.92%	1.94%	0.96%	1.3887
	-7.31%	-3.76%	-1.90%	-0.96%	

Table 2 The dimensionless temperature at the centre of the heating area

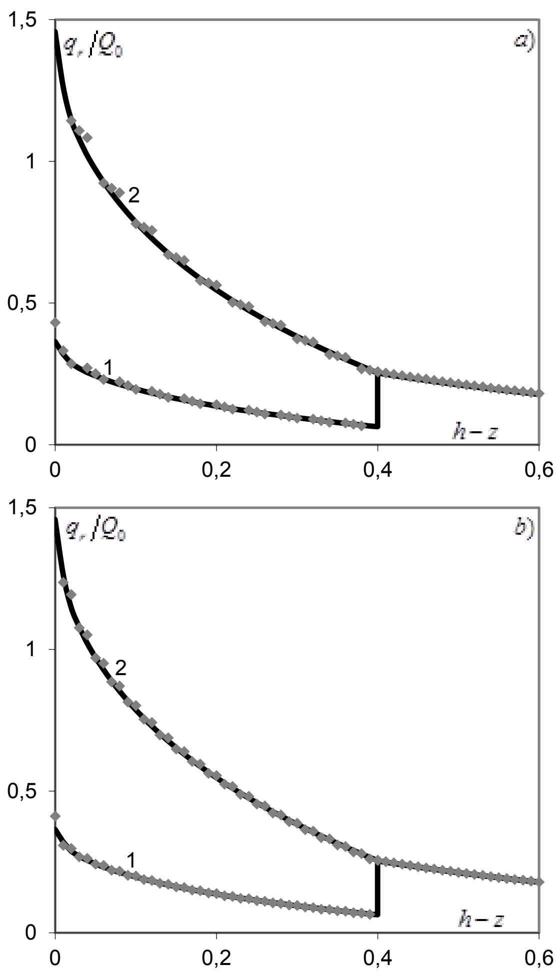

Figure 4 The dimensionless radial heat flux on the line r = 1: a) – n = 20; b) – n = 40; $K_0/K_I = 4$; $h = 0.4$; $h_I/h_{II} = 1$

The dimensionless temperature at the centre of the heating area for different thicknesses of the coating and different number of layers is presented in Table 2. The temperature calculated for the homogenized model is in the last column. The relative differences between the temperature in non-homogeneous coating and the temperature in homogenized coating are presented in columns with n = 10, 20, 40, and 80. The upper numbers in cells were calculated for the case $K_0/K_I = 4$, $K_0/K_{II} = 1$, the lower numbers were obtained for the case $K_0/K_I = 1$, $K_0/K_{II} = 4$ It can be seen that as number of layers becomes twice larger the errors become twice lower. For the same value of the parameter $\delta = h/n$ (for example: h = 0.2, n = 10; h = 0.4, n = 20; and h = 0.8, n = 40) these errors are in the same order of magnitude.

Figs. 4 show the dimensionless radial heat flux as functions of z for r=1 and two numbers of layers (n=20 and n=40). In Figs. 4, the rhombs mark the numerical results obtained for the non-homogeneous laminated coating, whereas the solid lines correspond to the homogenized coating. It should be emphasized that, in the case of homogenized coating, we do not know which layer of the slip of periodicity is located at the analyzed point of the coating. Hence, the radial heat flux at every point of the coating is described by the two curves. Curves 1 and 2 correspond to the heat flux acting in the layers with smaller and larger heat conductivity coefficients, respectively. In the homogeneous substrate curves 1 and 2 coincide.

Comparing the heat flux obtained in both analyzed problems, we conclude that, only in the case of the heat flux acting in the homogeneous substrate, we get deviations comparable with the deviations of the temperature. In the layers of the coating, the deviations of heat flux vary from 1–5% ($K_0/K_I \le 4$, n = 20) up to 10–20% for the heat flux acting on the boundary of the region of heating. The indicated deviations strongly depend on the gradient of the analyzed parameter in the investigated layer of the slip of periodicity, which explains the following observations: in the layers with lower heat conductivity coefficients, the deviations are much smaller (Figs. 4) and the maximum deviations are observed at the point (1, h). As could be expected, the agreement between the solutions improves as the number of layers in the coating increases.

CONCLUSIONS

This paper provides the solution to the problem of inhomogeneous half-space heated by the heat flux. It is shown that the solution of the problem for a package of 20–80 homogeneous layers is in good agreement compliant with the analytical solution to the problem for the coating whose the dependence of the heat conductivity coefficient on the coordinate z is described by a exponential (or power) function. This is a strong argument for the possibility of modeling of the gradient coating with continuous variation of thermal properties by a package of homogeneous layers.

It is shown that the solution of the axisymmetric problem of heat conductivity for the half-space with laminated coating of periodic structure heated by heat flux is in good agreement with the solution of the problem in which the coating is modeled by a homogenized coating. The smallest deviations are obtained in finding the temperature and heat flux in the homogeneous substrate.

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Appendix A.

A system of linear equations for determination of the functions $\breve{t}_i(s)$, $i = 0, 1, 2$:

case a):

$$
\begin{aligned}
&-\breve{t}_0(s)+\breve{t}_1(s)+\breve{t}_2(s)=0,\\
&-\breve{t}_0(s)+\alpha^{(-)}s^{-1}\breve{t}_1(s)+\alpha^{(+)}s^{-1}\breve{t}_2(s)=0,\\
&\alpha^{(-)}s^{-1}\breve{t}_1(s)\exp\left(\alpha^{(-)}h\right)+\alpha^{(+)}s^{-1}\breve{t}_2(s)\exp\left(\alpha^{(+)}h\right)=1;
\end{aligned}
\qquad \text{(A1)}
$$

case a′):

$$
\begin{aligned}
&-\breve{t}_0(s)+\breve{t}_1(s)c^pI_p(sc)+\breve{t}_2(s)c^pK_p(sc)=0,\\
&\breve{t}_0(s)\mp\breve{t}_1(s)c^pI_{p-1}(sc)\pm\breve{t}_2(s)c^pK_{p-1}(sc)=0,\\
&\breve{t}_1(s)(c\pm h)^p\,I_{p-1}\left(s(c\pm h)\right)+\\
&-\breve{t}_2(s)(c\pm h)^p\,K_{p-1}\left(s(c\pm h)\right)=\pm 1;
\end{aligned}
\qquad \text{(A2)}
$$

case c):

$$
\begin{aligned}
&-\breve{t}_0(s)+\breve{t}_1(s)\sinh(sp_1h)+\breve{t}_2(s)\cosh(sp_1h)=0,\\
&K_0\left(K_cp_1\right)^{-1}\breve{t}_0(s)+\breve{t}_1(s)\cosh(sp_1h)+\\
&+\breve{t}_2(s)\sinh(sp_1h)=0,\\
&\breve{t}_1(s)=-1;
\end{aligned}
\qquad \text{(A3)}
$$

A system of linear equations for determination of the functions $\breve{t}_i(s)$, $i = 0, 1, \ldots, 2n$ in case b):

$$
\begin{aligned}
&-\breve{t}_0(s)+\breve{t}_1(s)\sinh(sh_1)+\breve{t}_2(s)\cosh(sh_1)=0,\\
&K_0K_1^{-1}\breve{t}_0(s)+\breve{t}_1(s)\cosh(sh_1)+\breve{t}_2(s)\sinh(sh_1)=0,\\
&-\breve{t}_{2i}(s)+\breve{t}_{2i+1}(s)\sinh(sh_{i+1})+\\
&+\breve{t}_{2i+2}(s)\cosh(sh_{i+1})=0,\ i=1,\ldots,n-1,\\
&-K_iK_{i+1}^{-1}\breve{t}_{2i-1}(s)+\breve{t}_{2i+1}(s)\cosh(sh_{i+1})+\\
&+\breve{t}_{2i+2}(s)\sin(sh_{i+1})=0,\ i=1,\ldots,n-1\\
&\breve{t}_{2n-1}(s)=-1;
\end{aligned}
\qquad \text{(A4)}
$$